Considerations about Using Truncation Method to Treat the Singularities when Solving with Higher Order Boundary Elements the Boundary Integral Equation of the Compressible Fluid Flow

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Abstract.: **This paper presents a numerical solution for the problem of the 2D compressible fluid flow around obstacles based on solving the singular boundary integral equation with quadratic boundary elements of lagrangean type. A truncation method is used to evaluate the coefficients that arise due to the singular integrals. The singular boundary equation obtained with the direct boundary element technique is considered in this paper. Some considerations about the truncation method are also made.**

A computer code in MATHCAD is created based on the method proposed. Numerical results are compared with exact ones for some particular cases when exact solutions exist. The comparisons show that even for a small numbers of nodes on the boundary the numerical solutions are in good agreement with the exact ones.

Index Terms: **singular integral equation, quadratic boundary elements, compressible fluid flow, truncation method.**

I. Introduction

There are many papers regarding problems of fluid flows solved with the boundary element method (BEM) because it is a very powerful numerical technique for solving boundary value problems for systems of partial differential equations [1], [2], [3], [4].

The advantage of BEM over other methods arises from the fact that there is no need to mesh the whole domain of the problem but only its boundary and this brings an improvement to the computational efficiency. The advantage is greater when solving problems with unbounded domains, like those of fluid flows around obstacles, because when applying this method there is no need to introduce fictive boundaries at great distances as in case of other methods.

Two techniques can be used to get the boundary formulation of the problem: the direct and the indirect technique. In paper [5] a singular integral equation is obtained by applying the direct technique for the problem of the 2D compressible fluid flow around obstacles. The direct technique has the advantage that uses real unknowns, as the components of the perturbation velocity field.

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It doesn't use the potential or the stream function as in other approaches [1], [2], [3], [4], [6] and this brings a great advantage because there is no need of a derivation process in order to get the quantities of interest.

When solving singular boundary integrals big problems arise when handling the singularities. The boundary element method uses an exact equivalent formulation of the problem and so, when this method is applied, errors arise only because of the boundary discretization and of the numerical schemes use to evaluate the integrals that appear. For usual integrals numerical techniques have been implemented in many math software applications and so we can simply use them to evaluate such kind of integrals, but the evaluation of the singular ones represents a great challenge. Different methods can be used to treat singular integrals but some are easier than others. We mention only some of them (see for more details [7]): ignoring the singularity, the truncation method, the regularization method, the change of variables, etc.

II. Consideration about the Truncation Method

One of the simplest methods that can be applied is the truncation method. When applying it the errors are sometimes bigger than in other cases but, if using higher order boundary elements, it offers good numerical results too.

It can be applied in this case because the singular integral is considered in a Cauchy sense, which implies its definition with a limit process. This definition consists the support in choosing this method for evaluating the singular integrals.

This technique is advantageous because it doesn't need any manipulation or special treatment of the integral or of the kernel. It is suitable for the numerical evaluation of the integrals that don't oscillate near the singularities (see [7]), as in our case. It is easy to apply and it is also easy to implement in a computer code.

As shown in paper [8], in some cases, when the truncation method is used, the numerical results were not as good as we expected. As it is natural, higher order boundary elements offer better results then in case of solving the same boundary integral equation with constant or linear boundary elements.

Higher order boundary elements offer a better approximation for the geometry involved and for the unknown of the problem and so when using them for solving singular boundary integral equations, with singularities defined in the Cauchy sense we can use truncation method to evaluate the singular integrals.

For showing the truncation method efficiency in case of using higher order boundary elements for solving the singular boundary integral equation considered we made an analytical checking of the numerical results. We consider a particular case when exact solution of the problem exists. Comparisons between the numerical solution and the exact one are made and they show good agreement between them and made us to conclude that this method can be successfully applied in this case and in other similar.

For not introducing too many errors due to this treatment of the singularities the number of such evaluations has to be reduced at a minimum value. This aspect has been considered in this paper and it has influenced the accuracy of the numerical results.

III. The Singular Boundary Integral Equation

The problem we want to solve in this paper is that of finding the perturbation produced by an obstacle placed in a steady inviscid compressible fluid flow. Far from the obstacle it is assumed to have a subsonic velocity $U_\infty \bar{i}$, pressure p_∞ and density ρ_∞.

Using dimensionless variables and the fundamental solutions of the system involved and applying the direct boundary element method technique in [5] a singular boundary integral equation is obtained:

$$G(\bar{x}_0)+2\,{}^{/}\!\oint_C \left[u^* n_x^0 + v^* n_y^0\right] G ds + +2 \int_C^{/} M^2 \frac{n_x n_y}{n_x^2+\beta^2 n_y^2}\left(v^* n_x^0 - u^* n_y^0\right) G ds = 2\beta n_y^0 \qquad (1)$$

where $G=(\beta+u)n_y - v n_x$ is the unknown function, n_x, n_y are the components of the normal unit vector noted $\bar{n}$, u, v the components of the dimensionless perturbation velocity, $\beta=\sqrt{1-M^2}$ (*M*= Mach number for the unperturbed motion)., $\bar{x}_0 \in C$, $\bar{n}_0 = \bar{n}(\bar{x}_0)$ and u^*, v^* are the fundamental solutions given by the following relations:

$$u^*(\bar{x},\bar{x}_0)=\frac{1}{2\pi}\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2}, \quad v^*(\bar{x},\bar{x}_0)=\frac{1}{2\pi}\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2} \qquad (2)$$

The sign "′ "denotes the principal value in Cauchy sense of the integral (see[9]).

The principal value in Cauchy sense of an integral is defined as a limit of the implied integral:

$$\int_C^{`} = \lim_{\varepsilon\to 0} \oint_{C-c} \qquad (3)$$

where c represents a small piece of the boundary C situated in the vicinity of $\bar{x}_0 \in C$, given by:

$$c = C \cap D(\bar{x}_0,\varepsilon),$$

$D(\bar{x}_0,\varepsilon)$ being the disc centered in $\bar{x}_0 \in C$ of radius $\varepsilon > 0$.

In [7] the truncation method is presented and it is shown that it offers good results when evaluating singular integrals for which the integrand doesn't oscillate near the singularities.

In our case the integrand has this propriety so we can use this method because it offers good results. We group all terms with singularities into one coefficient in order to use only for one singular integral the truncation method.

Denoting by:

$$\tau = M^2 \frac{n_x n_y}{n_x^2+\beta^2 n_y^2} \qquad (4)$$

we get the following equivalent equation:

$$G(\bar{x}_0)+2\,{}^{/}\!\oint_C \left[u^*\left(n_x^0 - \tau n_y^0\right) + v^*\left(n_y^0 + \tau n_x^0\right)\right] G ds = 2\beta n_y^0 \qquad (5)$$

IV. Quadratic Isoparametric Boundary Elements for Solving the Singular Boundary Integral Equatiion

We use quadratic isoparametric boundary elements of lagrangean type to solve (5) and the truncation method for the treatment of the singularities.

The same boundary integral equation is solved in [10] by using linear boundary elements. A collocation method is used in paper [11] for the case when the ground effect is considered too. In paper [12] quadratic boundary elements are used, but for the singular boundary integrals that arise a regularization method is used. It offers very good results but it is not very easy to apply, and, as we shall see, the truncation method can be a good alternative at it, because it offers, in this case, good results too.

Same kind of boundary elements were used in papers [13] but for solving the singular integral equation obtained by an indirect method with sources distributions on the boundary.

We chose the quadratic boundary elements because they bring some advantages. They offer to the unknown of the problem a global continuity on the boundary, and a better approximation for it and for the geometry of the problem then in case of linear or constant boundary elements.

The boundary is divided into *N* one-dimensional quadratic boundary elements, noted L_i, with three nodes each of them: two extreme nodes and an interior one, noted $\bar{x}_1^i, \bar{x}_2^i, \bar{x}_3^i$, in a local numbering .We have the relations: $\bar{x}_1^{i+1} = \bar{x}_3^i$, $i=\overline{1, N-1}$, and $\bar{x}_3^N = \bar{x}_1^1$ contour *C* being closed. There are necessary 2N nodes on the boundary. We get:

$$G(\bar{x}_0) + 2\sum_{i=1}^{N} \oint_{L_i} \left[u^*\left(n_x^0 - \tau n_y^0\right) + v^*\left(n_y^0 + \tau n_x^0\right)\right] G ds^{/} = 2\beta n_y^0 \qquad (6)$$

Considering that equation (6) is satisfied in every node ($\bar{x}_0 = \bar{x}_j$ in the global notation), we obtain a system of linear equations. Its solution represents the set of the nodal values of the unknown of the problem.

We use two systems of notation: a global and a local one (global-G_j is the value of G for the node number $j, j=\overline{1,2N}$ and local-$G_l^i, l=\overline{1,3}, i=\overline{1,N}$ is the value for the node number l of element i).

When the local system of coordinates is considered to be the intrinsic system, we have for the shape functions the following expressions:

$$N_1(\xi)=\frac{\xi(\xi-1)}{2},\ N_2(\xi)=1-\xi^2,\qquad N_3(\xi)=\frac{\xi(\xi+1)}{2},\quad \xi\in[-1,1] \tag{7}$$

So $x=[N]\{x^i\},\ y=[N]\{y^i\},\ G=[N]\{G^i\}$ (8)

where $[N]$ is the line matrix, $[N]=(N_1\ N_2\ N_3)$, $\{x^i\},\{y^i\}$ the column matrices made with the nodes coordinates of element L_i, and $\{G^i\}$ the column matrix made with the nodal values of G on L_i, $\{G^i\}=(G_1^i\quad G_2^i\quad G_3^i)^t$.

Introducing in equation (6) the considered approximation models we get:

$$\pi G(\bar{x}_j)+\sum_{i=1}^{N}\int_{-1}^{1}\frac{\left(n_x^j-n_y^j\tau(\xi)\right)\left([N]\{x^i\}-x_j\right)}{\left\|\left([N]\{\bar{x}^i\}-\bar{x}_j\right)\right\|^2}[N]\{G^i\}J(\xi)d\xi+$$
$$+\sum_{i=1}^{N}\int_{-1}^{1}\frac{\left(n_y^j+n_x^j\tau(\xi)\right)\left([N]\{y^i\}-y_j\right)}{\left\|\left([N]\{\bar{x}^i\}-\bar{x}_j\right)\right\|^2}[N]\{G^i\}J(\xi)d\xi=$$
$$=2\pi Bn_y^j \tag{9}$$

where

$$J(\xi)=\sqrt{4a_i\xi^2+2b_i\xi+aa_i},\quad a_i=\frac{m_i^2+M_i^2}{4},$$
$$aa_i=\frac{n_i^2+N_i^2}{4},\quad b_i=\frac{m_in_i+M_iN_i}{2}$$

Making the following notations:

$$nx\,\tau_{ij}(\xi)=\left(n_x^j-n_y^j\tau(\xi)\right)\left([N]\{x^i\}-x_j\right)$$
$$Ny\,\tau_{ij}(\xi)=\left(n_y^j+n_x^j\tau(\xi)\right)\left([N]\{y^i\}-y_j\right) \tag{10}$$

we obtain:

$$\pi G(\bar{x}_j)+\sum_{i=1}^{N}\int_{-1}^{1}\frac{nx\,\tau_{ij}(\xi)+Ny\,\tau_{ij}(\xi)}{\left\|\left([N]\{\bar{x}^i\}-\bar{x}_j\right)\right\|^2}[N]\{G^i\}J(\xi)d\xi=$$
$$=2\pi\beta n_y^j \tag{11}$$

and further

$$\pi G(\bar{x}_j)+\sum_{i=1}^{N}\sum_{l=1}^{3}G_l^i\int_{-1}^{1}\frac{nx\,\tau_{ij}(\xi)+Ny\,\tau_{ij}(\xi)}{\left\|\left([N]\{\bar{x}^i\}-\bar{x}_j\right)\right\|^2}N_lJ(\xi)d\xi=$$
$$=2\pi\beta n_y^j \tag{12}$$

After evaluating the integrals in (12) we reduce the problem to the following system of equations:

$$\pi G_j+\sum_{i=1}^{N}\left(\sum_{l=1}^{3}G_l^ia_{ij}^l\right)=2\pi\beta n_y^j, \tag{13}$$

where

$$a_{ij}^l=\int_{-1}^{1}\frac{nx\,\tau_{ij}(\xi)+Ny\,\tau_{ij}(\xi)}{\left\|\left([N]\{\bar{x}^i\}-\bar{x}_j\right)\right\|^2}N_l(\xi)J(\xi)d\xi. \tag{14}$$

Some of the coefficients a_{ij}^l are given by usual integrals, which can be evaluated using ordinary rules or by any math software, but others are singular and arise from integrals considered in a Cauchy sense. For those we apply the truncation technique in order to evaluate them.

We first evaluate the coefficients that arise from the nonsingular integrals. For them we can use the expressions given in [13], but we prefer a uniform approach.

After doing some calculus, and denoting by:

$$m_i=x_1^i+x_3^i-2x_2^i,\ n_i=x_3^i-x_1^i,\ u_{ij}=x_2^i-x_j,$$
$$M_i=y_1^i+y_3^i-2y_2^i,\quad N_i=y_3^i-y_1^i$$
$$U_{ij}=y_2^i-y_j,$$
$$c_{ij}=aa_i+m_iu_{ij}+M_iU_{ij},\ d_{ij}=n_iu_{ij}+N_iU_{ij},$$
$$e_{ij}=u_{ij}^2+U_{ij}^2,\quad i,j=\overline{1,2N},$$
$$N_{ij}=a_i\xi^4+b_i\xi^3+c_{ij}\xi^2+d_{ij}\xi+e_{ij},$$
$$p_i=-4m_iM_i\ r_i=-2(m_iN_i+n_iM_i)\ s_i=-n_iN_i,$$
$$Nm_i(\xi)\underset{not}{=}4\left(M_i^2+\beta^2m_i^2\right)\xi^2+4\left(M_iN_i+\beta^2m_in_i\right)\xi+$$
$$+N_i^2+n_i^2 \tag{15}$$

we get:

$$\tau(\xi)=M^2\frac{p_i\xi^2+r_i\xi+s_i}{Nm_i(\xi)}, \tag{16}$$

and the expressions for the coefficients in (13).

For l=1 we obtain:

$$a_{ij}^1=\int_{-1}^{1}\left[ax\,\tau_{ij}^1(\xi)+ay\,\tau_{ij}^1(\xi)\right]J(\xi)d\xi, \tag{17}$$

where

$$ax\,\tau_{ij}^1(\xi)=\frac{\left[n_x^jNm_i-n_y^jM^2\left(p_i\xi^2+r_i\xi+s_i\right)\right]nr1_{ij}}{4Nm_i(\xi)N_{ij}(\xi)},$$
$$nr1_{ij}(\xi)=m_i\xi^4+(n_i-m_i)\xi^3+(2u_{ij}-n_i)\xi^2-2u_i\xi,$$
$$ay\,\tau_{ij}^1(\xi)=\frac{\left[n_y^jNm_i+n_x^jM^2\left(p_i\xi^2+r_i\xi+s_i\right)\right]Nr1_{ij}}{4Nm_i(\xi)N_{ij}(\xi)},$$
$$Nr1_{ij}(\xi)=M_i\xi^4+(N_i-M_i)\xi^3+(2U_{ij}-N_i)\xi^2-2U_i\xi$$

For l=2 we get:

$$a_{ij}^2=\int_{-1}^{1}\left[ax\,\tau_{ij}^2(\xi)+ay\,\tau_{ij}^2(\xi)\right]J(\xi)d\xi, \tag{18}$$

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where

$$ax\tau_{ij}^2(\xi)=\frac{\left[n_x^j Nm_i - n_y^j M^2\left(p_i\xi^2 + r_i\xi + s_i\right)\right]nr2_{ij}}{4Nm_i(\xi)N_{ij}(\xi)}$$

$$nr2_{ij}(\xi) = -m_i\xi^4 - (n_i)(\xi^3 - \xi) - (2u_{ij} - m_i)\xi^2 + 2u_{ij}$$

$$ay\tau_{ij}^2(\xi)=\frac{\left[n_y^j Nm_i + n_x^j M^2\left(p_i\xi^2 + r_i\xi + s_i\right)\right]Nr2_{ij}}{4Nm_i(\xi)N_{ij}(\xi)}$$

$$Nr2_{ij}(\xi) = -M_i\xi^4 - (N_i)(\xi^3 - \xi) + (2U_{ij} - M_i)\xi^2 + 2U_{ij}$$

The last coefficient arisen from nonsingular integrals is:

$$a_{ij}^3 = \int_{-1}^{1}\left[ax\tau_{ij}^3(\xi) + ay\tau_{ij}^3(\xi)\right]J(\xi)d\xi, \qquad (19)$$

where

$$ax\tau_{ij}^3(\xi)=\frac{\left[n_x^j Nm_i - n_y^j M^2\left(p_i\xi^2 + r_i\xi + s_i\right)\right]nr3_{ij}}{4Nm_i(\xi)N_{ij}(\xi)}$$

$$nr3_{ij}(\xi) = m_i\xi^4 + (n_i + m_i)\xi^3 + (2u_{ij} + n_i)\xi^2 + 2u_{ij}\xi$$

$$ay\tau_{ij}^3(\xi)=\frac{\left[n_y^j Nm_i + n_x^j M^2\left(p_i\xi^2 + r_i\xi + s_i\right)\right]Nr3_{ij}}{4Nm_i(\xi)N_{ij}(\xi)}$$

$$Nr3_{ij}(\xi) = M_i\xi^4 + (N_i + M_i)\xi^3 + (2U_{ij} + N_i)\xi^2 + 2U_{ij}\xi$$

For evaluating the singular coefficients, based on the truncation method, we consider a very small parameter eps > 0 in order to reduce the domain of integration by eliminating the singularity.

Taking into account the cases when the singularity arises we get the following three situations.

When j is the first node of element L_i ($j = 2i - 1$) the singularity arises when $\xi = -1$. The coefficients have in this case the same expressions but the inferior limit of the integral becomes -1+eps. So we have relations:

$$a_{ij}^l = \int_{-1+eps}^{1}\left[ax\tau_{ij}^l(\xi) + ay\tau_{ij}^l(\xi)\right]J(\xi)d\xi, \quad l = \overline{1,3} \qquad (20)$$

When j is the second node of element L_i ($j = 2i$) the singularity arises when $\xi = 0$. The coefficients have in this case expressions:

$$a_{ij}^l = \int_{-1}^{-eps/2}\left[ax\tau_{ij}^l(\xi) + ay\tau_{ij}^l(\xi)\right]J(\xi)d\xi + \int_{eps/2}^{1}\left[ax\tau_{ij}^l(\xi) + ay\tau_{ij}^l(\xi)\right]J(\xi)d\xi +, \quad l = \overline{1,3} \qquad (21)$$

Finally, when j is the third node of element L_i ($j = 2i + 1$), the singularity arises when $\xi = 1$, and so we have :

$$a_{ij}^l = \int_{-1}^{-eps}\left[ax\tau_{ij}^l(\xi) + ay\tau_{ij}^l(\xi)\right]J(\xi)d\xi, \quad l = \overline{1,3} \qquad (22)$$

Returning to the global system of notation we can write system (13) as:

$$\pi G_j + \sum_{i=1}^{2N+1} A_{ji}G_i = B_j, \quad j = \overline{1,2N} \qquad (23)$$

where

$$A_{ji} = \begin{cases} a_{kj}^2, \ if\ i = 2k,\ k = \overline{1,N} \\ a_{k+1\,j}^1 + a_{kj}^3, \ if\ i = 2k+1,\ k = \overline{1,N-1} \\ a_{1j}^1 + a_{Nj}^3, \ if\ i = 1 \end{cases}$$

$$B_j = 2\pi\beta n_y^j, \ j = \overline{1,2N} \qquad (24)$$

We finally get an equivalent form for the above system:

$$\sum_{i=1}^{2N+1} AA_{ji}G_i = B_j \ \ j = \overline{1,2N} \qquad (25)$$

where:

$$AA_{ji} = \begin{cases} A_{ji}, \ i \neq j \\ \pi + A_{jj}, \ i = j \end{cases} \qquad (26)$$

the unknowns being the nodal values of G: $G_1, G_2, \ldots G_{2N}$.

Solving this system we find the nodal values of function G, and then the velocity components with the following relations:

$$\beta + u_i = \frac{\beta^2 n_y(\bar{x}_i^0)G_i}{n_x^2(\bar{x}_i^0) + \beta^2 n_y^2(\bar{x}_i^0)},$$

$$v_i = -\frac{n_x(\bar{x}_i^0)G_i}{n_x^2(\bar{x}_i^0) + \beta^2 n_y^2(\bar{x}_i^0)}. \qquad (27)$$

V. Numerical Results and Conclusions

All coefficients in system (25) have analytic expressions and depend only on the nodes coordinates, so we can use a computer for their evaluation. The method presented in this paper has been implemented into a computer code made in MATHCAD.

In order to test the method we make an analytical checking. We consider an inviscid fluid flow in the presence of a circular obstacle. In this case there is known the exact solutions of the problem, which can be found in many books, as for example [14]. We consider the incompressible case, so $\beta = 1$.

The code we proposed, in MATHCAD, calculates the coefficients of the system, the unknowns, the components of the velocity and the pressure coefficient in the points situated on the boundary. For the boundary discretization we have used only 20 nodes. For the truncation parameter, eps, we have considered a very small value, eps=0.00001.

The results obtained are presented in the graphics bellow. The first two graphics make a comparison between the exact and numeric values of the components of the velocity field and the last one for the pressure coefficient.

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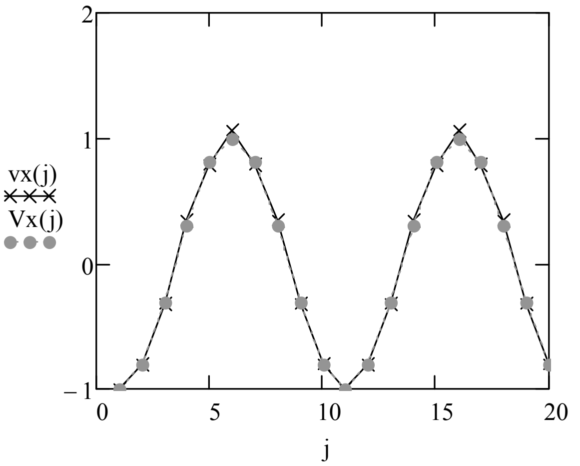

Fig.1. The velocity along Ox (vx- numerical solution, Vx- exact solution)- circular obstacle.

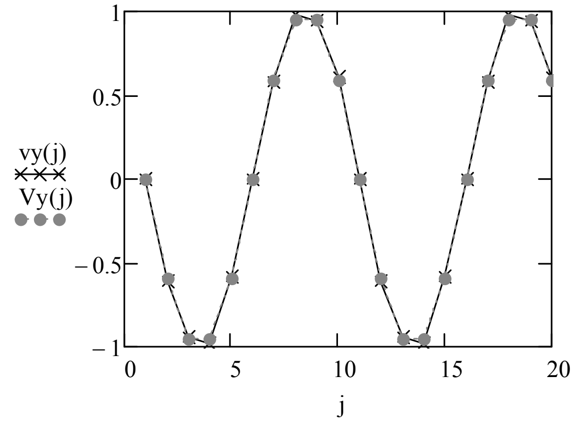

Fig.2. The velocity along Oy (vy- numerical solution, Vy- exact solution)-circular obstacle.

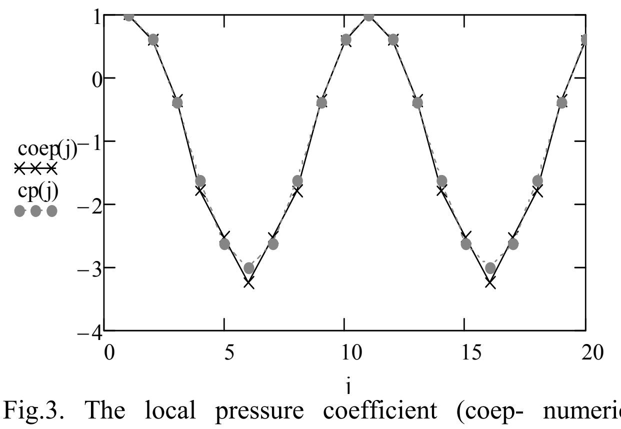

Fig.3. The local pressure coefficient (coep- numerical solution, cp-exact solution)-circular obstacle.

Even we have considered only 20 nodes on the boundary, the computer code we made can be adapted to run for different number of nodes chosen on the boundary, and also for other kind of smooth obstacles.

From figures 1, 2 and 3 we can observe that the calculated and the analytical values of the two components of the velocity field and the values of the pressure coefficient (Fig. 1 for the velocity along Ox, Fig 2 for the velocity along Oy, and Fig. 3 for the local pressure coefficient) are in good agreement, fact that shows the great accuracy of the method proposed.

Better results are expected when more nodes are chosen for the boundary discretization or when grater values for eps are considered.

So, even if the truncation method is a very simple method to treat the singularities, it can be successfully applied when using higher order boundary elements to solve problems with BEM, especially when the integrals that appear are defined in a Cauchy principal sense and when having integrands that don't oscillate near the singularities.

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