Proceedings of the World Congress on Engineering 2012 Vol II
WCE 2012, July 4 - 6, 2012, London, U.K.

Intrinsic Order and Hamming Weight

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***Abstract*—The intrinsic order is a partial order relation defined on the set $\{0,1\}^n$ of all binary n-tuples. This ordering enables one to automatically compare binary n-tuple probabilities without computing them, just looking at the relative positions of their 0s & 1s. In this paper, new relations between the intrinsic ordering and the Hamming weight (i.e., the number of 1-bits in a binary n-tuple) are derived. All theoretical results are rigorously proved and illustrated through the intrinsic order graph.**

***Index Terms*—complex stochastic Boolean system, Hamming weight, intrinsic order, intrinsic order graph.**

I. Introduction

MANY different phenomena, arising from scientific, technical or social areas, can be modeled by a system depending on a certain number n of random Boolean variables. The so-called complex stochastic Boolean systems (CSBSs) are characterized by the ordering between the occurrence probabilities $\Pr\{u\}$ of the 2^n associated binary strings $u=(u_1,\ldots,u_n)\in\{0,1\}^n$ of length n.

According to the usual terminology in Statistics, a CSBS on n variables $x_1,\ldots,x_n$ can be modeled by the n-dimensional Bernoulli distribution with parameters $p_1,\ldots,p_n$ defined by

$$\Pr\{x_i=1\}=p_i,\quad \Pr\{x_i=0\}=1-p_i.$$

Assuming that the n Bernoulli variables x_i are statistically independent, then the occurrence probability of a given binary string of length n, $u=(u_1,\ldots,u_n)\in\{0,1\}^n$, can be easily computed as

$$\Pr\{u\}=\prod_{i=1}^{n}p_i^{u_i}\left(1-p_i\right)^{1-u_i}, \tag{1}$$

that is, $\Pr\{u\}$ is the product of factors p_i if $u_i=1$, $1-p_i$ if $u_i=0$.

Example 1.1: Let $n=4$ and $u=(0,1,1,0)\in\{0,1\}^4$. Let $p_1=0.1$, $p_2=0.2$, $p_3=0.3$, $p_4=0.4$. Then using (1), we have

$$\Pr\{(0,1,1,0)\}=(1-p_1)\,p_2p_3\,(1-p_4)=0.0324.$$

The behavior of a CSBS is determined by the ordering between the current values of the 2^n associated binary n-tuple probabilities $\Pr\{u\}$. Computing all these 2^n probabilities –by using (1)– and ordering them in decreasing or increasing order of their values is only possible in practice for small values of the number n of basic variables. However, for large values of n, to overcome the exponential nature of this problem, we need alternative procedures for comparing the binary string probabilities. For this purpose, in [1] we have defined a partial order relation on the set $\{0,1\}^n$ of all the 2^n binary n-tuples, the so-called *intrinsic order* between binary n-tuples.

The intrinsic order enables one to easily compare (order) two given binary n-tuple probabilities $\Pr\{u\},\Pr\{v\}$, without computing them, simply looking at the positions of the 0s and 1s in the binary n-tuples u,v.

In this way, for those pairs (u,v) of binary n-tuples comparable by intrinsic order, the ordering between their occurrence probabilities is always the same for all sets of basic probabilities $\{p_i\}_{i=1}^n$. On the contrary, for those pairs (u,v) of binary n-tuples incomparable by intrinsic order, the ordering between their occurrence probabilities depends on the current values of the basic probabilities $\{p_i\}_{i=1}^n$.

The usual representation for a CSBS is the intrinsic order graph. This is a directed graph on 2^n nodes (denoted by I_n) that displays all the binary n-tuples from top to bottom in decreasing order of their occurrence probabilities. Formally, I_n is the Hasse diagram of the intrinsic partial order relation on $\{0,1\}^n$.

Let us recall that the Hamming weight of a binary n-tuple $u\in\{0,1\}^n$ is the sum of all its bits, that is, the number of 1-bits in u.

The aim of this paper is to derive new relations between the intrinsic order relation and the Hamming weight. For this purpose, this paper has been organized as follows. In Section II, we present all previous results about the intrinsic order required to make this paper self-contained. Section III is devoted to present the new relations between the intrinsic ordering and the Hamming weight. Finally, conclusions are presented in Section IV.

II. Intrinsic Order

Let $u=(u_1,\ldots,u_n)$ be a binary n-tuple. In the following, the decimal numbering of u is denoted by $u_{(10}$, i.e.,

$$u_{(10}=\sum_{i=1}^{n}2^{n-i}u_i.$$

The lexicographic order on $\{0,1\}^n$, denoted here by the symbol "$\leq_{lex}$", is the usual truth-table order, beginning with the n-tuple $(0,\ldots,0)$ and finishing with the n-tuple $(1,\ldots,1)$. As is well-known, it coincides with the natural order between the decimal representations of the bitstrings. That is,

$$u\leq_{lex}v\Leftrightarrow u \text{ precedes } v \text{ in the truth-table } \Leftrightarrow u_{(10}\leq v_{(10}.$$

Throughout this paper, we indistinctly denote the n-tuple $u\in\{0,1\}^n$ by its binary representation $(u_1,\ldots,u_n)$ or by its decimal representation $u_{(10}$, and we use the symbol "$\equiv$" to indicate the conversion between them, i.e.,

$$u=(u_1,\ldots,u_n)\equiv u_{(10}=\sum_{i=1}^{n}2^{n-i}u_i.$$

Manuscript received March 9, 2012; revised April 1, 2012. This work was supported in part by the Spanish Government, "Ministerio de Economía y Competividad", and FEDER, through Grant contract: CGL2011-29396-C03-01.

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ISBN: 978-988-19252-1-3
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

The Hamming weight –or simply the weight– of u is the sum of all its n bits. In other words, the Hamming weight of a binary n-tuple is the number of its 1-bits, and it will be denoted by

$$w_H(u) = \sum_{i=1}^{n} u_i.$$

Example 2.1: For $n = 5$, we have

$$u = (0,0,1,1,0) \equiv u_{(10} = 2^1 + 2^2 = 6.$$
$$v = (0,1,0,1,1) \equiv v_{(10} = 2^0 + 2^1 + 2^3 = 11.$$

$u \leq_{lex} v$ since u precedes v in the truth-table or since

$$u_{(10} = 6 \leq 11 = v_{(10}.$$

Finally,

$$w_H(u) = 2, \qquad w_H(v) = 3.$$

Given two binary n-tuples $u, v \in \{0,1\}^n$, the ordering between their occurrence probabilities $\Pr(u)$, $\Pr(v)$ obviously depends on the Bernoulli parameters p_i, as the following simple example shows.

Example 2.2: Let $n = 3$, $u = (0,1,1)$ and $v = (1,0,0)$. For $p_1 = 0.1$, $p_2 = 0.2$, $p_3 = 0.3$, using (1), we have:

$$\Pr\{(0,1,1)\} = 0.054 < \Pr\{(1,0,0)\} = 0.056,$$

for $p_1 = 0.2$, $p_2 = 0.3$, $p_3 = 0.4$, using (1), we have:

$$\Pr\{(0,1,1)\} = 0.096 > \Pr\{(1,0,0)\} = 0.084.$$

However, as mentioned in Section I, in [1] we have established an intrinsic, positional criterion to compare the occurrence probabilities of two given binary n-tuples without computing them. This criterion is presented in detail in Section II-*A*, while its graphical representation is shown in Section II-*B*.

A. The Intrinsic Order Relation

Theorem 2.1: (The intrinsic order theorem) Let $n \geq 1$. Let $x_1, \ldots, x_n$ be n mutually independent Bernoulli variables whose parameters $p_i = \Pr\{x_i = 1\}$ satisfy

$$0 < p_1 \leq p_2 \leq \cdots \leq p_n \leq 0.5. \tag{2}$$

Then the occurrence probability of the binary n-tuple v, i.e., $v = (v_1, \ldots, v_n) \in \{0,1\}^n$, is *intrinsically* less than or equal to the occurrence probability of the binary n-tuple u, i.e., $u = (u_1, \ldots, u_n) \in \{0,1\}^n$, (that is, for all set $\{p_i\}_{i=1}^n$ satisfying (2)) if and only if the matrix

$$M_v^u := \begin{pmatrix} u_1 & \ldots & u_n \\ v_1 & \ldots & v_n \end{pmatrix}$$

either has no $\binom{1}{0}$ columns, or for each $\binom{1}{0}$ column in M_v^u there exists (at least) one corresponding preceding $\binom{0}{1}$ column (IOC).

Remark 2.1: In the following, we assume that the parameters p_i always satisfy condition (2). Fortunately, this hypothesis is not restrictive for practical applications.

Remark 2.2: The $\binom{0}{1}$ column preceding each $\binom{1}{0}$ column is not required to be necessarily placed at the immediately previous position, but just at previous position.

Remark 2.3: The term *corresponding*, used in Theorem 2.1, has the following meaning: For each two $\binom{1}{0}$ columns in matrix M_v^u, there must exist (at least) two *different* $\binom{0}{1}$ columns preceding each other. In other words, for each $\binom{1}{0}$ column in matrix M_v^u the number of preceding $\binom{0}{1}$ columns must be strictly greater than the number of preceding $\binom{1}{0}$ columns.

The matrix condition IOC, stated by Theorem 2.1, is called the *intrinsic order criterion*, because it is independent of the basic probabilities p_i and it only depends on the relative positions of the 0s and 1s in the binary strings u and v. Theorem 2.1 naturally leads to the following partial order relation on the set $\{0,1\}^n$ [1], [2]. The so-called intrinsic order will be denoted by "$\preceq$", and when $v \preceq u$ we say that *v is intrinsically less than or equal to u* (or *u is intrinsically greater than or equal to v*).

Definition 2.1: For all $u, v \in \{0,1\}^n$

$$v \preceq u \ \text{ iff } \ \Pr\{v\} \leq \Pr\{u\} \ \text{ for all set } \{p_i\}_{i=1}^n \text{ s.t. (2)}$$
$$\text{iff matrix } M_v^u \text{ satisfies IOC.}$$

In the following, the partially ordered set (poset, for short) for n variables $(\{0,1\}^n, \preceq)$ will be denoted by I_n; see [10] for more details about posets.

Example 2.3: For $n = 3$:

$$3 \equiv (0,1,1) \npreceq (1,0,0) \equiv 4 \ \& \ (1,0,0) \npreceq (0,1,1) \ \text{ since}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

do not satisfy IOC (Remark 2.3). Therefore, $(0,1,1)$ and $(1,0,0)$ are incomparable by intrinsic order, i.e., the ordering between $\Pr\{(0,1,1)\}$ and $\Pr\{(1,0,0)\}$ depends on the basic probabilities p_i, as Example 2.2 has shown.

Example 2.4: For $n = 4$:

$$9 \equiv (1,0,0,1) \preceq (0,0,1,1) \equiv 3 \ \text{since}$$

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

satisfies IOC (Remark 2.2). For all $0 < p_1 \leq \cdots \leq p_4 \leq \frac{1}{2}$

$$\Pr\{(1,0,0,1)\} \leq \Pr\{(0,0,1,1)\}.$$

Example 2.5: For all $n \geq 1$, the binary n-tuples

$$\left(0, \overset{n}{\ldots}, 0\right) \equiv 0 \quad \text{and} \quad \left(1, \overset{n}{\ldots}, 1\right) \equiv 2^n - 1$$

are the maximum and minimum elements, respectively, in the poset I_n. Indeed, for all $(u_1, \ldots, u_n) \in \{0,1\}^n$, both matrices

$$\begin{pmatrix} 0 & \ldots & 0 \\ u_1 & \ldots & u_n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u_1 & \ldots & u_n \\ 1 & \ldots & 1 \end{pmatrix}$$

satisfy IOC, since they have no $\binom{1}{0}$ columns!.
Thus, for all $u \in \{0,1\}^n$ and for all $\{p_i\}_{i=1}^n$ s.t. (2)

$$\Pr\left\{\left(1, \overset{n}{\ldots}, 1\right)\right\} \leq \Pr\{(u_1, \ldots, u_n)\} \leq \Pr\left\{\left(0, \overset{n}{\ldots}, 0\right)\right\}.$$

Many different properties of the intrinsic order relation can be derived from its simple matrix description IOC (see, e.g., [1], [2], [3]). For the purpose of this paper, we must recall here the following two necessary (but not sufficient) conditions for intrinsic order.

ISBN: 978-988-19252-1-3
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Corollary 2.1: The intrinsic order respects the lexicographic order. That is, for all $u, v \in \{0,1\}^n$

$$u \succeq v \quad \Rightarrow \quad u_{(10} \leq v_{(10}.$$

Corollary 2.2: The intrinsic order respects the Hamming weight. That is, for all $u, v \in \{0,1\}^n$

$$u \succeq v \quad \Rightarrow \quad w_H(u) \leq w_H(v).$$

Let us briefly recall the simple idea of one the possible proofs of Corollary 2.2. If $u \succeq v$ then, according to IOC (see Definition 2.1 and Theorem 2.1), in matrix M_v^u the number of $\binom{0}{1}$ columns must be greater than or equal to the number of $\binom{1}{0}$ columns. Hence the total number of both $\binom{0}{1}$ & $\binom{1}{1}$ columns must be greater than or equal to the total number of both $\binom{1}{0}$ & $\binom{1}{1}$ columns in matrix M_v^u. This is equivalent to saying that the number of 1-bits in v must be greater than or equal to the number of 1-bits in u, i.e., $w_H(v) \geq w_H(u)$.

B. The Intrinsic Order Graph

In this subsection, the graphical representation of the poset $I_n = (\{0,1\}^n, \preceq)$ is presented. The usual representation of a poset is its Hasse diagram (see [10] for more details about these diagrams). Specifically, for our poset I_n, its Hasse diagram is a directed graph (digraph, for short) on 2^n vertices, namely the 2^n binary n-tuples of 0s and 1s.

In the Hasse diagram of I_n, u is intrinsically greater than v (i.e., $u \succ v$) if and only if u and v are connected either by one edge or by a longer descending path from u to v.

The Hasse diagram of the poset I_n will be also called the *intrinsic order graph* for n variables, denoted as well by I_n. In [3] we have developed a recursive algorithm for iteratively building up I_n (for all $n \geq 2$) from I_1 (depicted in Fig. 1). Fig. 2 illustrates this iterative process for the first few values of n, denoting all the binary n-tuples by their decimal equivalents.

Fig. 1. The intrinsic order graph for $n = 1$.

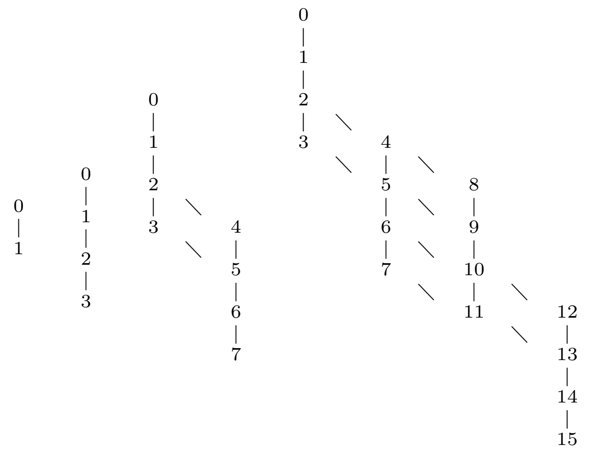

Fig. 2. The intrinsic order graphs for $n = 1, 2, 3, 4$.

Each pair (u, v) of vertices connected in I_n either by one edge or by a longer descending path from u to v, means that u is intrinsically greater than v, i.e., $u \succ v$. For instance, looking at the Hasse diagram of I_4, the right-most one in Fig. 2, we observe that $3 \equiv (0,0,1,1) \succ 9 \equiv (1,0,0,1)$, in accordance with Example 2.4.

On the contrary, each pair (u, v) of non-connected vertices in I_n either by one edge or by a longer descending path, means that u and v are incomparable by intrinsic order, i.e., $u \not\succ v$ and $v \not\succ u$. For instance, looking at the Hasse diagram of I_3, the third one from left to right in Fig. 2, we observe that $3 \equiv (0,1,1)$ and $4 \equiv (1,0,0)$ are incomparable by intrinsic order, in accordance with Example 2.3.

The edgeless graph for a given graph is obtained by removing all its edges, keeping its nodes at the same positions. In Fig. 3, the edgeless intrinsic order graph of I_5 is depicted.

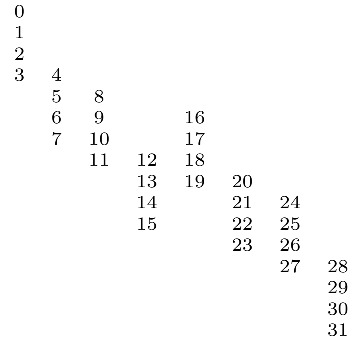

Fig. 3. The edgeless intrinsic order graph for $n = 5$.

For further theoretical properties and practical applications of the intrinsic order and the intrinsic order graph, we refer the reader to [4], [5], [6], [7], [8], [9].

III. Intrinsic Order and Hamming Weight

Now, we present some new relations between the intrinsic ordering and the Hamming weight. Our starting point is Corollary 2.2. This proposition has stated that a necessary condition for u being intrinsically greater than or equal to v is that the weight of u must be less than or equal to the weight of v. That is, let u be an arbitrary, but fixed, binary n-tuple. Then

$$u \succeq v \quad \Rightarrow \quad w_H(u) \leq w_H(v) \quad \text{for all } v \in \{0,1\}^n \tag{3}$$

or, equivalently,

$$w_H(u) > w_H(v) \quad \Rightarrow \quad u \not\succeq v.$$

For instance, looking at the digraph I_4, the right-most one in Fig. 2, we can confirm that

$$4 \equiv (0,1,0,0) \succ 13 \equiv (1,1,0,1),$$
$$w_H(4) = 1 < 3 = w_H(13)$$

and that

$$3 \equiv (0,0,1,1) \succ 12 \equiv (1,1,0,0),$$
$$w_H(3) = 2 = w_H(12).$$

However, the necessary condition for intrinsic order stated by Corollary 2.2 is not sufficient. That is,

$$w_H(u) \leq w_H(v) \nRightarrow u \succeq v,$$

as the following simple counter-example (indeed, the simplest one that one can find!) shows.

Example 3.1: For

$$n = 3, \quad u = 4 \equiv (1,0,0), \quad v = 3 \equiv (0,1,1),$$

ISBN: 978-988-19252-1-3
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

we have (see the digraph of I_3 in Fig. 2)

$$w_H(4) = 1 < 2 = w_H(3).$$

However $4 \not\succeq 3$, since matrix

$$M_3^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

does not satisfy IOC (or, more easily, since $4 > 3$; see Corollary 2.1).

Moreover, even assuming that the two necessary conditions stated by Corollaries 2.1 & 2.2 simultaneously hold, this does not imply intrinsic order. That is,

$$u_{(10} < v_{(10} \text{ and } w_H(u) \leq w_H(v) \nRightarrow u \succeq v,$$

as the following simple counter-example (indeed, the simplest one that one can find!) shows.

Example 3.2: For

$$n = 4, \quad u = 6 \equiv (0,1,1,0), \quad v = 9 \equiv (1,0,0,1),$$

we have (see the digraph of I_4 in Fig. 2)

$$u_{(10} = 6 < v_{(10} = 9 \text{ and } w_H(6) = 2 = w_H(9).$$

However $6 \not\succeq 9$, since matrix

$$M_9^6 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

does not satisfy IOC.

Moreover, even though assuming that the Hamming weight of u is strictly less than the Hamming weight of v, the two necessary conditions stated by Corollaries 2.1 & 2.2 do not imply intrinsic order. That is,

$$u_{(10} < v_{(10} \text{ and } w_H(u) < w_H(v) \nRightarrow u \succeq v,$$

as the following simple counter-example (indeed, the simplest one that one can find!) shows.

Example 3.3: For

$$n = 5, \quad u = 12 \equiv (0,1,1,0,0), \quad v = 19 \equiv (1,0,0,1,1),$$

we have

$$u_{(10} = 12 < v_{(10} = 19 \text{ and } w_H(12) = 2 < 3 = w_H(19).$$

However $12 \not\succeq 19$, since matrix

$$M_{19}^{12} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

does not satisfy IOC.

In this context, two dual questions naturally arise. They are posed in the two subsections of this section. First, we need to set the following notations.

Definition 3.1: For every binary n-tuple $u \in \{0,1\}^n$, C^u (C_u, respectively) is the set of all binary n-tuples v whose occurrence probabilities $\Pr\{v\}$ are always less (greater, respectively) than or equal to $\Pr\{u\}$, i.e., those n-tuples v intrinsically less (greater, respectively) than or equal to u, i.e.,

$$\begin{aligned} C^u &= \{v \in \{0,1\}^n \mid \Pr\{u\} \geq \Pr\{v\}, \forall \{p_i\}_{i=1}^n \text{ s.t. (2)}\} \\ &= \{v \in \{0,1\}^n \mid u \succeq v\}, \end{aligned}$$

$$\begin{aligned} C_u &= \{v \in \{0,1\}^n \mid \Pr\{u\} \leq \Pr\{v\}, \forall \{p_i\}_{i=1}^n \text{ s.t. (2)}\} \\ &= \{v \in \{0,1\}^n \mid u \preceq v\}. \end{aligned}$$

Definition 3.2: For every binary n-tuple $u \in \{0,1\}^n$, H^u (H_u, respectively) is the set of all binary n-tuples v whose Hamming weights are less (greater, respectively) than or equal to the Hamming weight of u, i.e.,

$$\begin{aligned} H^u &= \{v \in \{0,1\}^n \mid w_H(u) \geq w_H(v)\}, \\ H_u &= \{v \in \{0,1\}^n \mid w_H(u) \leq w_H(v)\}. \end{aligned}$$

A. When Greater Weight is Equivalent to Less Probability?

Looking at the implication (3), the following question immediately arises.

Question 3.1: Can we characterize the binary n-tuples u for which the necessary condition (3) is also sufficient? That is, we try to identify those bitstrings $u \in \{0,1\}^n$ for which the set of binary n-tuples v with weights greater than or equal to the one of u coincides with the set of binary n-tuples v with occurrence probabilities less than or equal to the one of u, i.e.,

$$u \succeq v \quad \Leftrightarrow \quad w_H(u) \leq w_H(v), \text{ i.e., } C^u = H_u.$$

The following theorem provides the answer to this question, in a very simple way.

Theorem 3.1: Let $n \geq 1$ and $u = (u_1, \ldots, u_n) \in \{0,1\}^n$ with Hamming weight $w_H(u) = m$ ($0 \leq m \leq n$). Then

$$C^u = H_u$$

if and only if either u is the zero n-tuple ($m = 0$) or the m 1-bits of u ($m > 0$) are placed at the m right-most positions, i.e., if and only if u has the general pattern

$$u = \left(0, \overset{n-m}{\ldots}, 0, 1, \overset{m}{\ldots}, 1\right) \equiv 2^m - 1, \quad 0 \leq m \leq n, \qquad (4)$$

where any (but not both!) of the above two subsets of bits grouped together can be omitted.

Proof: Sufficient condition. We distinguish two cases:
(i) If u is the zero n-tuple $0 \equiv \left(0, \overset{n}{\ldots}, 0\right)$, then u is the maximum element for the intrinsic order (as we have proved in Example 2.5). Then

$$\begin{aligned} C^0 &= \{v \in \{0,1\}^n \mid 0 \succeq v\} = \{0,1\}^n \\ &= \{v \in \{0,1\}^n \mid w_H(0) = 0 \leq w_H(v)\} = H_0. \end{aligned}$$

(ii) If u is not the zero n-tuple, then u has the pattern (4) with $m > 0$. Let $v \in H_u$, i.e., let v let a binary n-tuple with Hamming weight greater than or equal to m (the Hamming weight of u). We distinguish two subcases:
(ii)-(a) Suppose that the weight of v is

$$w_H(v) = m = w_H(u).$$

Then v has exactly m 1-bits and $n - m$ 0-bits. Call r the number of 1-bits of v placed among the m right-most positions ($\max\{0, 2m - n\} \leq r \leq m$). Obviously, v has r 1-bits and $m - r$ 0-bits placed among the m right-most positions, and also it has $m - r$ 1-bits and $n - 2m + r$ 0-bits placed among the $n - m$ left-most positions. These are the positions of the

$$r + (m - r) + (m - r) + (n - 2m + r) = m + (n - m) = n$$

bits of the binary n-tuple v.

Hence, matrix M_v^u has exactly $m - r$ $\binom{1}{0}$ columns (all placed among the m right-most positions) and exactly $m - r$

ISBN: 978-988-19252-1-3
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$\binom{0}{1}$ columns (all placed among the $n-m$ left-most positions). Thus, M_v^u satisfies IOC and then $u \succeq v$, i.e., $v \in C^u$.
So, for this case (ii)-(a), we have proved that

$$\{v \in \{0,1\}^n \mid w_H(v) = w_H(u) = m\} \subseteq C^u \tag{5}$$

(ii)-(b) Suppose that the weight of v is

$$w_H(v) = m + p > m = w_H(u) \quad (0 < p \leq n - m).$$

Then define a new binary n-tuple s as follows. First, select any p 1-bits in v (say, for instance, $v_{i_1} = \cdots = v_{i_p} = 1$). Second, s is constructed by changing these p 1-bits of v into 0-bits, assigning to the remainder $n-p$ bits of s the same values as the ones of v. Formally, $s = (s_1, \ldots, s_n)$ is defined by

$$s_i = \begin{cases} 0 & \text{if } i \in \{i_1, \ldots, i_p\}, \\ v_i & \text{if } i \notin \{i_1, \ldots, i_p\}. \end{cases}$$

On one hand, $u \succeq s$ since

$$w_H(s) = w_H(v) - p = m = w_H(u)$$

and then we can apply case (ii)-(a) to s.

On the other hand, $s \succeq v$ since matrix M_v^s has p $\binom{0}{1}$ columns (placed at positions $i_1, \ldots, i_p$), while its $n-p$ reminder columns are either $\binom{0}{0}$ or $\binom{1}{1}$. Hence M_v^s has no $\binom{1}{0}$ columns, so that it satisfies IOC.

Finally, from the transitive property of the intrinsic order, we derive

$$u \succeq s \text{ and } s \succeq v \quad \Rightarrow \quad u \succeq v, \text{ i.e., } v \in C^u.$$

So, for this case (ii)-(b), we have proved that

$$\{v \in \{0,1\}^n \mid w_H(v) > w_H(u) = m\} \subseteq C^u \tag{6}$$

From (5) & (6), we get

$$\{v \in \{0,1\}^n \mid w_H(v) \geq w_H(u) = m\} \subseteq C^u,$$

i.e., $H_u \subseteq C^u$, and this set inclusion together with the converse inclusion $C^u \subseteq H_u$ (which is always satisfied for every binary n-tuple u; see Corollary 2.2) leads to the set equality $C^u = H_u$. This proves the sufficient condition.

Necessary condition. Conversely, suppose that not all the m 1-bits of u are placed at the m right-most positions. In other words, suppose that

$$u \neq \left(0, \overset{n-m}{\smile\!\!\ldots}, 0, 1, \overset{m}{\smile\!\!\ldots}, 1\right).$$

Since, by assumption, $w_H(u) = m$ then simply using the necessary condition we derive that

$$\left(0, \overset{n-m}{\smile\!\!\ldots}, 0, 1, \overset{m}{\smile\!\!\ldots}, 1\right) \succ u,$$

and then

$$\left(0, \overset{n-m}{\smile\!\!\ldots}, 0, 1, \overset{m}{\smile\!\!\ldots}, 1\right) \in H_u - C^u$$

so that,

$$H_u \nsubseteq C^u.$$

This proves the necessary condition. ■

Corollary 3.1: Let $n \geq 1$ and let

$$u = \left(0, \overset{n-m}{\smile\!\!\ldots}, 0, 1, \overset{m}{\smile\!\!\ldots}, 1\right) \equiv 2^m - 1, \quad 0 \leq m \leq n,$$

where any (but not both!) of the above two subsets of bits grouped together can be omitted. Then the number of binary n-tuples intrinsically less than or equal to u is

$$|C^u| = \binom{n}{m} + \binom{n}{m+1} + \cdots + \binom{n}{n}.$$

Proof: Using Theorem 3.1, we have

$$\begin{aligned} C^u = H_u \Rightarrow |C^u| &= |H_u| \\ &= |\{v \in \{0,1\}^n \mid w_H(u) = m \leq w_H(v)\}| \\ &= |\{v \in \{0,1\}^n \mid w_H(v) = m, m+1, \ldots, n\}| \\ &= \binom{n}{m} + \binom{n}{m+1} + \cdots + \binom{n}{n}, \end{aligned}$$

as was to be shown. ■

B. When Less Weight is Equivalent to Greater Probability?

Interchanging the roles of u & v, (3) can be rewritten as follows. Let u be an arbitrary, but fixed, binary n-tuple. Then

$$v \succeq u \quad \Rightarrow \quad w_H(v) \leq w_H(u) \quad \text{for all } v \in \{0,1\}^n. \tag{7}$$

Looking at the implication (7), the following dual question of Question 3.1, immediately arises.

Question 3.2: Can we characterize the binary n-tuples u for which the necessary condition (7) is also sufficient? That is, we try to identify those bitstrings $u \in \{0,1\}^n$ for which the set of binary n-tuples v with weights less than or equal to the one of u coincides with the set of binary n-tuples v with occurrence probabilities greater than or equal to the one of u, i.e.,

$$v \succeq u \quad \Leftrightarrow \quad w_H(v) \leq w_H(u), \text{ i.e., } C_u = H^u.$$

The following theorem provides the answer to this question, in a very simple way. For a very short proof of this theorem, we use the following definition.

Definition 3.3: (i) The complementary n-tuple of a given binary n-tuple $u = (u_1, \ldots, u_n) \in \{0,1\}^n$ is obtained by changing its 0s into 1s and its 1s into 0s

$$u^c = (u_1, \ldots, u_n)^c = (1 - u_1, \ldots, 1 - u_n).$$

Obviously, two binary n-tuples are complementary if and only if their decimal equivalents sum up to

$$\left(1, \overset{n}{\smile\!\!\ldots}, 1\right)_{(10} = 2^n - 1.$$

(ii) The complementary set of a given subset $S \subseteq \{0,1\}^n$ of binary n-tuples is the set of the complementary n-tuples of all the n-tuples of S

$$S^c = \{u^c \mid u \in S\}.$$

Theorem 3.2: Let $n \geq 1$ and $u = (u_1, \ldots, u_n) \in \{0,1\}^n$ with Hamming weight $w_H(u) = m$ $(0 \leq m \leq n)$. Then

$$C_u = H^u$$

if and only if either u is the zero n-tuple ($m = 0$) or the m 1-bits of u ($m > 0$) are placed at the m left-most positions, i.e., if and only if u has the general pattern

$$u = \left(1, \overset{m}{\smile\!\!\ldots}, 1, 0, \overset{n-m}{\smile\!\!\ldots}, 0\right) \equiv 2^n - 2^{n-m}, \quad 0 \leq m \leq n, \tag{8}$$

ISBN: 978-988-19252-1-3
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

where any (but not both!) of the above two subsets of bits grouped together can be omitted.

Proof: Using Theorem 3.1 and the facts that (see, e.g., [4], [5], [7])

$$(C_u)^c = C^{u^c}, \qquad (H^u)^c = H_{u^c},$$

we get

$$\begin{aligned} C_u = H^u &\Leftrightarrow (C_u)^c = (H^u)^c \Leftrightarrow C^{u^c} = H_{u^c} \\ &\Leftrightarrow u^c \text{ has the pattern (4) } \Leftrightarrow u \text{ has the pattern (8)}, \end{aligned}$$

as was to be shown. ■

Corollary 3.2: Let $n \geq 1$ and let

$$u = \left(1, \overset{m}{\ldots}, 1, 0, \overset{n-m}{\ldots}, 0\right) \equiv 2^n - 2^{n-m}, \quad 0 \leq m \leq n,$$

where any (but not both!) of the above two subsets of bits grouped together can be omitted. Then the number of binary n-tuples intrinsically greater than or equal to u is

$$|C_u| = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{m}.$$

Proof: Using Corollary 3.1, we get

$$\begin{aligned} |C_u| &= |(C_u)^c| = \left|C^{u^c}\right| \\ &= \binom{n}{n-m} + \binom{n}{n-m+1} + \cdots + \binom{n}{n} \\ &= \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{m}, \end{aligned}$$

as was to be shown. ■

Example 3.4: Let $n = 5$.

According to Theorem 3.1, the 6 binary 5-tuples $u = 2^m - 1$ $(0 \leq m \leq 5)$, for which $C^u = H_u$ are:

$$\begin{gathered} (0,0,0,0,0) \equiv 0, \ \ (0,0,0,0,1) \equiv 1, \ \ (0,0,0,1,1) \equiv 3, \\ (0,0,1,1,1) \equiv 7, \ \ (0,1,1,1,1) \equiv 15, \ \ (1,1,1,1,1) \equiv 31. \end{gathered}$$

Note that obviously $\{2^n - 2^{n-m}\}_{m=0}^{n} = \{2^n - 2^m\}_{m=0}^{n}$. Then, according to Theorem 3.2, the 6 binary 5-tuples $u = 2^5 - 2^m$ $(0 \leq m \leq 5)$, for which $C_u = H^u$ are the complementary ones of the above 5-tuples :

$$\begin{gathered} (1,1,1,1,1) \equiv 31, \ \ (1,1,1,1,0) \equiv 30, \ \ (1,1,1,0,0) \equiv 28, \\ (1,1,0,0,0) \equiv 24, \ \ (1,0,0,0,0) \equiv 16, \ \ (0,0,0,0,0) \equiv 0. \end{gathered}$$

Example 3.4 is illustrated by Fig. 4, where the upper index or exponent on each node represents its Hamming weight. The reader can check easily that any of the nodes u inserted in a circle (box, respectively) is intrinsically greater (less, respectively) than or equal to all those nodes v with Hamming weights (exponents) greater (less, respectively) than or equal to the Hamming weight (exponent) of u. Just check that all the corresponding matrices M_v^u (M_u^v, respectively), satisfy IOC. Alternatively, it suffices to draw the omitted edges in Fig. 4! For instance,

$$7 \succ 26 \text{ since } M_{26}^{7} = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix} \text{ satisfies IOC,}$$

$$30 \prec 5 \text{ since } M_{30}^{5} = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \text{ satisfies IOC.}$$

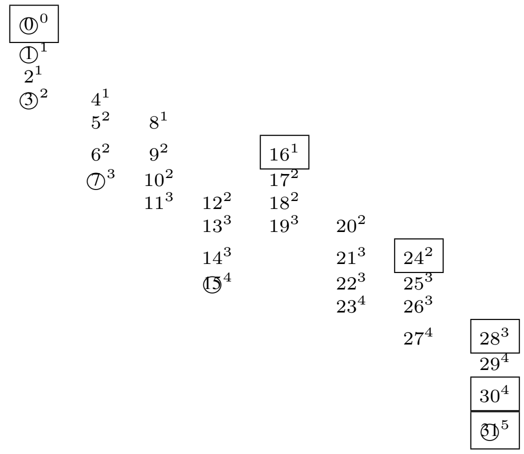

Fig. 4. The edgeless graph I_5. The nodes u for which all the bitstrings with weight (exponent) greater than or equal to the weight (exponent) of u exactly coincide with all the bitstrings whose probabilities are less than or equal the probability of u, are highlighted by a circle. The nodes u for which all the bitstrings with weight (exponent) less than or equal to the weight (exponent) of u exactly coincide with all the bitstrings whose probabilities are greater than or equal the probability of u, are highlighted by a box.

IV. CONCLUSION

Let u be an arbitrary, but fixed, binary n-tuple. It is known that the set C^u (C_u, respectively) of binary n-tuples v whose occurrence probabilities are always, i.e., intrinsically less (greater, respectively) than or equal to the occurrence probability of u is a subset of the set H_u (H^u, respectively) of binary n-tuples v whose Hamming weights are greater (less, respectively) than or equal to the Hamming weight of u. We have proved that $C^u = H_u$ ($C_u = H^u$, respectively) if and only if either u is the zero n-tuple or all the m 1-bits of u are placed at the m right-most (left-most, respectively) positions of the bitstring. The binary n-tuples u with such special patterns, can be identified in the intrinsic order graph I_n as the $n+1$ nodes $2^m - 1$ $(0 \leq m \leq n)$, i.e., the top-most node 0 and the bottom-most nodes of its subgraphs $I_1, I_2, \ldots, I_n$ (or as the symmetric nodes –with respect to the middle point– of the previous ones, respectively).

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ISBN: 978-988-19252-1-3
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)