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Convergence of Longest Edge n-Section of Triangles

Jose P. Suárez, Tania Moreno, Pilar Abad, and Ángel Plaza

***Abstract*—Let t be a triangle in $\mathbb{R}^2$. We find the Longest Edge (LE) of t, insert $n-1$ equally-space points in the LE and connect them to the opposite vertex. This yields the generation of n new sub-triangles whose parent is t. Now, continue this process iteratively. Proficient algorithms for mesh refinement using this method are known when $n = 2$, but less known when $n = 3$ and completely unknown when $n \geqslant 4$.**

We prove that the LE n-section of triangles for $n \geqslant 4$ of triangles produces a finite sequence of triangle meshes with guaranteed convergence of diameters. We give upper and lower bounds for the convergence speed in terms of diameter reduction. Then we fill the gap in the analysis of the diameters convergence for general Longest Edge based subdivision. In addition, we give a numerical study for the case of $n = 4$, the so-called LE quatersection, evidencing its utility in adaptive mesh refinement.

***Index Terms*—Longest-edge, Triangle partition, n-section, Mesh refinement, Triangulation.**

I. Introduction

Since the apparition of Finite Element Method in the 60th, many mesh partitions methods became popular. Mesh Refinement algorithms use such partition methods to refine a given mesh. One wishes to construct a sequence of nested conforming meshes which are adapted to a given criterion. Nested sequences of triangles where each element in the sequence is a child of parent triangle of same sequence are of quite interest in many areas as Finite Element Multigrid Methods, Image Multiresolutions etc., [1].

Although Delaunay triangulations maximize the minimum angle of all the angles of the triangles in any triangulation, some other competitive methods have emerged in the last decade specially those with cheaper computational cost as Longest Edge (LE) based subdivision. LE n-section based algorithms are surprisingly cheap. They are linear in the number of elements, as the only necessary calculations are: (i) Longest Edges and (ii) insertion of n points in the LE side, both of constant-time.

Before we discuss and analyze the general case of Longest Edge n-section of triangles, we first give a short overview of existing methods for LE bisection and LE trisection, n equal two and three respectively.

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A. *Longest Edge Bisection Partition*

Possible the first method to repeatedly subdivide a triangle mesh was the Longest Edge bisection of a triangle. Rosenberg and Stenger showed the non-degeneracy property for LE-bisection: $\alpha_n \geq \frac{\alpha_0}{2}$, [9] where α_n is the minimum interior angle in new triangles appeared at iteration n, and α_0 the minimum angle of initial given triangle.

For better understanding of LE based subdivision, other authors also study the longest edge (diameter) of successive triangle generation. Kearfott [5] proved a bound on the behavior the length of the longest edge of any triangle (diameter) obtained. Later Stynes [3] presented a better bound for certain triangles. After that, Stynes [4] and Adler [2] improved this bound for all triangles. From their studies they also derived that the number of classes of similarity of triangles generated is finite, which is a desirable property of triangle partitions.

B. *4T-LE: Four-Triangles Longest-Edge Partition.*

Further research on Longest-Edge bisection has been carried out since the ninety. Many other variants of LE-bisection have appeared in this period. For example, the *Four-Triangles Longest-Edge Partition, (4T-LE)* bisects a triangle into four subtriangles as follows: the original triangle is first subdivided by its longest edge and then the two resulting triangles are bisected by joining the new midpoint of the longest edge to the midpoints of the remaining two edges of the original triangle. The 4T-LE partition of a given triangle t never produces an angle smaller than half the minimum original angle and besides, it shows a remarkable mesh quality improvement between certain limits, as recently studied in [8]. In practice, Rivara refinement improves angles, and this improvement has been studied in depth, see [8] where sharper bounds for number of dissimilar triangles arising from the 4T-LE are given.

C. *7T-LE: Seven-Triangles Longest-Edge Partition.*

Superior quality improvement of the triangulation can be achieved by the 7-Triangle Longest-Edge (7T-LE) partition, [10]. This partition is constructed by positioning two equally spaced points per edge and join them, using parallel segments, to the edges, at the points closest to each vertex. Then joining the two interior points of the longest edge of the initial triangle to the base points of the opposite subtriangle in such a way that they do not intersect, and finally, triangulating the interior quadrangle by the shortest diagonal. Two of new triangles generated are similar to the new triangle also generated by the 4T-LE, and the other two triangles are, in general, better shaped. As a consequence, the area covered by better triangles is showed to be superior compared to the 4T-LE.

D. Longest Edge Trisection Partition

There is very little research so far on LE n-section methods other than bisection. Recently, a new class of triangle partitions based on the Longest-Edge Trisection has been presented by Plaza *et al.* [6]. It simply consists in inserting three equal points in the Longest Edge and then connecting them to the opposite vertex. Empirical evidence has been given of the non-degeneracy of the meshes obtained by iterative application of LE-trisection. In fact, if α_0 is the minimum interior angle of the initial triangle, and α_n, the minimum interior angle after n levels of LE-trisection, then $\alpha_n \geq \alpha_0/6.7052025350$, independently on the value of n [6]. To complete the study of non-degeneracy for LE-Trisection, it has been proved in [7] that for LE-trisection $\alpha_n \geq \frac{\alpha_0}{c}$ where $c = \frac{\pi/3}{\arctan(\frac{\sqrt{3}}{11})}$. This result confirms previous numerical research, [6].

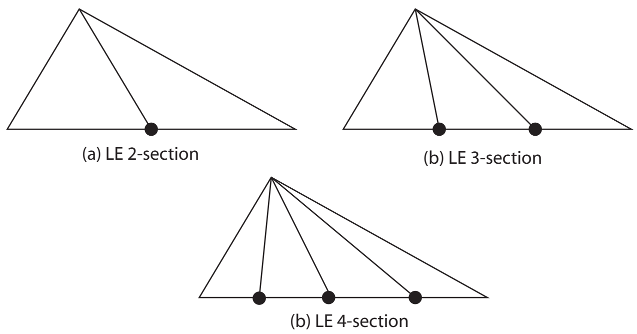

Fig. 1. Scheme for Longest Edge (LE) n-section of a triangle ($n = 2, 3, 4$)

In this paper we prove that the LE n-section of triangles for $n \geqslant 4$ of triangles produces a finite sequence of triangle meshes with guaranteed convergence of diameters. We give upper and lower bounds for the convergence speed in terms of diameter reduction. In addition, we explore in details the so-called LE quatersection ($n = 4$) of triangles by studying the triangles shapes emerging in that process.

The structure of this paper is as follows: Section II and III introduces and proves the upper and lower bound respectively. Section III gives a numerical study where shape quality is studied for the case of $n = 4$ of LE quatersection, evidencing its utility in adaptive mesh refinement. We close with some final conclusions in Section V.

II. Upper bound of diameters.

We prove following theorem:

Theorem 1. *Let d_k be the diameter in the k iterative application of Longest-Edge n-section ($n \geqslant 4$) to a given arbitrary triangle $\triangle ABC$, then:*

$$d_{2k} \leqslant \left(\frac{\sqrt{n^2 - n + 1}}{n} \right)^k d_0$$

where d_0 is the diameter of initial triangle and $k \geqslant 0$.

Before, we give some previous lemmas that are used in the proof:

Lemma 1. *(Theorem of Stewart.) Let $\triangle ABC$ be an arbitrary triangle and S be a point in $\overline{BC}$. Then:*

$$|\overline{AS}|^2|\overline{BC}| = |\overline{AB}|^2|\overline{SC}| + |\overline{AC}|^2|\overline{BS}| - |\overline{BS}||\overline{SC}||\overline{BC}|. \tag{1}$$

Lemma 2. *Let $\triangle ABC$ be an arbitrary triangle where $|\overline{AB}| \leqslant |\overline{AC}| \leqslant |\overline{BC}|$. Then:*

1) *$|\overline{AS}| \leqslant |\overline{AC}|$ for each $S \in \overline{BC}$, see Figure 2 (a).*
2) *Let X and Y be points in segment $\overline{BC}$ such that segments $\overline{BX}$ and $\overline{CY}$ are equal and have empty intersection, then $|\overline{AX}| \leqslant |\overline{AY}|$, see Figure 3 (b).*

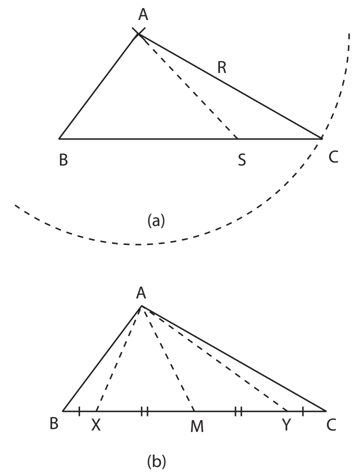

Fig. 2. (a) $|\overline{AS}| \leqslant |\overline{AC}|$ for each $S \in \overline{BC}$. (b) $|\overline{AX}| \leqslant |\overline{AY}|$

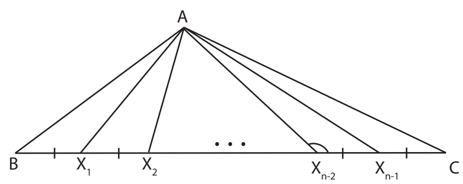

Fig. 3. $|\overline{BX_1}| = |\overline{X_1X_2}| = ... = |\overline{X_{n-1}C}| \leqslant |AX_{n-1}|$

Note that part two of Lemma 2 is straightforward as a consequence of part one, see Figure 2 .

If for an arbitrary triangle $\triangle ABC$ we have that $|\overline{AB}| \leqslant |\overline{AC}| \leqslant |\overline{BC}|$ then we say that length of shortest edge is $|\overline{AB}|$, length of medium edge is $|\overline{AC}|$ and length of longest edge is $|\overline{BC}|$.

Lemma 3. *Let $n \geqslant 4$ and $\triangle ABC$ such that $|\overline{AB}| \leqslant |\overline{AC}| \leqslant |\overline{BC}|$. Let X_1, X_2, ... X_{n-1} points of $\overline{BC}$ such that $|\overline{BX_1}| = |\overline{X_1X_2}| = ... = |\overline{X_{n-1}C}| = \frac{1}{n}|\overline{BC}|$. Then the length of medium size of triangle $\triangle BAX_1$, $\triangle X_1AX_2$,..., $\triangle X_{n-1}AC$ is less or equal than $|\overline{AX_{n-1}}|$.*

Proof: From Lemma 2 we have that $|\overline{AX_i}| \leqslant |\overline{AX_{n-1}}| \leqslant |\overline{AC}|$ for each $i \in \{1, 2, ...n-1\}$. Thus, it is clear that $\angle AX_{n-2}X_{n-1} \geqslant \frac{\pi}{2}$, and so $|\overline{BX_1}| = |\overline{X_1X_2}| = ... = |\overline{X_{n-1}C}| \leqslant |AX_{n-1}|$, see Figure 3.

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Suposse that $|\overline{AB}|$ is the length of medium edge of $\triangle BAX_1$, then the length of its longest edge is, either $|\overline{AX_1}|$ or $|\overline{BX_1}|$. Note that last two segments are less or equal than $|AX_{n-1}|$, as previously proven. □

Lemma 4. *Let a triangle $\triangle ABC$ such that $|\overline{AB}| \leqslant |\overline{AC}| \leqslant |\overline{BC}|$ and $P \in \overline{BC}$ such that $\frac{\overline{BP}}{\overline{PC}} = n-1$ for $n \geqslant 4$. Then:*

$$|\overline{AP}| \leqslant \frac{\sqrt{n^2-n+1}}{n}|\overline{AC}|. \qquad (2)$$

Proof: Let $|\overline{AB}| = c$, $|\overline{BC}| = a$, $|\overline{AC}| = b$. Then $|\overline{BP}| = \frac{n-1}{n}a$ and $|\overline{CP}| = \frac{1}{n}a$. By Lemma 1 we have:

$$|\overline{AP}|^2 a = c^2 \cdot \frac{1}{n}\,a + b^2 \cdot \frac{n-1}{n}\,a - (\frac{n-1}{n^2})a^3,$$

from where

$$|\overline{AP}|^2 = \frac{c^2+(n-1)b^2}{n} - (\frac{n-1}{n^2})a^2.$$

Note that $c \leqslant b \leqslant a$, and thus $s^2 \leqslant \frac{b^2(n-1)+b^2}{n} - (\frac{n-1}{n^2})b^2 = (\frac{n^2-n+1}{n^2})b^2$, from where we have inequality which proves the result of the Lemma. □

At this point, we follow with the proof of main result, Theorem 1.

Proof of Theorem 1: Let us consider the sequence $\{I_k\}_{k=1}^{\infty}$ such that I_k is the longest of the two medium edges of each triangle obtained after iteration k of LE n-section at ($n \geqslant 4$). Then:

$$d_{k+1} \leqslant I_k$$

Note that at iteration k, each longest edge previously obtained at iteration $(k+1)$ is subdivided in n equal parts. Each of these parts is the shortest edge of at least one of the triangle obtained at iteration $(k+1)$.

Using Lemmas 3 and 4 we have that: $I_k \leqslant \frac{\sqrt{n^2-n+1}}{n}d_k$. It is clear that $\frac{\sqrt{n^2-n+1}}{n} < 1$, and so $d_{k+2} \leqslant \frac{\sqrt{n^2-n+1}}{n}d_k$. Thus, we follow that $d_{2k} \leqslant \left(\frac{\sqrt{n^2-n+1}}{n}\right)^k d_0$. □

III. Lower bound for diameters.

We next provide a lower bound for the LE n-section of triangles.

Theorem 2. *Let d_k be the diameter in the k iterative application ($k \geqslant 1$) of Longest-Edge n-section ($n \geqslant 4$) to a given arbitrary triangle with edges a, b and c such that $c \leqslant b \leqslant a$. Then, there exists constants p, q, r, s, t and u only dependent on a, b, c and n, such that it holds:*

1) *For $n = 4$, $d_k^2 \geqslant \left(\frac{1}{4}\right)^k(pk^2+qk+r) > 0$.*
2) *For $n \geqslant 5$, $d_k^2 \geqslant s\frac{1}{n^k} + t\left(\frac{n^2-2n+n^{\frac{3}{2}}\sqrt{n-4}}{2n^2}\right)^k + u\left(\frac{n^2-2n-n^{\frac{3}{2}}\sqrt{n-4}}{2n^2}\right)^k$.*

Proof:
Let $n \geqslant 4$ and $\triangle ABC$ be an arbitrary triangle with $|\overline{AB}| \leqslant |\overline{AC}| \leqslant |\overline{BC}|$, $|\overline{AB}| = c$, $|\overline{BC}| = a$ and $|\overline{AC}| = b$. Let us consider the triangle sequence $\{\Delta_k\}_{k=0}^{\infty}$ such that $\Delta_0 = \triangle A_0B_0C_0$, $A_0 = A$, $B_0 = B$, $C_0 = C$, and for each $k \geqslant 0$ let $\Delta_{k+1} = \triangle A_{k+1}B_{k+1}C_{k+1}$ where $A_{k+1} \in \overline{B_kC_k}$ such that $|\overline{A_{k+1}C_k}| = \frac{1}{n}|\overline{B_kC_k}|$, $B_{k+1} = C_k$ y $C_{k+1} = A_k$. It can be noted that for each $k \geqslant 1$, $|\overline{A_kB_k}| \leqslant |\overline{A_kC_k}| \leqslant |\overline{B_kC_k}|$; from where we have that for each $k \geqslant 1$, Δ_k is one of the n triangles obtained by applying the LE n-section to triangle Δ_{k-1}.

Let now consider the sequence $\{a_k\}_{k=0}^{\infty}$ where $a_k = |\overline{B_kC_k}|$. Using Lemma 1, following recurrence equation can be obtained:

$$a_{k+3}^2 - \frac{n-1}{n}a_{k+2}^2 + \frac{n-1}{n^2}a_{k+1}^2 - \frac{1}{n^3}a_k^2 = 0,$$

where $a_0 = nc$, $a_1 = a$ y $a_2 = b$. Stating $y_k = a_k^2$, we have:

$$y_{k+3} - \frac{n-1}{n}y_{k+2} + \frac{n-1}{n^2}y_{k+1} - \frac{1}{n^3}y_k = 0,$$

where $a_0 = n^2c^2$, $a_1 = a^2$ and $a_2 = b^2$. It can be noted that from the construction of sequence y_k it is deduced that each terms of the sequence is positive. The characteristic equation of such recurrence equation is as follows:

$$\lambda^3 - \frac{n-1}{n}\lambda^2 + \frac{n-1}{n^2}\lambda - \frac{1}{n^3} = 0.$$

At this point, two separated situations can be given: (i) $n = 4$, where a square root of multiplicity three appears, and (ii) $n \geqslant 5$ where three real roots appears.

(i) Case $n = 4$. The solution of the characteristic equation is $\lambda = \frac{1}{4}$, of multiplicity 3, and then:

$$y_k = \left(\frac{1}{4}\right)^k(pk^2+qk+r),$$

where p, q y r are real constants only dependent on a, b y c. Such constants are solutions of an equation system obtained from the initial conditions, omitted here for brevity.

(ii) Case $n \geqslant 5$. The characteristic equation has three real roots: $\lambda_1 = \frac{1}{n}$, $\lambda_2 = \frac{n^2-2n+n^{\frac{3}{2}}\sqrt{n-4}}{2n^2}$ and $\lambda_3 = \frac{n^2-2n-n^{\frac{3}{2}}\sqrt{n-4}}{2n^2}$. Thus:

$$y_k = \left(\frac{1}{n}\right)^k s + \left(\frac{n^2-2n+n^{\frac{3}{2}}\sqrt{n-4}}{2n^2}\right)^k t + $$
$$+\left(\frac{n^2-2n-n^{\frac{3}{2}}\sqrt{n-4}}{2n^2}\right)^k u.$$

where s, t and u are real constants only dependent on a, b, c and n. Such constants are solutions of an equation system obtained from the initial conditions, omitted here for brevity.

It should be noted that $d_k \geqslant a_k$, and so $d_k^2 \geqslant y_k = (\frac{1}{n})^k s + (\frac{n^2-2n+n^{\frac{3}{2}}\sqrt{n-4}}{2n^2})^k t + (\frac{n^2-2n-n^{\frac{3}{2}}\sqrt{n-4}}{2n^2})^k u$. □

Among the LE n-section methods, the LE 4-section or *LE quatersection* this point forwards, has not been explored yet as far as we know. We are dealing with LE quatersection in the rest of the paper. In Figure 4 it is graphed the bound diameters evolution when repeated LE quatersection is applied to three initial triangles with initial diameter equals 1. The coordinates (x, y) of the targeted triangles are:

$$\begin{array}{llll} \Delta 1 = (0,0) & (0.5, \sqrt{3/2}) & (1,0) \\ \Delta 2 = (0,0) & (0.1, 0.1) & (1,0) \\ \Delta 3 = (0,0) & (0.4, 0.01) & (1,0) \end{array}$$

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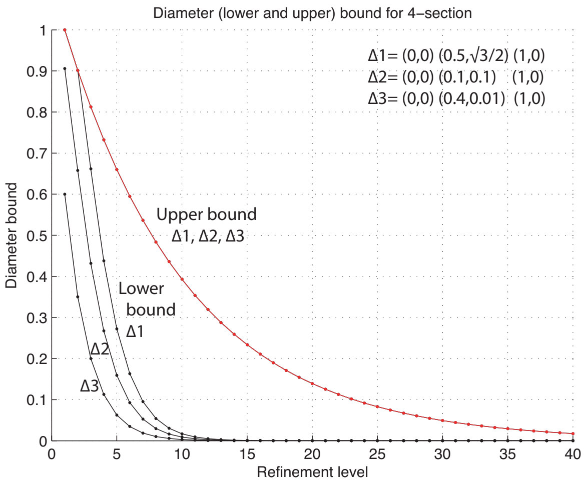

Fig. 4. Upper and Lower bound for several triangles.

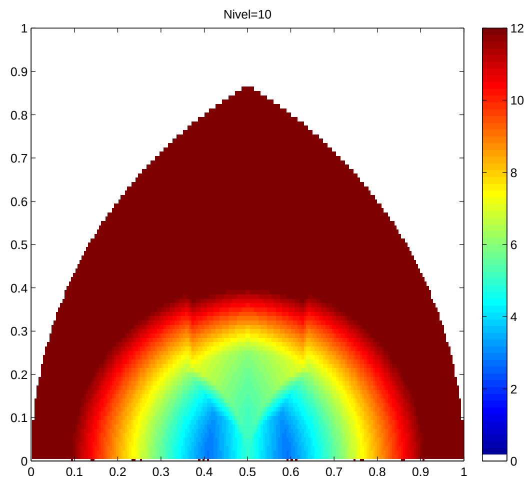

Fig. 5. Mapping diagram for two levels of quatersection. Shaded colors are $\frac{\tau_0}{\tau_2}$, being τ the minimum angle.

TABLE I

MINIMUM ANGLES RATIOS α_0/α_n FOR THREE TRIANGLE CASES AND THREE REFINEMENT LEVELS

	α_0/α_1	α_0/α_2	α_0/α_3
$\Delta 1$	11.5191	22.3865	37.1927
$\Delta 2$	5.0247	8.1670	12.0620
$\Delta 3$	1.8685	2.4038	3.0064

IV. A MAPPING DIAGRAM TO REPRESENT TRIANGLE SHAPES

A mapping diagram is constructed as follows, see [11] to visually represent triangle shapes in LE quatersection refinements: (1) for a given triangle or subtriangle the longest edge is scaled to have unit length. This forms the base of the diagram, (2) it follows that the set of all triangles is bounded by this horizontal segment (longest edge) and by two bounding exterior circular arcs of unit radius. The diagram is then defined by the set:

$$\{(x, y) :\ x^2 + y^2 \geqslant 1\} \cap \{(x, y) :\ (x-1)^2 + y^2 \geqslant 1\} \cap \ldots$$
$$\{(x, y) :\ x \geqslant 0, y \geqslant 0\ \}$$

In this manner, a point within the diagram univocally represents a triangle, whose apex is this point itself and the other two vertices are $(0, 0)$ and $(1, 0)$ respectively. This lead us to an easy and simple way to uniformly represent triangle shapes. For instance, a degenerated triangle in which its three vertices are collinear is represented by and apex over the base, the segment defined by coordinates $(0, 0)$ and $(1, 0)$, of the diagram. The equilateral triangle corresponds to the apex at $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. As the vertex of a triangle moves from this point along either boundary arc, the maximum angle increases from $\frac{\pi}{3}$ to approach a right angle at the degenerate 'needle triangle' limit near $(0, 0)$ or $(1, 0)$.

In order to show the behaviour of LE quatersection for some refinement iterations, we employ the so described mapping diagram where shaded values within the diagram represents the quotient $\frac{\tau_0}{\tau_2}$, being τ_0 the minimum initial angle and τ_2 the minimum angle after two levels of refinement, see Figure 5. Approximately five thousand of triangles are targeted for refinement, covering uniformly the interior area of the diagram. It can be noted in the diagram two interesting areas around coordinates $x = 0.4$ and $x = 0.6$. These focused areas are triangles with greatest minimum angles, whereas upper zones correspond to lower values.

The mapping diagram can be also used to analyze the shapes of subsequent triangles generated in iterative LE quatersection. For example, Figures 6 (a), (b) (c) and (d) show separate cases for iterative refinement of given initial triangles, corresponding respectively to triangles $\Delta 1$, $\Delta 2$, $\Delta 3$ and $\Delta 4$. It should be noted as rapidly new subdivided triangles move down close to the limit base segment of the diagram, evidencing so the degeneracy trend of LE quatersection.

Finally, reported values of α_0/α_n, $n = 1, 2, 3$ for same triangles cases are reported in Table I. Note that LE quatersection generally deteriorates minimum angles. Note also that shape quality is clearly dependent of the initial triangle considered for refinement. Thus, the regular triangle, as expected, poses the worst case $\alpha_0/\alpha_3 = 37.1927$, which is in agreement with the behaviour of LE bisection and LE trisection. In the case of initial triangles of poor quality, for example triangle $\Delta 3$, LE quatersection leads to reasonable output values for the minimum angles and then an adaptive mesh refinement algorithm that uses this method can be a valuable option for special narrowed or skinny triangle like this type, as those appearing in meshes from Fluid Dynamic, Electromagnetism etc.

V. CONCLUSIONS

Nested sequences of triangles where each element in the sequence is a child of parent triangle has shown to be critical in Multigrid Methods or Finite Element. In this paper we generalized a class of triangle mesh refinement based on the Longest Edge and introduced the so-called Longest Edge n-section methods. Proficient algorithms for mesh refinement using this method are known when $n = 2$, but less known when $n = 3$ and completely unknown when $n \geqslant 4$. LE n-section based algorithms are surprisingly cheap. They are linear in the number of elements, as the only necessary calculations are: (i) Longest Edges and (ii) insertion of n points in LE sides, which is of constant-time.

In this paper we proved upper and lower bounds for the sequence of diameters generated by iterative application of

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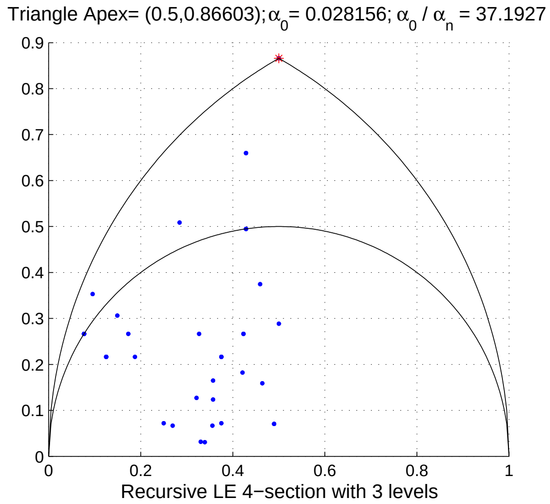
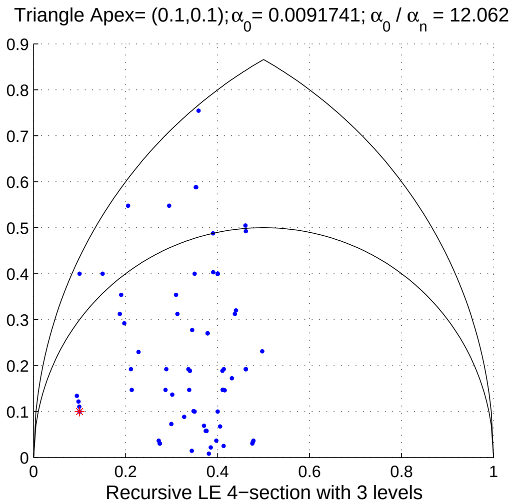
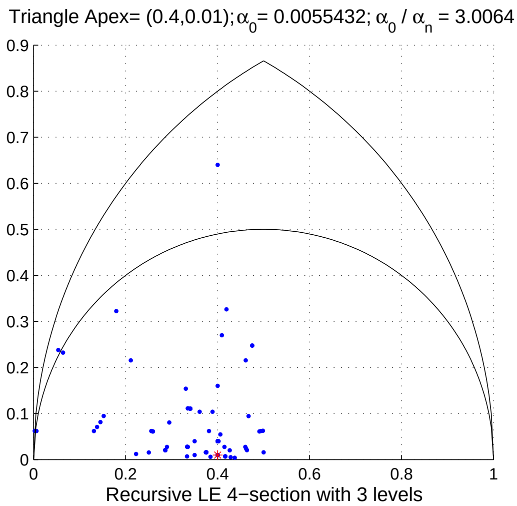


(a) Case of the regular initial triangle, which marks the worst case for minimum angles. Shape of all the new triangles are worse than the regular.



(b) Superior triangles in terms of shape quality are detected above the red apex. Some others with inferior quality appear below the red apex.



(c) A very bad initial triangle leading to some new improved subtriangles.

Fig. 6. Triangle generation resulting after 3 levels of LE quatersection applied to an initial triangle (marked in red). Studied cases for initial triangles: (a) $\Delta 1 = (0,0)(0.5,\sqrt{3/2})(1,0)$, (b) $\Delta 2 = (0,0)(0.1,0.1)(1,0)$ and (c) $\Delta 3 = (0,0)(0.4,0.01)(1,0)$.

LE n-section partition. We gave upper and lower bounds for the convergence speed in terms of diameter reduction. In addition, we have explored in details the LE quatersection ($n = 4$) of triangles by studying the triangles shapes that emerge in that process. We then evidence its utility in adaptive mesh refinement specially in meshes with narrowed or skinny triangles as those appearing in Fluid Dynamic and Electromagnetism.

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