# Applying a Technique of Identification for Computing Fourier Series Coefficients

A. Maddi, A. Guessoum, and D. Berkani

***Abstract*—Fourier series provides a way of representing the signal by how much information is contained at different frequencies. In this paper, we propose a new application method based on the technique of identification. This method shows how to estimate the Fourier series coefficients for general periodic functions. The signal reference estimator is deduced directly from an infinite series of sine and cosine signals. The method is tested and compared to analytical method, for examples: square-wave, triangular-wave and parabolic-wave function; the results obtained confirm the originality and the effectiveness of the method. Moreover this method is numerically stable, fast and can be used for on-line applications.**

***Index Terms*—Fourier series, periodic function, recursive estimation, least squares, sinusoidal disturbances.**

## I. Introduction

THE Fourier series is one of the principal methods of analysis for mathematical physics, engineering, and signal processing. While the original theory of Fourier series applies to periodic functions occurring in wave motion, such as with light and sound, its generalizations often relate to wider settings, such as the time-frequency analysis underlying the recent theories of wavelet.

Rademacher used the circle method to prove a formula which expresses the complex Fourier coefficients as a convergent infinite series in terms of Bessel functions and Kloosterman sums. He realized that the convergence of the series is rather slow.

For this reason, [Mahler, 1979], proved a system of recursive formulas. [Zagier and Kaneko, 1999] introduce the main formula to compute the complex Fourier coefficients of a specific function. [Balogh and Kollar, 2002] obtained theoretical results only for the independent Gaussian white noise, [Baier and Köhler, 2003] discussed run times and the complexity of the corresponding method to compute the coefficients in practice, and they concluded an approach due to Kaneko and Zagier turns out to be most efficient.

In this work, we present a new application method based on the technique of identification. This method can be applied for computing Fourier series coefficients in general cases of periodic functions. Furthermore, we give evidence that the method proposed here is much simpler and use less arithmetic operations, this method is especially suitable for digital signal processing.

The paper is organized as follows. We start by definitions and proprieties of Fourier series in section II. We introduce a new application method in Section III. In section IV, we illustrate the feasibility and precision obtained with the proposed method for several periodic functions. Concluding remarks are given in section V.

## II. Theoretical Background

### A. Definition of Fourier series

Let $f$ be a continuous function on $[0\,,T]$. Then the Fourier series of $f$ is the series:

$$f(t)=a_0+\sum_{n=1}^{\infty}a_n\cos(n\frac{2\pi}{T}t)+\sum_{n=1}^{\infty}b_n\sin(n\frac{2\pi}{T}t) \qquad (1)$$

where the coefficients $a_n$ and $b_n$ defined by,

$$a_0=\frac{1}{T}\int_{-T/2}^{T/2}f(t)\,dt \qquad (2)$$

$$a_n=\frac{2}{T}\int_{-T/2}^{T/2}f(t)\cos(n\frac{2\pi}{T}t)\,dt \qquad (3)$$

$$b_n=\frac{2}{T}\int_{-T/2}^{T/2}f(t)\sin(n\frac{2\pi}{T}t)\,dt \qquad (4)$$

The series in (Eq.1) is usually written in the complex form:

$$f(t)=\sum_{n=-\infty}^{\infty}c_n e^{jn\frac{2\pi}{T}t} \qquad (5)$$

where

$$c_0=a_0\quad ,$$

$$c_n=\frac{a_n-jb_n}{2}\ ,$$

$$c_{-n}=\frac{a_n+jb_n}{2}\quad ,\ (n\geq 1)$$

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### B. Properties of Fourier series

-- If $f(t)$ is an odd function, then $f(t)\cos(n\frac{2\pi}{T}t)$ is an odd function and $a_n = 0$ for $n \geq 1$. Also $f(t)\sin(n\frac{2\pi}{T}t)$ is an even function so that $b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin(n\frac{2\pi}{T}t)\,dt$.

-- If $f(t)$ is an even function, then $f(t)\sin(n\frac{2\pi}{T}t)$ is an odd function and $b_n = 0$ for $n \geq 1$. Also $f(t)\cos(n\frac{2\pi}{T}t)$ is an even function so that $a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos(n\frac{2\pi}{T}t)\,dt$.

## III. FORMULATION OF THE PROPOSED METHOD

### A. Problem Statement

If one is interested in the computation of Fourier series coefficients for all $n \in N$, then the problem is very difficult and more complex signals may not have a solution. In these cases, we propose a new application method for computing of Fourier series coefficients.

Recall that the equation (1) can be written in the form of

$$f(t) = \theta^T x(t) \tag{6}$$

where $x(t)$ is the data vector $(2n+1)\times 1$ and $\theta^T$ is a vector $1\times(2n+1)$ of unknown parameters which are defined by,

$$\theta^T = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_n, & b_1 & b_2 & \dots & b_n \end{bmatrix}$$

$$x^T(t) = \begin{bmatrix} 1 & \cos(\omega_0 t)\dots\cos(nw_0 t) & \sin(w_0 t)\dots\sin(nw_0 t) \end{bmatrix}$$

or $\omega_0 = \frac{2\pi}{T}$, is the fundamental frequency in radians/sec.

A model given with (6) presents an accurate description of the system. However, in this expression the vector of system parameter $\theta$ is not known. It is important to determine it by using available data in reference signal samples $x(t)$. For that purpose a model of a system is supposed,

$$f(t) = \hat{f}(t) + \varepsilon(t) \tag{7}$$

where $\varepsilon(t)$ is an error of identification at moment $t$, and $\hat{f}(t)$ is the estimator model equation which can be written in the form as,

$$\hat{f}(t) = \hat{\theta}^T x(t) \tag{8}$$

with $\hat{\theta}^T$ is a vector $1\times(2n+1)$ of estimated parameters defined by,

$$\hat{\theta}^T = \begin{bmatrix} \hat{a}_0 & \hat{a}_1 & \hat{a}_2 & \dots & \hat{a}_n, & \hat{b}_1 & \hat{b}_2 & \dots & \hat{b}_n \end{bmatrix}$$

We define the error of identification as the difference between the signal $f(t)$ and the estimated signal $\hat{f}(t)$:

$$\varepsilon(t) = f(t) - \hat{f}(t) \tag{9}$$

Formally, the scheme of recursive identification can be summarized by the Fig.1.

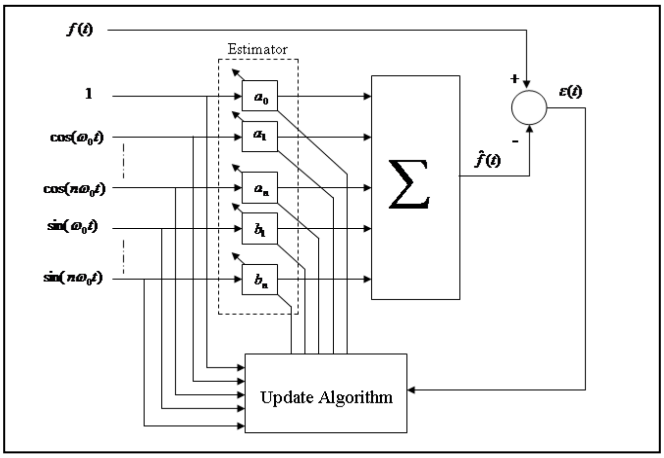


**Fig.1**: Scheme of Recursive Method of Identification

### B. Update Algorithm

A Recursive Least Squares (RLS) algorithm is an efficient method for finding linear estimators minimizing the least squared error over the training data.

In this context, we are often interested in on-line applications, where the estimator is updated following the arrival of each new sample. A recursive way to estimate the parameters in a linear regression model described in (Eq.1) can be evaluated by the RLS algorithm, minimizing the least squares criterion:

$$J_N(\theta) = \frac{1}{N}\sum_{t=1}^{N}\varepsilon^2(t) \tag{10}$$

where $N$ represents the number of samples.

The RLS algorithm is given by the following form:

$$\hat{\theta}(t) = \hat{\theta}(t-1) + P(t)\varphi(t)\left(y(t) - \hat{\theta}^T(t-1)\varphi(t)\right) \tag{11}$$

$$P(t) = P(t-1) - \frac{P(t-1)\varphi(t)\varphi^T(t)P(t-1)}{1+\varphi^T(t)P(t-1)\varphi(t)} \tag{12}$$

If there is no primary knowledge about the system, the choice of the starting values $\hat{\theta}(0)$ is null and a great starting value of covariance matrix $P(0)$ is recommended.

## IV. SIMULATION STUDIES

Recursive Method of Identification for Fourier series coefficients using the RLS algorithm was implemented and tested by means of MATLAB for several periodic functions. To illustrate the feasibility and precision obtained with the proposed method, consider the following examples:

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Example 1: Square-wave function *f* defined by

$$f(t) = \begin{cases} +0.5 & if\ 0 \le t < \pi \\ -0.5 & if\ \pi \le t < 2\pi \end{cases}$$ and

$$f(t+2\pi) = f(t)$$

So $f$ is a periodic function with period $T = 2\pi$ . The results are given on Table I. And we can plot in Fig.2 the Fourier series representation and see how the series does at reproducing the original signal. If we plot the first twelve terms in the sum, we see the general shape of the original function.

Example 2: Triangular-wave function *f* defined by

$$f(t) = \begin{cases} \pi + t & if\ -\pi \le t < 0 \\ \pi - t & if\ 0 \le t < \pi \end{cases}$$ and

$$f(t+2\pi) = f(t)$$

So $f$ is a periodic function with period $T = 2\pi$ .

Fourier series coefficients are given on Table II and its graph is reconstructed in Fig.3

Example 3: Parabolic-wave function *f* defined by

$$f(t) = \frac{1}{2} t^2 \quad For\ 0 \le t \le \pi$$ and

$$f(t+\pi) = f(t)$$

So $f$ is a periodic function with period $T = \pi$ . The results obtained by Recursive Application Method of Identification are illustrated on Table III and its graph is shown in Fig.4.

TABLE I: FOURIER SERIES COEFFICIENTS FOR SQUARE-WAVE FUNCTION

| $n$ | $a_n$ | $b_n$ | $a_0$ |
|---|---|---|---|
| 1 | 0.0000 | 0.6366 | |
| 2 | 0.0000 | 0.0000 | |
| 3 | 0.0000 | 0.2122 | |
| 4 | 0.0000 | 0.0000 | |
| 5 | 0.0000 | 0.1273 | 0.0000 |
| 6 | 0.0000 | 0.0000 | |
| 7 | 0.0000 | 0.0909 | |
| 8 | 0.0000 | 0.0000 | |
| 9 | 0.0000 | 0.0707 | |
| 10 | 0.0000 | 0.0000 | |
| 11 | 0.0000 | 0.0579 | |
| 12 | 0.0000 | 0.0000 | |

TABLE II: FOURIER SERIES COEFFICIENTS FOR TRIANGULAR-WAVE FUNCTION

| $n$ | $a_n$ | $b_n$ | $a_0$ |
|---|---|---|---|
| 1 | 1.2732 | 0.0000 | |
| 2 | 0.0000 | 0.0000 | |
| 3 | 0.1415 | 0.0000 | |
| 4 | 0.0000 | 0.0000 | |
| 5 | 0.0509 | 0.0000 | 1.5708 |
| 6 | 0.0000 | 0.0000 | |
| 7 | 0.0260 | 0.0000 | |
| 8 | 0.0000 | 0.0000 | |
| 9 | 0.0157 | 0.0000 | |
| 10 | 0.0000 | 0.0000 | |
| 11 | 0.0105 | 0.0000 | |
| 12 | 0.0000 | 0.0000 | |

TABLE III: FOURIER SERIES COEFFICIENTS FOR PARABOLIC-WAVE FUNCTION

| $n$ | $a_n$ | $b_n$ | $a_0$ |
|---|---|---|---|
| 1 | 0.4995 | -1.5708 | |
| 2 | 0.1245 | -0.7854 | |
| 3 | 0.0551 | -0.5236 | |
| 4 | 0.0308 | -0.3927 | |
| 5 | 0.0195 | -0.3142 | 1.6447 |
| 6 | 0.0134 | -0.2618 | |
| 7 | 0.0097 | -0.2244 | |
| 8 | 0.0073 | -0.1963 | |
| 9 | 0.0057 | -0.1745 | |
| 10 | 0.0045 | -0.1571 | |
| 11 | 0.0036 | -0.1428 | |
| 12 | 0.0030 | -0.1309 | |

In general Fourier series can reconstruct a signal with a small number of modes if the original signal is smooth. Discontinuities require many high frequency components to construct the signal accurately.

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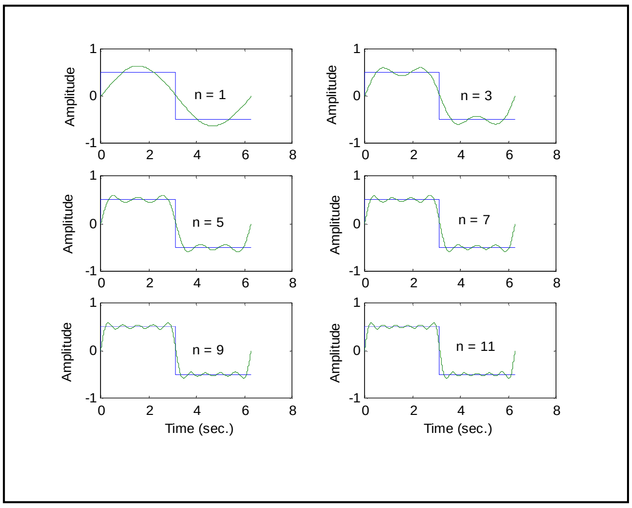


**Fig. 2**: Fourier reconstruction of a square-wave function

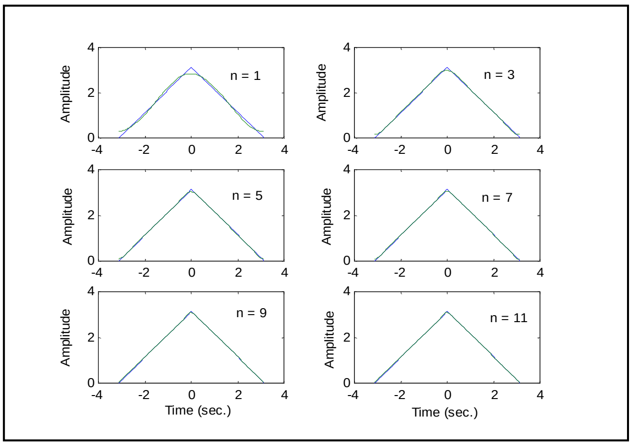


**Fig. 3**: Fourier reconstruction of a triangular-wave function

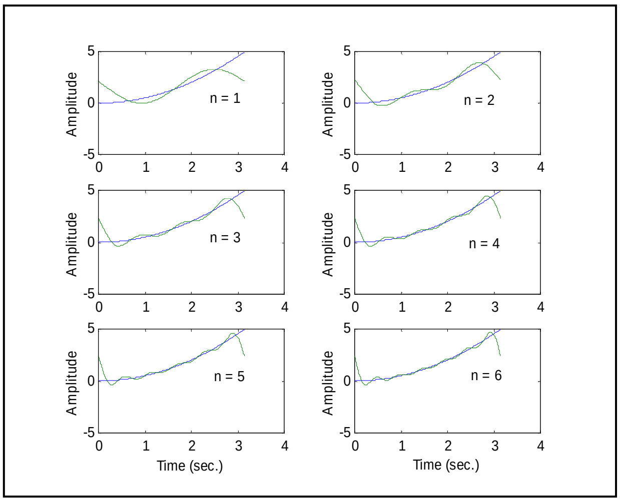


**Fig. 4**: Fourier reconstruction of a parabolic-wave function

## V. CONCLUSION

A new application method based on the technique of identification for Fourier series coefficients has been developed. The method is implemented and tested on MATLAB for several examples. Simulations results confirmed the analytical methods, and the error of estimation can be more accurate with the Recursive Method of Identification. Furthermore, the simulations showed that the estimated Fourier coefficients tend to zero for more terms, and we see the original function is represented quite accurately. In addition, this method is numerically stable, fast and can be used for on-line applications.

Finally, the proposed method can be implemented on Digital Signal Processing (DSP) cards for analysis and synthesis of musical sounds.

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