Double Exponential Sinc Nyström Solution of the Urysohn Integral Equations

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***Abstract*—In this paper, we study the numerical approximation of the Urysohn integral equation by means of the sinc approximation with the Double Exponential (DE) transformations. This numerical method combine a sinc Nyström method with the Newton iterative process that involves solving a nonlinear system of equations. We provide an error analysis for the method. These method improves conventional results and achieve exponential convergence. Some numerical examples are given to confirm the accuracy and the ease of implementation of the method.**

***Index Terms*—Urysohn integral equation, sinc approximation, Nyström method.**

I. Introduction

IN this paper, we consider the sinc Nyström method for the numerical solution of the Urysohn integral equations of the Fredholm type

$$u(t) - \int_a^b k(t,s,u(s))\mathrm{d}s = g(t), \qquad t \in [a,b], \tag{1}$$

where $u(t)$ is an unknown function to be determined and $k(t,s,u)$ and $g(t)$ are given functions. Equation (1) was introduced for the first time by Pavel Urysohn in [20]. The Urysohn integral equation includes the Hammerstein equation and many other equations. Equations of these types appear in many applications. For example, they arise as a reformulation of two-point boundary value problems with certain nonlinear boundary conditions [2], [5]. Several authors have written a number of papers which establish numerical techniques for finding an approximation of the nonlinear Fredholm integral equations. These methods can be categorized into two major types. The first types are those lead to solve a system of nonlinear equations, and the other use iterative methods to solve the nonlinear equation directly. The aim of this work is to present a numerical scheme for a Nyström method based on sinc quadrature formulas. This method is derived by replacing the smoothing transformation, with the so-called double exponential transformation. Such replacement improves the order of convergence to $O(\exp(-C\frac{N}{\log N}))$. For a comprehensive study of double exponential sinc approximation to [6], [7], [8], [9], [10], [11].

Equation (1) can be expressed in the operator form as

$$(I-\mathcal{K})u = g, \tag{2}$$

where $(\mathcal{K}u)(t) = \int_a^b k(t,s,u(s))\mathrm{d}s$. The operator is defined on the Banach space $X = \mathbf{Hol}(D)\bigcap C(\overline{D})$. In this notation, $D \subset \mathbb{C}$ is a simply connected domain which satisfies $(a,b) \subset D$ and $\mathbf{Hol}(D)$ denotes the family of all functions f that are analytic in the domain D. Furthermore, assume (2) has at least one solution, and note that the right side of (1) is a completely continuous opertaor [12]. Let $\|u\| = sup\{|u(t)| : t \in [0,1]\}$. Additionally, suppose that the solution $u^*(t)$ to be determined is geometrically isolated [13], in the other words, there is some ball

$$\mathfrak{B}(u^*, r) = \{u \in X : \|u - u^*\| \leq r\},$$

with $r > 0$, that contains no solution of (1) other than u^*. It is assumed that the linear operator $\mathcal{K}'(u^*)$ does not have 1 as an eigenvalue. Then there is a geometrically isolated solution for (1) [5]. This paper is organized in five sections. In section II we will review the basic properties of the sinc quadrature rule which has been used in our approximation and analysis. The numerical method based on sinc approximation are considered in Section III. We provide in Section IV a complete convergence analysis for the proposed methods. Finally, in section V, we present several numerical experiments. The numerical results are consistent with the theoretical estimates on order of convergence.

II. The Quadrature Formulae

The sinc function is defined on the whole real line by

$$sinc(t) = \begin{cases} \frac{\sin(\pi t)}{\pi t}, & t \neq 0, \\ 1, & t = 0. \end{cases}$$

The sinc numerical methods are based on approximation over the infinite interval $(-\infty, \infty)$, written as

$$f(t) \approx \sum_{j=-N}^{N} f(jh)S(j,h)(t), \qquad t \in \mathbb{R},$$

where the basis function $S(j,h)(t)$ is defined by

$$S(j,h)(t) = sinc(\frac{t}{h} - j),$$

and h is a step size appropriately chosen depending on a given positive integer N, and j is an integer. The sinc approximation and numerical integration are closely related through the following identity

$$\begin{aligned} &\int_{-\infty}^{\infty}(\textstyle\sum_{j=-N}^{N} f(jh)S(j,h)(t) - f(t))\mathrm{d}t \\ &= h\textstyle\sum_{j=-N}^{N} f(jh) - \int_{-\infty}^{\infty} f(t)\mathrm{d}t. \end{aligned} \tag{3}$$

On the other hand, this is a relation between the approximation error of the sinc approximation and the one of integration

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by the trapezoidal rule [6]. The equation (3) can be adapted to approximate on general intervals with the aid of appropriate variable transformations $t = \varphi(x)$. As the transformation function $\varphi(x)$ double exponential (DE) transformations are applied. In order to define a convenient function space, the strip domain

$$D_d = \{z \in \mathbb{C} : |Imz| < d\},$$

for some $d > 0$ is introduced. The DE-transformation and its inverse are

$$\varphi_{DE}(x) = \frac{b-a}{2}\tanh(\frac{\pi}{2}\sinh(x)) + \frac{b+a}{2},$$

$$\phi_{DE}(t) = \log[\frac{1}{\pi}\log(\frac{t-a}{b-t}) + \sqrt{1+\{\frac{1}{\pi}\log(\frac{t-a}{b-t})\}^2}].$$

This transformation maps D_d onto the domain

$$\varphi_{DE}(D_d) = \{z \in \mathbb{C} : |\arg[\frac{1}{\pi}\log(\frac{z-a}{b-z}) + \sqrt{1+\{\frac{1}{\pi}\log(\frac{z-a}{b-z})\}^2}]| < d\}. \quad (4)$$

Definition 2.1: Let D be a simply connected domain which satisfies $(a, b) \subset D$, and let α and C be positive constants. Then $\mathcal{L}_\alpha(D)$ denotes the family of all functions $f \in \mathbf{Hol}(D)$ which satisfy

$$|f(z)| \leq C|Q(z)|^\alpha,$$

for all z in D where $Q(z) = (z-a)(b-z)$.
The following theorem involves bounding the error of $(2N+3)$-point sinc quadrature for f on (a, b). When incorporated with the DE-transformation, the quadrature rule is designated.

Theorem 2.2: ([7]) Let $(fQ) \in \mathcal{L}_\alpha(\varphi_{DE}(D_d))$ for d with $0 < d < \frac{\pi}{2}$. Assume that N is a positive integer and h is selected by the formula

$$h = \frac{\log(\frac{2dN}{\alpha})}{N}.$$

Then there exists a constant C which is independent of N, such that

$$|\int_a^b f(t)\,\mathrm{d}t - h\sum_{j=-N}^{N} f(\varphi_{DE}(jh))\varphi'_{DE}(jh)| \leq C\exp(\frac{-2\pi dN}{\log(\frac{2dN}{\alpha})}). \quad (5)$$

III. Sinc Nyström method

In the DE-sinc Nyström method we approximate the integral operator in (1) by the quadrature formula (5). Let $u \in \mathbf{Hol}(\varphi_{DE}(D_d))$ and $k(t, ., u(.))Q(.) \in \mathcal{L}_\alpha(\varphi_{DE}(D_d))$ for all $t \in [a, b]$ and $u \in \mathfrak{B}$. Then the integral in (1) can be approximated by Theorem 2.2 and the following discrete DE-operator can be defined,

$$\mathcal{K}_N^{DE}(u)(t) = h\sum_{j=-N}^{N} k(t, t_j^{DE}, u(t_j^{DE}))\varphi'_{DE}(jh).$$

The Nyström method applied to (1) is to find u_N^{DE} such that

$$u_N^{DE}(t) - h\sum_{j=-N}^{N} k(t, t_j^{DE}, u(t_j^{DE}))\varphi'_{DE}(jh) = g(t), \quad (6)$$

where the points t_j^{DE} are defined by the formula

$$t_j^{DE} = \varphi_{DE}(jh), \quad j = -N, \ldots, N.$$

Solving (6) reduces to solving a finite dimensional nonlinear system. For any solution of (6) the values $u_N^{DE}(t_j^{DE})$ at the quadrature points satisfies the nonlinear system

$$u_N^{DE}(t_i^{DE}) - h\sum_{j=-N}^{N} k(t_i^{DE}, t_j^{DE}, u(t_j^{DE}))\varphi'_{DE}(jh) = g(t_i^{DE}), \qquad i = -N, \ldots, N. \quad (7)$$

Conversely, given a solution $u_N^{DE}(t_i^{DE})$, $i = -N, ..., N$, of the system (7), then the function u_N^{DE} defined by

$$u_N^{DE}(t) = h\sum_{j=-N}^{N} k(t, t_j^{DE}, u(t_j^{DE}))\varphi'_{DE}(jh) + g(t),$$

is readily seen to satisfy (6).

We rewrite the (6) in operator notation as

$$(I - \mathcal{K}_N^{DE})u_N^{DE} = g. \quad (8)$$

Atkinson in [3] by using the Leray-Schauder theorem proved that under certain differentiability assumptions on $\mathcal{K}$ and $\mathcal{K}_N^{DE}$, (8) has a unique solution in a neighborhood of an isolated solution of (1) and these approximation solutions converge to an isolated solution for sufficiently large N. We assume that $k_u(t, s, u) \equiv \frac{\partial k(t,s,u)}{\partial u}$ is continuous for all $t, s \in [a, b]$ and $u \in \mathfrak{B}$. This assumption implies that $\mathcal{K}$ is Fréchet differentiable [3] with

$$\mathcal{K}'(u)x(t) = \int_a^b k_u(t, s, u(s))x(s)\,\mathrm{d}s, \quad t \in [a, b], \quad x \in X.$$

Furthermore, the continuity assumption is considered for second partial derivative of the kernel, $k_{uu}(t, s, u)$, leading to the existence and the boundedness of the second Fréchet derivative with

$$\mathcal{K}''(u)(x, y)(t) = \int_a^b k_{uu}(t, s, u(s))x(s)y(s)\,\mathrm{d}s, \quad t \in [a, b], \quad x, y \in X.$$

Similar to $\mathcal{K}_N^{DE}$, $(\mathcal{K}_N^{DE})'$ and $(\mathcal{K}_N^{DE})''$ can be defined by the DE-sinc quadrature formula as follow

$$(\mathcal{K}_N^{DE})'(u)x(t) = h\sum_{j=-N}^{N} k_u(t, t_j^{DE}, u(t_j^{DE}))\varphi'_{DE}(jh)x(t_j^{DE}), \quad (9)$$

and

$$(\mathcal{K}_N^{DE})''(u)(x, y)(t) = h\sum_{j=-N}^{N} k_{uu}(t, t_j^{DE}, u(t_j^{DE}))\varphi'_{DE}(jh)x(t_j^{DE})y(t_j^{DE}). \quad (10)$$

IV. Convergence Analysis

The convergence of the sinc Nyström method which is introduced in the previous sections is discussed in the present section. For the following lemma D represents $\varphi_{DE}(D_d)$. In this lemma, the sufficient conditions to have a completely continuous operator have been investigated.

Lemma 4.1: ([12]) Let the kernel $k(t, s, u)$ be continuous and have a continuous partial derivative $\frac{\partial k(t,s,u)}{\partial u}$ for all $t, s \in$

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D and $u \in \mathfrak{B}$. Then $\mathcal{K} : X \to X$ is a completely continuous operator and is differentiable at each point of $\mathfrak{B}$.

Our basic assumption is that the equation (1) has an analytic solution. The sufficient conditions to have such a solution have been mentioned in [12, p. 83]. We supposed that those conditions are satisfied here. Our idea for deriving the order of convergence is based on collectively compact operator theory [14]. For ease of referencing, the following required conditions are mentioned from [3], [15].

$\mathcal{C}_1$. $\{\mathcal{K}_N^{DE} : N \geq 1\}$ is a collectively compact family on X.

$\mathcal{C}_2$. $\mathcal{K}_N^{DE}$ is pointwise convergent to $\mathcal{K}$ on X.

$\mathcal{C}_3$. For $N \geq 1$, $\mathcal{K}_N^{DE}$ possesses continuous first and bounded second Fréchet derivatives on $\mathfrak{B}$. Moreover,

$$\|(K_N^{DE})''\| \leq \alpha < \infty,$$

where α is a constant.

It is more convenient to rewrite the quadrature rule defined in Theorem 2.2 in the following notation. Let $Q_N^{DE} : X \to \mathbb{R}$ be a discrete operator defined by

$$Q_N^{DE} f = h \sum_{j=-N}^{N} f(t_i^{DE}) \varphi'_{DE}(jh), \tag{11}$$

and $Q : X \to \mathbb{R}$ be an integral operator defind by $Qf = \int_a^b f(t)\,\mathrm{d}t$. Kress et al. in [16] have concluded from Steklov's theorem that $Q_N^{DE} f \to Qf$ for all $f \in C[a, b]$. Additionally, it is easily proven by the Banach-Steinhaus that Q_N^{DE} is uniformly bounded [8]. Now, the following theorem is stated to prove that $\mathcal{K}_N^{DE}$ satisfies the conditions $\mathcal{C}_1$-$\mathcal{C}_3$.

Theorem 4.2: Assume that $k(t, ., u(.))Q(.) \in \mathcal{L}_\alpha(\varphi_{DE}(D_d))$ for $0 < d < \pi$ and $k_{uu}(t, s, u)$ is continuous for all $t, s \in [a, b]$ and $u \in \mathfrak{B}$, then the conditions $\mathcal{C}_1$-$\mathcal{C}_3$ are fulfilled.

Proof: From the continuity of the kernel and the above discussion, the family

$$S = \{\mathcal{K}_N^{DE} u \mid N \geq 1,\ u \in \mathfrak{B}\},$$

is uniformly bounded. Furthermore, note that the function $k(t, s, u)$ is uniformly continuous on $[a, b] \times [a, b] \times \mathfrak{B}$, and therefore we can conclude from the uniform boundedness of Q_N^{DE} that S is a family of equicontiuous functions. So $\mathcal{C}_1$ follows from the Arzelà-Ascoli theorem.

Due to the Theorem 2.2 and the relevant discussion to (11), the condition $\mathcal{C}_2$ holds. By considering (10) on $\mathfrak{B}$ and the continuity of $k_{uu}(t, s, u)$, $\mathcal{C}_3$ is easily concluded. ■

Lemma 4.3: Let $I - \mathcal{K}'(u^*)$ be nonsingular and the assumptions of Theorem 4.2 be fulfilled. Then for sufficiently large N, the linear operators $I - (\mathcal{K}_N^{DE})'(u^*)$ are nonsingular; furthermore,

$$\|(I - (\mathcal{K}_N^{SE})'(u^*))^{-1}\| \leq M,$$

where M is a constant independent of N.

Proof: Condition $\mathcal{C}_1$ is satisfied and $\{\mathcal{K}_N^{DE}(u^*) \mid N \geq N_1\}$ is equidifferentiable. Therefore, according to Theorem 6.10 in [14], $\{(\mathcal{K}_N^{DE})'(u) \mid N \geq N_1\}$ is a collectively compact family of operators. Moreover, from condition $\mathcal{C}_3$ and Theorem 6.11 of [14], we can conclude that $(\mathcal{K}_N^{DE})'(u)$ is pointwise convergent to $\mathcal{K}'(u)$ for all $u \in \mathfrak{B}$. So, the final result has been obtained from the existence of $(I - \mathcal{K}'(u^*))^{-1}$ and the theory of collectively compact operators. ■

Now we are ready to formulate the main result.

Theorem 4.4: Suppose that the assumptions of Lemma 4.3 hold. Then there exists a positive integer N_1 such that for all $N \geq N_1$, (8) has a unique solution $u_N^{DE} \in X$. Furthermore, there exists a constant C independent of N such that

$$\|u^* - u_N^{DE}\| \leq C \exp\left(\frac{-2\pi d N}{\log(\frac{2dN}{\alpha})}\right).$$

Proof: By subtracting (1) from (8) and adding the term $\mathcal{K}'(u^*)(u^* - u_N^{DE})$ on both sides, the following term has been obtained

$$(I - (\mathcal{K}_N^{DE})'(u^*))(u^* - u_N^{DE}) = \mathcal{K}(u^*) - \mathcal{K}_N^{DE}(u^*) -$$
$$[\mathcal{K}_N^{SE}(u_N^{DE}) - \mathcal{K}_N^{DE}(u^*) - (\mathcal{K}_N^{DE})'(u^*)(u_N^{DE} - u^*)]. \tag{12}$$

By applying $\|.\|$ on both sides of (12) and Lemma 4.3, we achieve the following relation

$$\|u^* - u_N^{DE}\| \leq M\{\|\mathcal{K}(u^*) - \mathcal{K}_N^{DE}(u^*)\| +$$
$$\|\mathcal{K}_N^{DE}(u_N^{DE}) - \mathcal{K}_N^{DE}(u^*) - (\mathcal{K}_N^{DE})'(u^*)(u_N^{DE} - u^*)\|\}.$$

The second term on the right-hand side has been bounded by the term $\frac{1}{2}\alpha\|u^* - u_N^{DE}\|$ by condition $\mathcal{C}_3$, and the finite result has been obtained from Theorem 2.2 ■

V. Numerical Experiments

In this section, the theoretical results of the previous sections are used for some numerical examples. The numerical experiments are implemented in *Mathematica* 7. The programs are executed on a PC with 2.00 GHz Intel Core 2 dual processor with 2 GB RAM. In order to analyze the error of the method the following notations are introduced:

$$e_{max} = max\{|u(t_i) - u_N(t_i)| : t_i = \frac{i}{1000}, i = 1(1)1000\},$$

and e_{max} approximate $\|u - u_N\|_\infty$. For the solution of the nonlinear system which arises in the formulation of the methods, one may use the steepest descent method, the Newton method or a mathematical software package. In our experiments we have used Mathematica's routine `FindRoot`. This routine needs an initial guess to solve the nonlinear systems. If the initial guess is selected badly, this routine may fail to converge to the desired solution. In these examples, an initial point is selected by the steepest descent method [17]. As we saw in Section IV, the convergence of the method depends on two parameters α and d. In fact the parameter d indicates the size of the holomorphic domain of u, and α is the order of the Hölder constant of kQ [7]. So due to the smoothness of the kernels, it is assumed that $\alpha = 1$ for all examples. The important parameter d values is 1.57 for the DE-sinc methods. e_{max} is reported for $N = 10\,(10)\,100$. In tables, we present the computing time T_N^{DE} measured in seconds when DE-sinc is used. Additionally, DESN is the abbreviations for Double Exponential Sinc Nyström methods. These tables show that by increasing N, the error is reduced significantly.

TABLE I
NUMERICAL RESULTS FOR EXAMPLE 5.1

N	DESN Method	T_N^{DE}
10	1.91E − 11	23.978
20	2.77E − 16	24.539
30	2.77E − 16	24.897
40	2.22E − 16	25.069
50	2.77E − 16	24.741
60	2.22E − 16	25.958
70	2.77E − 16	27.253
80	3.33E − 16	27.097
90	2.77E − 16	28.173
100	2.77E − 16	29.437

TABLE II
NUMERICAL RESULTS FOR EXAMPLE 5.2

N	DESN Method	T_N^{DE}
10	4.19E − 06	23.556
20	2.01E − 10	24.493
30	2.02E − 14	25.069
40	4.44E − 16	24.227
50	4.44E − 16	25.427
60	4.44E − 16	27.112
70	4.44E − 16	27.238
80	6.66E − 16	31.574
90	4.44E − 16	32.258
100	4.44E − 16	37.643

Example 5.1: The following Urysohn integral equation is considered

$$u(t) - \int_0^1 \frac{\mathrm{d}s}{2+t+u(s)} = g(t), \quad t \in [0,1], \tag{13}$$

where $g(t)$ is chosen so that $u^*(t) = \cos(0.3\,\pi t)$ is a solution of (13). This equation has been solved in [18] by three algorithms based on multigrid method. The best error obtained with the described multigrid method is 3.02×10^{-11} with 16.7 seconds computational time. TABLE I shows the error results achieved for the DE-sinc Nyström method.

Example 5.2: Consider

$$u(t) - \int_0^1 \frac{\mathrm{d}s}{t+s+\,u(s)} = g(t), \quad t \in [0,1],$$

with $g(t)$ is chosen so that $u^*(t) = \frac{1}{1+t}$. This Urysohn integral equation has been introduced and solved in [1] by the projection and iterated projection methods. TABLE 1 and TABLE 2 in [1] report the Galerkin and iterated Galerkin solution based on a piecewise polynomial space. TABLE II shows the DE-sinc Nyström results.

Example 5.3: We consider the following integral equation

$$u(t) + \int_0^1 \frac{t}{2}\cos(u(s))\,\mathrm{d}s = t, \quad t \in [0,1],$$

introduced by Döring in [19]. Its exact solution is $u^*(t) = \mathrm{q}t$, where q is a solution of the nonlinear equation

$$2t^2 - 2t + \sin(t) = 0.$$

In [4] the Chebyshev-Newton Type Method (CNTM) is considered. This method constructs a family of iterative processes free of derivatives, such as the classic secant method. By comparing TABLE III with Table 2 in [4], it is concluded that the presented methods are as efficient as the CNTM.

TABLE III
NUMERICAL RESULTS FOR EXAMPLE 5.3

N	DESN Method	T_N^{DE}
10	4.38E − 07	0.343
20	4.14E − 13	0.437
30	1.11E − 16	0.453
40	1.11E − 16	0.500
50	1.11E − 16	0.733
60	1.11E − 16	0.718
70	0.00E − 00	1.935
80	0.00E − 00	2.901
90	0.00E − 00	4.352
100	0.00E − 00	5.382

VI. CONCLUSION

Finding exact solutions for nonlinear the Fredholm integral equations are often not available. So approximating these solutions are very important. Many authors have proposed different methods. In this research, a numerical method based on sinc quadrature, the DE-sinc Nyström method has been suggested. It has been shown theoretically and numerically that the scheme is extremely accurate and achieve exponential convergence with respect to N.

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