Orthonormal Decomposition of Third Rank Tensors and Applications

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Abstract—A new procedure for representation of third rank tensors in terms of its orthonormal irreducible decomposed parts, namely as irreducible decomposition is presented. Orthonormal tensor basis method is developed by using the results of existing theory in the literature. As an example to third rank tensors, piezoelectricity tensor is decomposed by each method and results of this decomposition methods are compared for this tensor in hexagonal symmetry. As a result of comparison process, it is stated that the results for new method and other one are consistent and each decomposed parts have physical meaning. Moreover, the norm concept of piezoelectricity tensor is used to study the piezoelectric effect of some materials. It is also shown that one can determine in which material the piezoelectric effect is stronger by using the norm for different materials with the same symmetries.

Index Terms—third rank tensor, piezoelectricity tensor, irreducible decomposition method, orthonormal tensor basis method.

I. INTRODUCTION

Tensors are the most significant mathematical entities to describe direction dependent physical properties of solids and the tensor components characterizing physical properties which must be specified without reference to any coordinate system.

Piezoelectricity is an interaction between electrical and mechanical systems. The direct piezoelectric effect is that electric polarization is produced by mechanical stress. Closely related to it is the converse effect, whereby a crystal becomes strained when an electric field is applied. Both effects are manifestations of the same fundamental property of the crystal.

In the continuum approach, it is well known that certain physical properties can be represented by tensors. The polarization of a crystal produced by an electric field is an example of an anisotropic material property that is represented by tensors. If a stress is applied to certain crystals they develop an electric moment whose magnitude is proportional to the applied stress; known as piezoelectric effect.

The piezoelectric effect in materials has not attracted much attention until after the Second World War, since when the applications and the research of piezoelectric materials have advanced greatly. Piezoelectric materials nowadays have been widely used to manufacture various sensors, conductors, and actuators have been, extensively, applied in electronics, laser, ultrasonic, naval and space navigation as well as biology, smart structures and many other high-tech areas. They also play an important role in the so-called smart structures.

The direct piezoelectric effect comprises a group of phenomena in which the mechanical stresses or strains induce in crystals an electric polarization (electric field) proportional to these factors. Besides, the mechanical and electrical quantities are found to be linearly related [1]:

\[ P_i = d_{ijk} \sigma_{jk} \]  

(1)

Where \( P_i \) and \( \sigma_{jk} \) denote the components of the electric polarization vector and the components of the mechanical stress tensor respectively and \( d_{ijk} \) are the piezoelectric coefficient forming a rank-three tensor. The coefficients \( d_{ijk} \) are usually referred to as piezoelectric moduli. The piezoelectric tensor is a third rank tensor symmetric with respect to the last two indices which means that

\[ d_{ijk} = d_{ikj} \]  

(2)

is reduced from 27 to 18 independent coefficients for the triclinic system. For the monoclinic system of class 2, for example, the number of independent coefficients is reduced to eight, for the orthotropic system of class mm2 is reduced to five coefficients and for the hexagonal system of class 6mm is reduced to three independent coefficients [2].

The indices are abbreviated according to the replacement rule given in the following TABLE:

| TABLE I  
| ABBREVIATION OF INDICES FOR THREE AND DOUBLE INDEX NOTATIONS |
| Three index notation | 11 22 33 23 32 13 31 12 12 |
| Double index notation | 1 2 3 4 5 6 |

Decomposition of tensor is not new (see, for instance, in [2]-[4]) such as canonical tensor and single value decompositions. In fact the methods presented here, provides new perspectives for decomposition of third rank tensors. This work is an extension of the work [5] by means of applying the orthonormal basis method to third rank tensors.

One of the aims of this work is to present a new method based on orthonormal irreducible representations for third rank tensors such as piezoelectricity tensors and compare this method with the orthonormal tensor basis method developed by the existing theory[5] in the literature.
Another one is to elaborate on the norm concept for different materials in order to determine the degree of anisotropy and the piezoelectric effect of these materials.

Main outline of the paper is listed as a brief description for irreducible decomposition and presentation of orthonormal tensor basis method explicitly. Both methods are compared. Next the concept of norm is revealed to measure the overall effect of material properties. Numerical engineering applications are presented for several piezoelectric materials like semiconductor compounds and piezoelectric ceramics. Finally, conclusions pertinent to this work are also stated.

II. IRREDUCIBLE DECOMPOSITION METHOD

In this section a procedure of decomposing third rank tensors into orthonormal parts which are irreducible under the three dimensional rotation group is given. Explicit results for third rank tensor are produced.

Any rank-n cartesian tensor can be written as the direct sum of irreducible tensors in the cartesian representation. The term irreducible indicates sets that cannot be resolved into subsets with separate linear transformations.

The irreducible tensors of the first five ranks have special names; Scalar (zero-rank tensor of valence 0), vector (first-rank tensor of valence 1.), deviator (second-rank tensor of valence 2.), sektor (third-rank tensor of valence 3.), nonor (a fourth-rank tensor of valence 4). The irreducible decomposition method can be investigated under the title of groups and reflection symmetries. The group of rotations associated with elastic symmetry provides an irreducible representation. There are various related ways of considering elasticity tensors in terms of rotational group properties of tensors for example based on subgroups of O(3) or SO(3). These ideas are closely related to definitions of elastic symmetry in terms of a single symmetry element: reflection about a plane.

For second-rank tensor, there are three irreducible parts which are 1 scalar, 1 vector and 1 deviator. For third-rank tensor, there are seven irreducible parts which are 1 scalar, 3 vectors, 2 deviators and one septor. These irreducible parts are defined explicitly through the relation

\[ \delta_{pq} E^{(j)}_{k_{1}k_{2}...k_{j};i_{1}l_{1}...l_{j}} Q^{(p)}_{l_{1}...i_{1},k_{1}...k_{j}} Q^{(q)}_{l_{2}...i_{2},k_{2}...k_{j}} - Q^{(p)}_{l_{1}...i_{1},k_{1}...k_{j}} Q^{(q)}_{l_{2}...i_{2},k_{2}...k_{j}} E^{(j)}_{k_{1}k_{2}...k_{j};i_{1}l_{1}...l_{j}}. \]

The mappings \( Q^{(p)} \) and \( Q^{(q)} \) are dual to \( Q^{(p)} \) and \( Q^{(q)} \) defined by the relation

\[ Q^{(p)} Q^{(q)} = \delta_{pq}. \]

The dual mappings extract the natural forms \( T^{(j)}_{i_{1}i_{2}...i_{j}} \) from the tensor \( T_{i_{1}i_{2}...i_{j}} \) as

\[ T^{(j)}_{i_{1}i_{2}...i_{j}} = Q^{(0)}_{i_{1}i_{2}...i_{j}} T_{i_{1}i_{2}...i_{j}}. \]

These tensors can be embedded in the tensor space of order n through the mapping

\[ T^{(j)}_{i_{1}i_{2}...i_{j}} = Q^{(0)}_{i_{1}i_{2}...i_{j}} k_{1}k_{2}...k_{j} T_{i_{1}i_{2}...i_{j}} T_{k_{1}k_{2}...k_{j}}. \]

There are total of seven irreducible parts (one scalar, 3 vectors, 2 deviators and one septor). These irreducible parts can be obtained by using (11) as:

\[ T^{(0)}_{i_{1}i_{2}...i_{j}} = \frac{1}{6} \epsilon_{ijk} \epsilon_{jkl} T_{kij}. \]

(12)

\[ T^{(1)}_{i_{1}i_{2}...i_{j}} = \frac{1}{2} \delta_{ijk} T_{kij}. \]

(13)

\[ T^{(2)}_{i_{1}i_{2}...i_{j}} = \frac{1}{2} \delta_{ijk} (T_{jip} + T_{jpi}) + \frac{1}{4} \delta_{ij} (T_{kpp} - T_{ppk}). \]

(14)

\[ T^{(3)}_{i_{1}i_{2}...i_{j}} = \frac{1}{2} \delta_{ijk} (T_{jip} + T_{jpi} - 6 T_{jpp}) + \delta_{ij} (T_{9pp} + 9 T_{pp9} - 9 T_{pp9}). \]

(15)

\[ T^{(2)}_{ijk} = \frac{1}{2} \epsilon_{ijk} \epsilon_{jkl} \left( \frac{1}{2} (\delta_{ij} T_{kij} - \delta_{ik} T_{kij}) - \frac{1}{2} \delta_{j} T_{jkl} \right). \]

(16)

\[ T^{(2)}_{ijk} = \frac{1}{2} \epsilon_{ijk} \epsilon_{jkl} \left( \frac{1}{2} (\delta_{ij} T_{kij} + \delta_{ik} T_{kij}) - \frac{1}{2} \delta_{j} T_{jkl} \right) - \frac{1}{2} \epsilon_{ijk} \epsilon_{jkl} \left( \frac{1}{2} (\delta_{ij} T_{kij} + \delta_{ik} T_{kij}) + \frac{1}{2} \delta_{j} T_{jkl} \right). \]
Decomposition of the third rank cartesian tensor into irreducible parts was given by the work of [8] are not the same as these results. In this work, irreducible parts are orthonormal to each other but theirs are not. The only similarity is that the sum of the irreducible parts for certain weight are the same, e.g. for weight \( j=1 \): The sum is the same but the individual parts are different.

As an application of decomposition of third rank tensors, piezoelectric tensor \( \mathbf{d} \) is represented in terms of its orthonormal irreducible parts.

The following irreducible parts for the piezoelectric tensor \( \mathbf{d} \) are obtained by the application of the index symmetry condition (2) to (12)-(18).

\[
\begin{align*}
\mathbf{d}_{1}^{(1)} &= \frac{1}{3} \delta_{jkl} d_{iss} \\
\mathbf{d}_{1}^{(3)} &= \frac{1}{20} (9 \delta_{dss} - 3d_{dss} + \delta_{ij}(9d_{dsk} - 3d_{kss})) \\
\mathbf{d}_{1}^{(5)} &= \frac{1}{15} \delta_{jkl} (2d_{dss} + d_{dss}) \\
\mathbf{d}_{1}^{(7)} &= \frac{1}{15} \delta_{jkl} (2d_{dss} + d_{dss})
\end{align*}
\]

So we have two vectors and one septor part which are the same in number predicted by group theoretical methods for this internal tensor symmetry. Here these decomposed parts are orthonormal to each other but not those in the work of [8].

III. ORTHONORMAL TENSOR BASIS METHOD

This method comprises of two basic steps which are constructing form-invariant and orthonormal basis elements. [5] The form invariant expressions are derived for many classes of piezomagnetic and piezoelectric coefficients [9]. Although such constitutive equations are form invariant with respect to arbitrary orthogonal coordinate transformations, the coefficients, \( d_{ijk} \), do not determine directly the material constants since their values vary with the direction of the coordinate axes.

The form-invariant expressions [9] for the piezoelectric coefficients is respectively.

\[
d_{ijk} = v_{aij} y_{jk} c_{a} A_{abc}
\]

where summation is implied by repeated indices and this convention is followed throughout. This expression is referred to a Cartesian system \( Oxyz \); \( v_{aij} \) are the components of the unit vectors \( \mathbf{v}_{a} \) (\( a = 1, 2, 3 \)) along the crystallographic axes. The quantity \( A_{abc} \) is invariant in the sense that when the Cartesian system is rotated to a new orientation \( O'x'y'z' \), then (22) takes the form

\[
d'_{ijk} = v'_{aij} y'_{jk} c'_{a} A_{abc}
\]

It should be remembered that \( v_{1}, v_{2}, v_{3} \) form a linearly independent basis in three dimensions but are not necessarily always orthogonal. Let us consider the hexagonal symmetry as an example. The form invariant expression for the hexagonal system class \( 6mm \) is [9]

\[
d_{ijk} = d_{12} v_{ij} v_{jk} + d_{2} (v_{3} \delta_{ij} + v_{3} \delta_{jk}) + d_{3} v_{3} \delta_{ik} \]

where \( v_{3} \) is the sixfold axis. A similar form can be derived from tetragonal symmetry (class \( 4mm \)) [11]

The first step in the generation of orthonormal tensor basis is one of writing the \( \delta_{ai} \) in the place of \( v_{ai} \) in (22). It will assume respectively the form

\[
d_{ijk} = \delta_{ai} \delta_{bj} \delta_{ck} A_{abc}
\]

One can subject the expression (25) to the symmetry of any crystal and then derive the elements of the basis appropriate to that class. Instead the form-invariant expression for any given class can be taken and straightforward replaced the \( v_{ai} \) by the \( \delta_{ai} \) to obtain the elements of the basis. As an illustration, let us consider the simplest example, namely, the expression (24). According to the present scheme, the elements of the basis are

\[
\begin{align*}
\delta_{3} \delta_{3} \delta_{3} & \delta_{jkl} \\
\delta_{3} \delta_{jkl} + \delta_{3} \delta_{jkl} & = \delta_{ij}
\end{align*}
\]

This is a particular case of a more general identity.

\[
v_{ai} v_{bj} + v_{ai} v_{bj} - \cos \theta (v_{ai} v_{bj} + v_{bj} v_{ai}) + \sin^{2} \theta v_{aij} = \sin^{2} \theta \delta_{ij}
\]

with \( v_{ai} \) is replaced by \( \delta_{ai} \) and \( \theta = 90^{\circ} \). On subjecting these elements to the Gram-Schmidt process, we obtain,

\[
\begin{align*}
T_{ijk}^{(1)} & = \delta_{3} \delta_{3} \delta_{3} \delta_{3} \\
T_{ijk}^{(2)} & = \frac{1}{\sqrt{2}} (\delta_{3} \delta_{3} \delta_{3} \delta_{3}) \\
T_{ijk}^{(3)} & = \frac{1}{\sqrt{2}} (\delta_{3} \delta_{3} \delta_{3} \delta_{3})
\end{align*}
\]

These are the elements of the basis for the most general case, namely, the noncentrosymmetric triclinic case.

In the actual exercise, starting with (25) and following the recipe to construct the orthonormal tensor basis spans the space of the third-rank tensor representing the piezoelectric effect and having the index symmetry \( d_{ijk} = d_{i(kj)} \). To so the use of the identity (27) is understood.

In terms of this basis, the representation of \( d_{ijk} \) is given by

\[
d_{ijk} = \sum_{k} (d, T^{k}) T_{ijk}^{k}
\]

where

\[
(d, T^{k}) = d_{ijk} T_{ijk}^{k}
\]

represents the inner product of \( T_{ijk}^{k} \) and the \( k \)th element \( T_{ijk}^{k} \); of the basis. The expressions for the inner product of \( d_{ijk} \) with each element of the basis are listed as

\[
\begin{align*}
(d, T^{1}) & = \frac{1}{\sqrt{2}} (d_{1}(d_{1} + d_{2})) \\
(d, T^{2}) & = \frac{1}{\sqrt{2}} (d_{1}(d_{1} + d_{2})) \\
(d, T^{3}) & = \frac{1}{\sqrt{2}} (d_{1}(d_{1} + d_{2})) \\
(d, T^{4}) & = \frac{1}{\sqrt{2}} (d_{1}(d_{1} + d_{2})) \\
(d, T^{5}) & = \frac{1}{\sqrt{2}} (d_{1}(d_{1} + d_{2})) \\
(d, T^{6}) & = \frac{1}{\sqrt{2}} (d_{1}(d_{1} + d_{2}))
\end{align*}
\]

These are the elements of the basis for the most general case, namely, the noncentrosymmetric triclinic case.
IV. COMPARISON OF THE METHODS

In previous sections, for piezoelectric tensor as an example for third rank tensors irreducible decomposition method is introduced and orthonormal tensor basis method is described in detail.

Let us compare these two methods for hexagonal symmetric materials. By using the symmetry condition in (2) and applying the formula in (30), piezoelectric tensor for hexagonal symmetry is represented in terms of the following orthonormal decomposed parts:

\[
\begin{align*}
(d_{ijk}) = & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2}(d_{31} + d_{32}) & \frac{1}{2}(d_{31} + d_{32}) & 0 & 0 & 0 & 0 \\
\end{bmatrix} \frac{1}{2}(d_{15} + d_{24}) \begin{bmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} \\
& + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2}(d_{15} + d_{24}) & \frac{1}{2}(d_{15} + d_{24}) & 0 & 0 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} \\
& + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} \\
\end{align*}
\]

(33) indicates that the hexagonal symmetric third rank tensor, \(d_{ijk}\) is a subset of the general symmetric third rank tensor and decomposed into three terms, each of which has a distinct physical meaning. It is easy to verify that the three decomposed parts form an orthogonal set and their sum is the hexagonal symmetric third rank tensor, \(d_{ijk}\) of class \(6\text{mm}\) which is identical to following matrix:

\[
\begin{align*}
(d_{ijk}) = & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2}(d_{31} + d_{32}) & \frac{1}{2}(d_{31} + d_{32}) & 0 & 0 & 0 & 0 \\
\frac{1}{2}(d_{15} + d_{24}) & \frac{1}{2}(d_{15} + d_{24}) & 0 & 0 & 0 & 0 \\
\end{bmatrix} \\
\end{align*}
\]

(34)

So \(d_{31} = d_{32}\) and \(d_{24} = d_{15}\)

(35)

Physically \(d_{ijk}\) is decomposed into three independent tensors, each has an independent piezoelectric coefficient and physical piezoelectric effect. If a tensile stress \(\sigma_3\) is applied parallel to \(x_3\) which is a diad axis of the crystal. The first matrix in (33) shows that the components of polarization are given by the moduli in the third column of the first matrix; so

\[
P_1 = 0, \quad P_2 = 0, \quad P_3 = d_{33}\sigma_3
\]

(37)

The polarization therefore directed along \(x_3\).

On the other hand, a tensile stress \(\sigma_1\) along \(x_1\) produces no polarization parallel to itself, but it produces a polarization along \(x_3\) which is introduced in the second matrix of (33), the tensile stress \(\sigma_1\) along \(x_1\) produces

\[
P_1 = 0, \quad P_2 = 0, \quad P_3 = d_{31}\sigma_1
\]

(38)

Similarly, for a tensile stress \(\sigma_2\) along \(x_2\), produces no polarization parallel to itself, but it produces a polarization along \(x_3\).

The tensile stress \(\sigma_2\) along \(x_1\) produces

\[
P_1 = 0, \quad P_2 = 0, \quad P_3 = d_{31}\sigma_2
\]

(39)

For the third matrix in (33), the polarization along \(x_2\) can be produced by a shear stress \(\sigma_{12}\) about \(x_2\), so, for this stress

\[
P_1 = 0, \quad P_2 = d_{12}\sigma_{12}, \quad P_3 = 0
\]

and the polarization along \(x_1\) can be produced by a shear stress \(\sigma_{12}\) about \(x_2\), so, for this stress

\[
P_1 = 0, \quad P_2 = d_{12}\sigma_{12}, \quad P_3 = 0
\]

(40)

Thus orthonormal tensor basis method presented, is decomposing the polarization along orthonormal axes into three parts; the first part is the polarization along the diad axes due to normal stress, the second part is the polarization along nondiad orthogonal axes due to normal stress and the third part is the polarization due to the shear stresses.

By applying the symmetry condition in (2) and applying the formulas in (19)-(21), piezoelectric tensor for hexagonal symmetry is represented in terms of the following irreducible orthonormal decomposed parts:

\[
\begin{align*}
(d_{ijk}) = & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3}d_{31} + \frac{1}{3}d_{32} & \frac{1}{3}d_{31} + \frac{1}{3}d_{32} & \frac{1}{3}d_{31} + \frac{1}{3}d_{32} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \\
+ & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\frac{6d_{11}}{15} & 0 & 0 & 0 & 0 & 0 \\
-6d_{11} & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \\
+ & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \\
\end{align*}
\]

(42)

Where \(a = \frac{6d_{11} + 6d_{12} + 6d_{13}}{15}\) and \(b = \frac{6d_{12} + 6d_{13}}{15}\).

The decomposed parts for hexagonal symmetry in (42) are different those in (33). So polarization decomposition is not valid in irreducible decomposition method. However, decomposed parts obtained from both methods are orthonormal.

Furthermore, the irreducible cartesian tensor parts that obtained in this work are different than the irreducible cartesian tensor parts appeared in the literature [6],[7],[8]. mappings \(Q_{ijk\ldots n}\) have been chosen such that they are orthonormal but theirs are not.

It should be pointed out that there may be some arbitrariness in choosing the first mappings for certain weight. Not paying attention to this point may lead to different physically meaningless orthonormal irreducible parts. In irreducible decomposition procedure, to obtain a unique physically meaningful orthonormal irreducible parts set, the restriction imposed is that the indices of different deltas and epsilon of the first mapping should be chosen in the same order as they are written in the cartesian tensor itself (it can be called as it the first natural choice or the first choice). For instance, for the third rank tensor \(n = 3, j = 1\):

\(T_{ijk} \rightarrow \delta_{ir}\delta_{jk}.\) Here the index \(r\) is inserted to make the number of indices and the number of mapping indices equal to each other.

For example, considering the vector part, in this case, there are three choices for the mappings (\(j = 1\), each alternative will produce different orthonormal irreducible sets of three in number.

On the other hand, application of the internal symmetry of the piezoelectric tensor (which is symmetric with respect to
its last two indices, i.e., \( d_{ijk} = d_{(k)} \) to the first choice mapping set will produce two irreducible parts which is the same number as predicted by group theoretical methods [6] and application of the same symmetry condition to the other two mapping choices will not produce the same number (they produce three), so they are rejected.

V. NORM CONCEPT

Norm is an invariant of the material. Generalizing the concept of the modulus of a vector, norm of a Cartesian tensor (or the modulus of a tensor) is defined as the square root of the contracted product over all indices with itself:

\[
N = \| \mathbf{T} \| = \left( T_{ijkl} T_{ijkl} \right)^{1/2}
\]

(43)

Denoting rank \( n \) Cartesian tensor \( T_{ijkl} \)...... by \( T_n \), the square of the norm is expressed as Jerphagnon et al.

This definition is consistent with the reduction of the tensor in Cartesian formulation when all the irreducible parts are embedded in the original rank \( n \) tensor space.

Since the norm of a Cartesian tensor is an invariant quantity, following rule is suggested:

The norm of a Cartesian tensor may be used as a criterion for representing and comparing the overall effect of a certain property of piezoelectric materials of the same or different symmetry. At this stage, the norm ratios:

\[
\frac{N_v}{N} \quad \text{for vector part and} \quad \frac{N_{sp}}{N} \quad \text{for septor part}
\]

are defined. In this work, norms and norm ratios of the irreducible parts are used as a criterion. The larger the norm ratio value exists, the stronger the material property is.

A. Applications

Among semiconductors crystals, a family of quartzite-type belongs to the 6mm class, which is piezoelectric active. Piezoelectric tensor data and norm are tabulated, norm ratio calculations for semiconductors in TABLE II and III respectively.

<table>
<thead>
<tr>
<th>Material</th>
<th>( N_v )</th>
<th>( N_{sp} )</th>
<th>( N )</th>
<th>( N_v/N )</th>
<th>( N_{sp}/N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>ZnO</td>
<td>4.05</td>
<td>19.337</td>
<td>19.757</td>
<td>0.205</td>
<td>0.979</td>
</tr>
<tr>
<td>CdS</td>
<td>11.635</td>
<td>24.695</td>
<td>27.299</td>
<td>0.426</td>
<td>0.905</td>
</tr>
<tr>
<td>CdSe</td>
<td>8.059</td>
<td>18.000</td>
<td>19.722</td>
<td>0.409</td>
<td>0.913</td>
</tr>
</tbody>
</table>

By taking into account the rule, the most piezoelectric effective among these three materials is CdS which has a very important feature in the thin films of semiconductors. Piezoelectric ceramic is the most potential piezoelectric material because of its higher strength, high rigidity and more importantly, the better piezoelectricity. TABLES IV and V include the piezoelectric coefficients and calculated norms for these materials.

<table>
<thead>
<tr>
<th>Material</th>
<th>( d_{31} )</th>
<th>( d_{33} )</th>
<th>( d_{15} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>PZT-4</td>
<td>-5.2</td>
<td>15.1</td>
<td>12.7</td>
</tr>
<tr>
<td>PZT-5</td>
<td>-5.4</td>
<td>15.8</td>
<td>12.3</td>
</tr>
<tr>
<td>PZT-5H</td>
<td>-6.5</td>
<td>23.3</td>
<td>17</td>
</tr>
<tr>
<td>PZT-8</td>
<td>-6</td>
<td>23.3</td>
<td>10.4</td>
</tr>
</tbody>
</table>

TABLE II

<table>
<thead>
<tr>
<th>Materials</th>
<th>( d_{31} )</th>
<th>( d_{33} )</th>
<th>( d_{15} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>ZnO</td>
<td>-5.0</td>
<td>12.4</td>
<td>-8.3</td>
</tr>
<tr>
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<td>-5.2</td>
<td>10.3</td>
<td>-14</td>
</tr>
<tr>
<td>CdSe</td>
<td>-3.9</td>
<td>7.8</td>
<td>-10</td>
</tr>
</tbody>
</table>
Among these piezoceramics the piezoelectric effect in PZT-5H is the strongest.

VI. CONCLUSION

Any physical property is characterized by n-rank tensors and decomposition methods presented in this work are capable for decomposing these tensors with intrinsic symmetry which is derived from the nature of the physical property itself into orthonormal parts. In other words, these methods of constructing orthonormal decomposed parts can be easily extended to (physical property) tensor of any rank.

In this work, they are applied to piezoelectric tensor. The third rank tensor, like piezoelectric tensors are of interest in engineering.

To summarize, irreducible decomposition method is a new procedure in literature which gives orthonormal parts and decomposition methods presented in this work are both qualitatively and quantitatively different from isotropic materials.

REFERENCES


<table>
<thead>
<tr>
<th>Material</th>
<th>$N_1$</th>
<th>$N_{op}$</th>
<th>$N$</th>
<th>$N/N$</th>
<th>$N_{op}/N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>PZT-4</td>
<td>25.462</td>
<td>3.557</td>
<td>25.709</td>
<td>0.990</td>
<td>0.138</td>
</tr>
<tr>
<td>PZT-5</td>
<td>25.360</td>
<td>2.892</td>
<td>25.525</td>
<td>0.994</td>
<td>0.113</td>
</tr>
<tr>
<td>PZT-5H</td>
<td>35.507</td>
<td>3.522</td>
<td>35.681</td>
<td>0.995</td>
<td>0.099</td>
</tr>
<tr>
<td>PZT-8</td>
<td>26.822</td>
<td>4.010</td>
<td>27.120</td>
<td>0.989</td>
<td>0.148</td>
</tr>
</tbody>
</table>

TABLE V

THE NORMS AND NORM RATIOS (THE ANISOTROPY DEGREES) FOR PIEZOELECTRIC CERAMICS