Orthonormal Decomposition of Third Rank Tensors and Applications

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Abstract—**A new procedure for representation of third rank tensors in terms of its orthonormal irreducible decomposed parts, namely as irreducible decomposition is presented. Orthonormal tensor basis method is developed by using the results of existing theory in the literature. As an example to third rank tensors, piezoelectricity tensor is decomposed by each method and results of this decomposition methods are compared for this tensor in hexagonal symmetry. As a result of comparison process, it is stated that the results for new method and other one are consistent and each decomposed parts have physical meaning. Moreover, the norm concept of piezoelectricity tensor is used to study the piezoelectric effect of some materials. It is also shown that one can determine in which material the piezoelectric effect is stronger by using the norm for different materials with the same symmetries.**

***Index Terms*—third rank tensor, piezoelectricity tensor, irreducible decomposition method, orthonormal tensor basis method.**

I. Introduction

TENSORS are the most significant mathematical entities to describe direction dependent physical properties of solids and the tensor components characterizing physical properties which must be specified without reference to any coordinate system.

Piezoelectricity is an interaction between electrical and mechanical systems. The direct piezoelectric effect is that electric polarization is produced by mechanical stress. Closely related to it is the converse effect, whereby a crystal becomes strained when an electric field is applied. Both effects are manifestations of the same fundamental property of the crystal.

In the continuum approach, it is well known that certain physical properties can be represented by tensors. The polarization of a crystal produced by an electric field is an example of an anisotropic material property that is represented by tensors. If a stress is applied to certain crystals they develop an electric moment whose magnitude is proportional to the applied stress; known as piezoelectric effect.

The piezoelectric effect in materials has not attracted much attention until after the Second World War, since when the applications and the research of piezoelectric materials have advanced greatly. Piezoelectric materials nowadays have been widely used to manufacture various sensors, conductors, and actuators have been, extensively, applied in electronics, laser, ultrasonic, naval and space navigation as well as biology, smart structures and many other high-tech areas. They also play an important role in the so-called smart structures.

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The direct piezoelectric effect comprises a group of phenomena in which the mechanical stresses or strains induce in crystals an electric polarization (electric field) proportional to these factors. Besides, the mechanical and electrical quantities are found to be linearly related [1]:

$$P_i = d_{ijk}\sigma_{jk} \tag{1}$$

Where P_i and t_{jk} denote the components of the electric polarization vector and the components of the mechanical stress tensor respectively and d_{ijk} are the piezoelectric coefficient forming a rank-three tensor. The coefficients d_{ijk} are usually referred to as piezoelectric moduli. The piezoelectric tensor is a third rank tensor symmetric with respect to the last two indices which means that

$$d_{ijk} = d_{ikj} \tag{2}$$

is reduced from 27 to 18 independent coefficients for the triclinic system. For the monoclinic system of class 2, for example, the number of independent coefficients is reduced to eight, for the orthotropic system of class mm2 is reduced to five coefficients and for the hexagonal system of class 6mm is reduced to three independent coefficients [2].

The indices are abbreviated according to the replacement rule given in the following TABLE:

TABLE I
Abbreviation of Indices for Three and Double Index Notations

Three index notation	11	22	33	23, 32	13, 31	12, 12
Double index notation	1	2	3	4	5	6

Decomposition of tensor is not new (see, for instance, in [2]-[4]) such as canonical tensor and single value decompositions. In fact the methods presented here, provides new perspectives for decomposition of third rank tensors. This work is an extension of the work [5] by means of applying the orthonormal basis method to third rank tensors.

One of the aims of this work is to present a new method based on orthonormal irreducible representations for third rank tensors such as piezoelectricity tensors and compare this method with the orthonormal tensor basis method developed by the existing theory[5] in the literature.

Another one is to elaborate on the norm concept for different materials in order to determine the degree of anisotropy and the piezoelectric effect of these materials.

Main outline of the paper is listed as a brief description for irreducible decomposition and presentation of orthonormal tensor basis method explicitly. Both methods are compared. Next the concept of norm is revealed to measure the overall effect of material properties. Numerical engineering applications are presented for several piezoelectric materials like semiconductor compounds and piezoelectric ceramics. Finally, conclusions pertinent to this work are also stated.

II. Irreducible Decomposition Method

In this section a procedure of decomposing third rank tensors into orthonormal parts which are irreducible under the three dimensional rotation group is given. Explicit results for third rank tensor are produced.

Any rank-n cartesian tensor can be written as the direct sum of irreducible tensors in the cartesian representation. The term irreducible indicates sets that cannot be resolved into subsets with seperate linear transformations.

The irreducible tensors of the first five ranks have special names; Scalar (zero-rank tensor of valance 0), vector (first-rank tensor of valence 1.), deviator (second-rank tensor of valence 2.), septor (third-rank tensor of valence 3.), nonor (a fourth-rank tensor of valence 4). The irreducible decomposition method can be investigated under the title of groups and reflection symmetries. The group of rotations associated with elastic symmetry provides an irreducible representation. There are various related ways of considering elasticity tensors in terms of rotational group properties of tensors for example based on subgroups of O(3) or SO(3). These ideas are closely related to definitions of elastic symmetry in terms of a single symmetry element: reflection about a plane.[6], [7]

For second-rank tensor, there are three irreducible parts which are 1 scalar, 1 vector and 1 deviator. For third-rank tensor, there are seven irreducible parts which are 1 scalar, 3 vectors, 2 deviators, 1 septor and for fourth-rank tensor, there are 3 scalars, 6 vectors, 6 deviators, 3 septors and 1 nonor.

The reduction of a (rank-n) Cartesian tensor $\mathbf{T}_{(n)}$ generally results in a sum of irreducible tensors, with some weights (j) represented more than once. (where $0 \le j \le n$), it can be accomplished by the formula:

$$\mathbf{T}_{(n)} = \sum_{j=0}^{n} \sum_{q=1}^{N_n^{(j)}} \mathbf{T}_{(n)}^{(j;q)}, \tag{3}$$

where q is called the seniority index of the irreducible tensor $\mathbf{T}_{(n)}^{(j;q)}$ (irreducible cartesian tensor which is symmetric and traceless) and $N_n^{(j)}$ is the multiplicity of weight j in this reduction, it denotes the number of independent weight-j irreducible tensor parts. (See for instance, Jerphagnon et al. [6] and Andrews and Ghoul [5])

$$N_n^{(j)} = \sum_k (-1)^k \binom{n}{k} \binom{2n-3k-j-2}{n-2}, \tag{4}$$

Each irreducible tensor has (2j+1) independent components. So that the total number of components in the reduction is

$$\sum_{j=0}^{n} (2j+1) N_n^{(j)} = 3^n. \tag{5}$$

The natural projection of x^j onto the irreducible subspace H_j^j of traceless symmetric tensors of order j is denoted by $E^{(j)} = E^{(j)}_{k_1k_2...k_j;l_1l_2...l_j}$. The principal element in the reduction procedure is the mappings $Q^{(0;q)}_{i_1i_2...i_n;k_1k_2...k_j}$ of the minimal rank tensor subspace $H^j_{j,q}$ onto $H^n_{j,q}$, we have chosen the mappings $Q^{(0;q)}_{i_1i_2...i_n;k_1k_2...k_j}$ such that they are orthonormal and g_{pq} will be reduced to identity matrix, δ_{ij}, where g_{pq} is a symmetric matrix which was used and defined in Andrews and Ghoul [7] through the relation

$$g_{pq} E^{(j)}_{k_1k_2...k_j;l_1l_2...l_j} = Q^{(0;p)}_{i_1i_2...i_n;k_1k_2...k_j} Q^{(0;q)}_{i_1i_2...i_n;l_1l_2...l_j}. \tag{6}$$

In this work, this relation is reduced to

$$\delta_{pq} E^{(j)}_{k_1k_2...k_j;l_1l_2...l_j} = Q^{(0;p)}_{i_1i_2...i_n;k_1k_2...k_j} Q^{(0;q)}_{i_1i_2...i_n;l_1l_2...l_j}. \tag{7}$$

The mappings $Q'^{(0;p))}_{k_1k_2...k_j;i_1i_2...i_n}$ dual to $Q^{(0;p)}_{i_1i_2...i_n;k_1k_2...k_j}$ are defined by the relation

$$Q'^{(0;p)}_{k_1k_2...k_j;i_1i_2...i_n} = Q^{(0;p)}_{i_1i_2...i_n;k_1k_2...k_j} \tag{8}$$

The dual mappings extract the natural forms $t^{(j;p)}_{\lambda_1\lambda_2...\lambda_j}$ from the tensor $T_{i_1i_2...i_n}$ as

$$t^{(j;p)}_{\lambda_1\lambda_2...\lambda_j} = Q'^{(0;p)}_{\lambda_1\lambda_2....\lambda_j;i_1i_2....i_n} T_{i_1i_2.....i_n} \tag{9}$$

These tensors can be embedded in the tensor space of order n through the mapping

$$T^{(j;q)}_{i_1i_2...i_n} = Q^{(0;q)}_{i_1i_2...i_n;k_1k_2...k_j} t^{(j;p)}_{k_1k_2...k_j}, \tag{10}$$

or

$$T^{(j;q)}_{i_1i_2...i_n} = Q^{(0;q)}_{i_1i_2...i_n;k_1k_2...k_j} Q'^{(0;p)}_{k_1k_2....k_j;l_1l_2....l_n} T_{l_1l_2...l_n}, \tag{11}$$

There are total of seven irreducible parts (one scalar, 3 vectors, 2 deviators and one septor). These irreducible parts can be obtained by using (11) as:

$$T^{(0;1)}_{ijk} = \frac{1}{6}\epsilon_{ijk}\epsilon_{rst}T_{rst} \tag{12}$$

$$T^{(1;1)}_{ijk} = \frac{1}{3}\delta_{jk}T_{ipp} \tag{13}$$

$$T^{(1;2)}_{ijk} = \frac{1}{4}\delta_{ik}\left(T_{pjp} - T_{ppj}\right) + \frac{1}{4}\delta_{ij}\left(T_{ppk} - T_{pkp}\right) \tag{14}$$

$$T^{(1;3)}_{ijk} = \frac{1}{60}\left[\delta_{ik}\left(9T_{pjp} + 9T_{ppj} - 6T_{jpp}\right) + \delta_{ij}\left(9T_{pkp} + 9T_{ppk} - 6T_{kpp}\right)\right] \\ \delta_{jk}\left(6T_{pip} + 6T_{ppi} - 4T_{ipp}\right) \tag{15}$$

$$T^{(2;1)}_{ijk} = \frac{1}{2}\epsilon_{ljk}\epsilon_{rst}\left[\frac{1}{2}\left(\delta_{lr}T_{ist} - \delta_{ir}T_{lst}\right) - \frac{1}{3}\delta_{il}T_{rst}\right] \tag{16}$$

$$T^{(2;2)}_{ijk} = \left[\frac{1}{6}\epsilon_{ljk}\epsilon_{rst}\left[\frac{1}{2}\left(\delta_{lr}T_{jst} + \delta_{jr}T_{lst}\right) - \frac{1}{3}\delta_{jl}T_{rst}\right]\right] \\ -2\epsilon_{ljk}\epsilon_{rst}\left[\frac{1}{2}\left(\delta_{lr}T_{sit} + \delta_{ir}T_{slt}\right) - \frac{1}{3}\delta_{il}T_{srt}\right]$$

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$$-2\epsilon_{lik}\epsilon_{rst}\left[\frac{1}{2}(\delta_{lr}T_{jst}+\delta_{jr}T_{lst})-\frac{1}{3}\delta_{jl}T_{rst}\right]+ 4\epsilon_{ljk}\epsilon_{rst}\left[\frac{1}{2}(\delta_{lr}T_{sjt}+\delta_{jr}T_{slt})-\frac{1}{3}\delta_{jl}T_{srt}\right] \quad (17)$$

$$T_{ijk}^{(3;1)}=\frac{1}{6}(T_{ijk}+T_{ikj}+T_{jik}+T_{jki}+T_{kij}+T_{kji})-\frac{1}{15}[\delta_{ij}(T_{ppk}+T_{pkp}+T_{kpp})+\delta_{ik}(T_{ppj}+T_{pjp}+T_{jpp})]+\frac{1}{15}\delta_{jk}(T_{ppi}+6T_{pip}-4T_{ipp}) \quad (18)$$

Decomposition of the third rank cartesian tensor into irreducible parts was given by the work of [8] are not the same as these results. In this work, irreducible parts are orthonormal to each other but theirs are not. The only similarity is that the sum of the irreducible parts for certain weight are the same, e.g. for weight j=1: The sum ${T_{ijk}}^{(1;1)}+{T_{ijk}}^{(1;2)}+{T_{ijk}}^{(1;3)}$ is the same but the individual parts ${T_{ijk}}^{(1;1)},{T_{ijk}}^{(1;2)},{T_{ijk}}^{(1;3)}$ are different.

As an application of decomposition of third rank tensors, piezoelectric tensor d is represented in terms of its orthonormal irreducible parts.

The following irreducible parts for the piezoelectric tensor d are obtained by the application of the index symmetry condition (2) to (12)-(18).

$$d_{ijk}^{(1;1)}=\frac{1}{3}\delta_{jk}d_{iss} \quad (19)$$

$$d_{ijk}^{(1;3)}=\frac{1}{30}\Big(\delta_{ik}(9d_{ssj}-3d_{jss})+\delta_{ij}(9d_{ssk}-3d_{kss})\Big) -\delta_{jk}(9d_{ssi}-3d_{iss}) \quad (20)$$

$$d_{ijk}^{(3;1)}=\frac{(d_{ijk}+d_{jik}+d_{kij})}{3}-\frac{1}{15}\{\delta_{ik}(2d_{ssj}+d_{jss})\} -\frac{1}{15}\delta_{ij}(2d_{ssk}+d_{kss})-\frac{1}{15}\delta_{jk}(2d_{ssi}+d_{iss}) \quad (21)$$

So we have two vectors and one septor part which are the same in number predicted by group theoretical methods for this internal tensor symmetry. Here these decomposed parts are orthonormal to each other but those are not in the work of [8].

III. Orthonormal Tensor Basis Method

This method comprises of two basic steps which are constructing form-invariant and orthonormal basis elements. [5] The form invariant expressions are derived for many classes of piezomagnetic and piezoelectric coefficients[9]. Although such constitutive equations are form invariant with respect to arbitrary orthogonal coordinate transformations, the coefficients, d_{ijk}, do not determine directly the material constants since their values vary with the direction of the coordinate axes.

The form-invariant expressions[9] for the piezoelectric coefficients is respectively,

$$d_{ijk}=\nu_{ai}\nu_{bj}\nu_{ck}A_{abc} \quad (22)$$

where summation is implied by repeated indices and this convention is followed throughout. This expression is referred to a Cartesian system Oxyz; ν_{ai} are the components of the unit vectors ν_a $(a=1,2,3)$ along the crystallographic axes. The quantity A_{abc} is invariant in the sense that when the Cartesian system is rotated to a new orientation O$x'y'z'$, then (22) takes the form

$$d'_{ijk}=\nu'_{ai}\nu'_{bj}\nu'_{ck}A_{abc} \quad (23)$$

It should be remembered that ν_1,ν_2,ν_3 form a linearly independent basis in three dimensions but are not necessarily always orthogonal. Let us consider the hexagonal symmetry as an example. The form invariant expression for the hexagonal system class 6mm is [9]

$$d_{ijk}=d_1\nu_{3i}\nu_{3j}\nu_{3k}+d_2(\nu_{3k}\delta_{ij}+\nu_{3j}\delta_{ik})+d_3\nu_{3i}\delta_{jk} \quad (24)$$

where ν_3 is the sixfold axis. A similar form can be derived from tetragonal symmetry (class 4mm)[1]

The first step in the generation of orthonormal tensor basis is one of writing the δ_{ai} in the place of ν_{ai} in (22). It will assume respectively the form

$$d_{ijk}=\delta_{ai}\delta_{bj}\delta_{ck}A_{abc} \quad (25)$$

One can subject the expression (25) to the symmetry of any crystal and then derive the elements of the basis appropriate to that class. Instead the form-invariant expression for any given class can be taken and straightaway replaced the ν_{ai} by the δ_{ai} to obtain the elements of the basis. As an illustration, let us consider the simplest example, namely, the expression (24). According to the present scheme, the elements of the basis are

$$\delta_{3i}\delta_{3j}\delta_{3k} \qquad \delta_{3i}\delta_{jk} \qquad (\delta_{3j}\delta_{ik}+\delta_{3k}\delta_{ij}) \quad (26)$$

In constructing this basis we have made use of the identity:

$$\delta_{1i}\delta_{1j}+\delta_{2i}\delta_{2j}+\delta_{3i}\delta_{3j}=\delta_{ij} \quad (27)$$

This is a particular case of a more general identity.

$$\nu_{1i}\nu_{1j}+\nu_{2i}\nu_{2j}-\cos\theta(\nu_{1i}\nu_{2j}+\nu_{2i}\nu_{1j})+sin^2\theta\nu_{3i}\nu_{3j}=sin^2\theta\delta_{ij} \quad (28)$$

with ν_{ai} is replaced by δ_{ai} and $\theta=90^o$. On subjecting these elements to the Gram-Schmidt process, we obtain,

$$T_{ijk}^{I}=\delta_{3i}\delta_{3j}\delta_{3k},$$
$$T_{ijk}^{II}=\frac{1}{\sqrt{2}}(\delta_{3i}\delta_{jk}+\delta_{3i}\delta_{3j}\delta_{3k}),$$
$$T_{ijk}^{III}=\frac{1}{2}(\delta_{3j}\delta_{ik}+\delta_{3k}\delta_{ij}-2\delta_{3i}\delta_{3j}\delta_{3k}),$$
$$T_{ijk}^{IV}=\frac{1}{\sqrt{2}}(2\delta_{3i}\delta_{1j}\delta_{1k}-\delta_{3i}\delta_{jk}-\delta_{3i}\delta_{3j}\delta_{3k}),$$
$$T_{ijk}^{V}=\frac{1}{2}[2(\delta_{1i}\delta_{3j}\delta_{1k}+\delta_{1i}\delta_{1j}\delta_{3k}-\delta_{3i}\delta_{3j}\delta_{3k})-\delta_{3j}\delta_{ik}-\delta_{3k}\delta_{ij}],$$
$$T_{ijk}^{VI}=\frac{1}{\sqrt{2}}(\delta_{1i}\delta_{2j}\delta_{3k}+\delta_{1i}\delta_{3j}\delta_{2k}) \quad (29)$$

These are the elements of the basis for the most general case, namely, the noncentrosymmetric triclinic case.

In the actual exercise, starting with (25) and following the recipe to construct the orthonormal tensor basis spans the space of the third-rank tensor representing the piezoelectric effect and having the index symmetry $d_{ijk}=d_{ikj}$; so the use of the identity (27) is understood.

In terms of this basis, the representation of d_{ijk} is given by

$$d_{ijk}=\sum_K(d,T^K)\,T_{ijk}^K \quad (30)$$

where

$$(d,T^K)=d_{ijk}T_{ijk}^K \quad (31)$$

represents the inner product of T_{ijk}^K and the Kth element T_{ijk}^K; of the basis. The expressions for the inner product of d_{ijk} with each element of the basis are listed as

$$(d,T^I)=d_{33},\quad (d,T^{II})=\frac{1}{\sqrt{2}}(d_{31}+d_{32}),\quad (d,T^{II})=\frac{1}{\sqrt{2}}(d_{31}+d_{32})$$
$$(d,T^{III})=\frac{1}{2}(d_{15}+d_{24}),\quad (d,T^{IV})=\frac{1}{\sqrt{2}}(d_{31}-d_{32}),$$
$$(d,T^{V})=\frac{1}{2}(d_{15}-d_{24}),\quad (d,T^{VI})=d_{11},\quad (d,T^{VII})=\frac{1}{\sqrt{2}}(d_{12}+d_{13}),$$
$$(d,T^{VIII})=\frac{1}{2}(d_{26}+d_{35}),\quad (d,T^{IX})=\frac{1}{\sqrt{2}}(d_{12}-d_{13}),$$
$$(d,T^{X})=\frac{1}{2}(d_{26}-d_{35}),\quad (d,T^{XI})=d_{22},\quad (d,T^{XII})=\frac{1}{\sqrt{2}}(d_{23}+d_{21}),$$

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$$(d,T^{XIII}) = \frac{1}{2}(d_{34}+d_{16}),\quad (d,T^{XIV}) = \frac{1}{\sqrt{2}}(d_{23}-d_{21}),$$
$$(d,T^{XV}) = \frac{1}{2}(d_{34}-d_{16}),\quad (d,T^{XVI}) = \frac{1}{\sqrt{2}}d_{14},\quad (d,T^{XVII}) = \frac{1}{\sqrt{2}}d_{25},$$
$$(d,T^{XVII}) = \frac{1}{\sqrt{2}}d_{36}. \tag{32}$$

IV. Comparison of the Methods

In previous sections, for piezoelectric tensor as an example for third rank tensors irreducible decomposition method is introduced and orthonormal tensor basis method is described in detail.

Let us compare these two methods for hexagonal symmetric materials. By using the symmetry condition in (2) and applying the formula in (30), piezoelectric tensor for hexagonal symmetry is represented in terms of the following orthonormal decomposed parts:

$$[d_{ijk}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_{33} & 0 & 0 & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}(d_{31}+d_{32}) & \frac{1}{2}(d_{31}+d_{32}) & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{2}(d_{15}+d_{24}) & 0 \\ 0 & 0 & 0 & \frac{1}{2}(d_{15}+d_{24}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{33}$$

(33) indicates that the hexagonal symmetric third rank tensor, d_{ijk} is a subset of the general symmetric third rank tensor and decomposed into three terms, each of which has a distinct physical meaning. It is easy to verify that the three decomposed parts form an orthogonal set and their sum is the hexagonal symmetric third rank tensor, d_{ijk} of class 6*mm* which is identical to following matrix:

$$[d_{ijk}] = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{2}(d_{15}+d_{24}) & 0 \\ 0 & 0 & 0 & \frac{1}{2}(d_{15}+d_{24}) & 0 & 0 \\ \frac{1}{2}(d_{31}+d_{32}) & \frac{1}{2}(d_{31}+d_{32}) & d_{33} & 0 & 0 & 0 \end{bmatrix} \tag{34}$$

Since

$$d_{31} = d_{32} \; and \; d_{24} = d_{15} \tag{35}$$

So $[d_{ijk}]$ is

$$[d_{ijk}] = \begin{bmatrix} 0 & 0 & 0 & 0 & d_{15} & 0 \\ 0 & 0 & 0 & d_{15} & 0 & 0 \\ d_{31} & d_{31} & d_{31} & 0 & 0 & 0 \end{bmatrix} \tag{36}$$

Physically d_{ijk} is decomposed into three independent tensors, each has an independent piezoelectric coefficient and physical piezoelectric effect. If a tensile stress σ_3 is applied parallel to x_3 which is a diad axis of the crystal. The first matrix in (33) shows that the components of polarization are given by the moduli in the third column of the first matrix; so

$$P_1 = 0,\; P_2 = 0,\qquad P_3 = d_{33}\sigma_3 \tag{37}$$

The polarization therefore directed along x_3.

On the other hand, a tensile stress σ_1 along x_1 produces no polarization parallel to itself, but it produces a polarization along x_3 which is introduced in the second matrix of (33), the tensile stress σ_1 along x_1 produces

$$P_1 = 0,\; P_2 = 0,\qquad P_3 = d_{31}\sigma_1 \tag{38}$$

Similarly, for a tensile stress σ_2 along x_2, produces no polarization parallel to itself, but it produces a polarization along x_3.

The tensile stress σ_2 along x_1 produces

$$P_1 = 0,\; P_2 = 0,\qquad P_3 = d_{31}\sigma_2 \tag{39}$$

For the third matrix in (33), the polarization along x_2 can be produced by a shear stress σ_4 about x_2 so, for this stress

$$P_1 = 0,\; P_2 = d_{15}\sigma_4,\qquad P_3 = 0 \tag{40}$$

and the polarization along x_1 can be produced by a shear stress σ_5 about x_2 so, for this stress

$$P_1 = d_{15}\sigma_5,\; P_2 = 0,\qquad P_3 = 0 \tag{41}$$

Thus orthonormal tensor basis method presented, is decomposing the polarization along orthonormal axes into three parts: the first part is the polarization along the diad axes due to normal stress, the second part is the polarization along nondiad orthogonal axes due to normal stress and the third part is the polarization due to the shear stresses.

By applying the symmetry condition in (2) and applying the formulas in (19)-(21), piezoelectric tensor for hexagonal symmetry is represented in terms of the following irreducible orthonormal decomposed parts:

$$[d_{ijk}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3}(d_{31}+d_{33}) & \frac{1}{3}(d_{31}+d_{33}) & \frac{1}{3}(d_{31}+d_{33}) & 0 & 0 & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} \frac{6d_{31}}{15} & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & a & 0 & 0 \\ \frac{-6d_{15}-3d_{33}+12d_{31}}{15} & \frac{-6d_{15}-3d_{33}+12d_{31}}{15} & \frac{6d_{15}-9d_{33}-3d_{31}}{15} & 0 & 0 & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} -\frac{6d_{31}}{15} & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & b & 0 & 0 \\ \frac{6d_{15}+3d_{33}+3d_{31}}{15} & \frac{6d_{15}+3d_{33}+3d_{31}}{15} & d_{33}-\frac{6d_{15}-9d_{33}-3d_{31}}{15} & 0 & 0 & 0 \end{bmatrix} \tag{42}$$

Where $a = \frac{9d_{15}+3d_{33}-3d_{31}}{15}$ and $b = \frac{6d_{15}-3d_{33}+3d_{31}}{15}$.

The decomposed parts for hexagonal symmetry in (42) are different those in (33). So polarization decomposition is not valid in irreducible decomposition method. However, decomposed parts obtained from both methods are orthonormal.

Furthermore, the irreducible cartesian tensor parts that obtained in this work are different than the irreducible cartesian tensor parts appeared in the literature [6],[7],[8]. mappings $Q_{i_1 i_2 i_3 \ldots i_n; k_1 k_2 k_3 \ldots k_n}$ have been chosen such that they are orthonormal but theirs are not.

It should be pointed out that there may be some arbitrariness in choosing the first mappings for certain weight. Not paying attention to this point may lead to different physically meaningless orthonormal irreducible parts. In irreducible decomposition procedure, to obtain a unique physically meaningful orthonormal irreducible parts set, the restriction imposed is that the indices of different deltas and epsilon of the first mapping should be chosen in the same order as they are written in the cartesian tensor itself (it can be called as it the first natural choice or the first choice). For instance, for the third rank tensor n = 3; j = 1: $T_{ijk} \to \delta_{ri}\delta_{jk}$. Here the index r is inserted to make the number of indices and the number of mapping indices equal to each other.

For example, considering the vector part, in this case, there are three choices for the mappings (j = 1), each alternative will produce different orthonormal irreducible sets of three in number.

On the other hand, application of the internal symmetry of the piezoelectric tensor (which is symmetric with respect to

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its last two indices, i.e., $d_{ijk} = d_{ikj}$) to the first choice mapping set will produce two irreducible parts which is the same number as predicted by group theoretical methods [6] and application of the same symmetry condition to the other two mapping choices will not produce the same number (they produce three), so they are rejected.

V. Norm Concept

Norm is an invariant of the material. Generalizing the concept of the modulus of a vector, norm of a Cartesian tensor (or the modulus of a tensor) is defined as the square root of the contracted product over all indices with itself:

$$N = \|T\| = \{T_{ijkl...}T_{ijkl...}\}^{\frac{1}{2}} \tag{43}$$

Denoting rank n Cartesian T_{ijkl}...... by T_n the square of the norm is expressed as Jerphagnon et al.

This definition is consistent with the reduction of the tensor in Cartesian formulation when all the irreducible parts are embedded in the original rank n tensor space.

Since the norm of a Cartesian tensor is an invariant quantity, following rule is suggested:

The norm of a Cartesian tensor may be used as a criterion for representing and comparing the overall effect of a certain property of piezoelectric materials of the same or different symmetry. At this stage, the norm ratios:

N_v/N for vector part and N_{sp}/N for septor part are defined. In this work, norms and norm ratios of the irreducible parts are used as a criterion. The larger the norm ratio value exists, the stronger the material property is.

Here, the square of the norm of the piezoelectric tensor is found as

$$N^2 = \sum_{mn} (d_{mn}^{(1;1)})^2 + \sum_{mn} \left(d_{mn}^{(1;3)}\right)^2 + 2\sum_{mn} \left(d_{mn}^{(1;1)}\right)\left(d_{mn}^{(1;3)}\right) + \sum_{mn} \left(d_{mn}^{(3;1)}\right)^2 \tag{44}$$

A. Applications

Among semiconductors crystals, a family of quartzite-type belongs to the 6mm class, which is piezoelectric active. Piezoelectric tensor data and norm are tabulated, norm ratio calculations for semiconductors in TABLE II and III respectively.

TABLE II
PIEZOELECTRIC COEFFICIENTS OF SEMICONDUCTORS, d_{ijk} (10^{-12} CN^{-1}) [9]

Materials	d_{31}	d_{33}	d_{15}
ZnO	-5.0	12.4	-8.3
CdS	-5.2	10.3	-14
CdSe	-3.9	7.8	-10

TABLE III
THE NORMS AND NORM RATIOS (THE ANISOTROPY DEGREES) FOR SEMICONDUCTORS

Material	N_v	N_{sp}	N	N_v/N	N_{sp}/N
ZnO	4.05	19.337	19.757	0.205	0.979
CdS	11.635	24.695	27.299	0.426	0.905
CdSe	8.059	18.000	19.722	0.409	0.913

By taking into account the rule, the most piezoelectric effective among these three materials is Cds which has a very important feature in the thin films of semiconductors. Piezoelectric ceramic is the most potential piezoelectric material because of its higher strength, high rigidity and more importantly, the better piezoelectricity. TABLES IV and V include the piezoelectric coefficients and calculated norms for these materials.

TABLE IV
PIEZOELECTRIC COEFFICIENTS, d_{ijk} (10^{-12} CN^{-1}) OF PIEZOELECTRIC CERAMICS [10]

Material,class symmetry	d_{31}	d_{33}	d_{15}
PZT-4	-5.2	15.1	12.7
PZT-5	-5.4	15.8	12.3
PZT-5H	-6.5	23.3	17
PZT-8	-4	23.3	10.4

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TABLE V
THE NORMS AND NORM RATIOS (THE ANISOTROPY DEGREES) FOR PIEZOELECTRIC CERAMICS

Material	N_v	N_{sp}	N	N_v/N	N_{sp}/N
PZT-4	25.462	3.557	25.709	0.990	0.138
PZT-5	25.360	2.892	25.525	0.994	0.113
PZT-5H	35.507	3.522	35.681	0.995	0.099
PZT-8	26.822	4.010	27.120	0.989	0.148

Among these piezoceramics the piezoelectric effect in PZT-5H is the strongest.

VI. CONCLUSION

Any physical property is characterized by n-rank tensors and decomposition methods presented in this work are capable for decomposing these tensors with intrinsic symmetry which is derived from the nature of the physical property itself into orthonormal parts. In other words, these methods of constructing orthonormal decomposed parts can be easily extended to (physical property) tensor of any rank.

In this work, they are applied to piezoelectric tensor. The third rank tensor, like piezoelectric tensors are of interest in engineering.

To summarize, irreducible decomposition method is a new procedure in literature which gives orthonormal parts and orthonormal tensor basis method is developed to be applied to third rank tensors.

As a result of orthonormal tensor basis method, it is possible to decide the stress in which direction the polarization will be produced and what type of stress required. By use of norm concept for different materials with the same symmetries, it is feasible to determine in which material the piezoelectric effect is stronger. Both irreducible and orthonormal tensor basis methods yield orthonormal decomposed parts.

Besides, decomposition of third rank tensors into elementary tensors by both methods, are to be undertaken, to offer valuable insight into the tensor structure and at the same time, simplifying immensely the calculations of sums, products, inverses and inner products.

Nevertheless, those representations are introducing new forms of decomposition that have more featured and transparent physical information. Criteria to measure the overall effect of the material properties proposed and the norms which represent the piezoelectricity effect in the material like piezoceramics are computed. Through these methods it is possible to study the effect of angle orientation of fibers and the material properties of fiber and matrix on the stiffness of the composite. One can determine in which material the piezoelectric effect is stronger by using the norm for different materials with the same symmetries.

The method developed in this work for tensors has many engineering applications in anisotropic elastic materials which are both qualitatively and quantitatively different from isotropic materials.

REFERENCES

[1] A. Safari A., E.K. Akdogan, *Piezoelectric and Acoustic Materials for Transducer Applications*, Springer, 2008.
[2] M.E.Kilmer, M. C. D. Moravitz, “Decomposing a Tensor”, *SIAM News*, vol.37, no.9, 2004.
[3] P. Comon, “Canonical Tensor Decomposition”, *I3S Report RR-*2004-17, 1-16.
[4] N. Dimitri and L. Lieven De, “Decomposing a Third-Order Tensor in Rank-(L,L,1) Terms by Means of Simultaneous Matrix, Diagonalization”, *SIAM Conference on Applied Linear Algebra*, October 26-29, 2009.
[5] Ç. Dinçkal, “Orthonormal Decomposition of Symmetric Second Rank Tensors”, *International Journal of Pure and Applied Mathematics*, vol. 65, no.2, pp. 225-241, 2010.
[6] J. Jerphagnon, D. Chemla and R. Bonneville, “The description of the physical properties of condensed matter using irreducible tensors”, *Advances in Physics*, vol.27, pp. 609-650, 1978.
[7] D. L. Andrews, W. A. Ghoul, “Irreducible fourth-rank cartesian tensors”, *Physical Review A*, vol. 25, pp. 2647-2657, 1982.
[8] D. L. Andrews, N. P. Blake, “Three-dimensional rotational averages in radiation-molecule interactions: an irreducible cartesian tensor formulation”, *Journal of Physics A: Mathematical and General*, vol. 22, pp. 49-60, 1989.
[9] T. P. Srinivasan, “Invariant piezoelectric coe¢ cients for crystals”, *Phys. Stat. Sol*., vol. 41, pp. 615-620, 1970.
[10] Sh. M. Butabaev, Yu. I. Sirotin, “Anisotropy in Crystal Physical Properties Represented by Surfaces of Rotation”, *Sov. Phys. Crystallography*, vol. 17, pp. 1037-1040, 1973.
[11] D. Haojiang, C. Weiqiu, *Three Dimensional Problems of Piezoelectricity*, Nova Science Publishers, 2001.