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On the Stability of Piecewise Genetic Regulatory Networks

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Abstract—**The hybrid dynamics of the genetic regulatory networks (GRNs) have attracted much research attention in recent years. This paper is concerned with the stability analysis of piecewise GRNs. Depending on whether the state partitions and mode transitions are known or unknown as a priori, the proposed networks are divided into two categories, i.e., switched GRNs and hybrid GRNs. It is shown that, by using common polynomial Lyapunov functions and piecewise polynomial Lyapunov functions, two conditions are established to ensure the globally asymptotically stability for switched and hybrid GRNs, respectively. Moreover, it is shown that, by using the sum of squares (SOS) techniques, stability conditions in form of linear matrix inequalities (LMIs) for both models can be obtained. An example with synthetic hybrid GRN model is provided to illustrate the use of the proposed methodology.**

Index Terms—**piecewise genetic regulatory networks, stability, sum of squares.**

I. Introduction

THE study of GRNs has attracted much research attention in recent years, and the advances of experimental techniques in the past decades have helped the researchers to make the quantitative analysis of such complicated networks to be realistic. In the past years, both theoretical and experimental results are fruitful, and a variety of constructive approaches for the modeling of GRNs have been proposed [1]–[4], which make it possible for us to have a deeper insight of how genes and proteins interact with each other and influence each other's behavior. But, researches never end, how to understand the underlying principles of the gene regulation processes in living organism still remains to be a great challenge.

Theoretically, GRNs are considered as biological dynamic systems [5]–[8], so it is reasonable to consider the network dynamics from the control systems point of view. It is now well known that, among many natural biological and biomedical systems, hybrid dynamics indeed exist, and such systems can be mathematically described as hybrid systems, see [9]–[12] for example and references therein. Furthermore, in the literature, it is found that both discrete and continuous are common behaviors in GRNs [13], [14], thus, it is natural and reasonable to construct the mathematical models of GRNs by using the hybrid systems model, which will provide us a powerful tool for better understanding of the real world gene regulation processes. In the past years, many contributions have been made in constructing the hybrid model of GRNs, with which the fundamental research approaches have been established. In [15], a method for the hybrid modeling and simulation of GRNs based on a class of piecewise-linear differential equations was proposed. In [16], the authors proposed minimal hybrid models of GRNs, in that paper, how the concepts of robustness and minimality can be used was detailed discussed. In [17], a hybrid model of GRN with cell division cycle was investigated. Besides the establishment of the hybrid GRNs' models, the mathematical tools which are adopted in analyzing the systems' characteristics also play a key role. Fortunately, pioneers have established many theoretical foundations in this area. In [18], the authors provided an overview of the recent progresses and research activities in the field of stability analysis of switched systems. In [19], a methodology for robust stability analysis of nonlinear hybrid systems was presented, in which the SOS was used to decompose the multivariate polynomials. For other research approaches used in the analysis of hybrid and switched systems, see [20]–[25] for example and references therein.

In this paper, we foucs on the piecewise GRN models. Specifically, the state variables are corresponded to the concentrations of the metabolites of a GRN, such as proteins and mRNAs, and the evolution of such entities are described by differential equations. It is worth noting that, the regulation function we choose in this paper is generally nonlinear, which is different from the existing model [15], [16], where the regulation functions they used are linear. In our model, if the concentrations of the transcription factor are above some thresholds, they will control the production of other transcription factor in GRNs, and this process leads to the discrete switching between active and dormant states, and different models will be activated at different levels of concentrations, such processes we call them mode transitions. In many real biomedical systems, such switching phenomenons are indeed existing, please see [16], [26], [27] for example. One thing worths mentioning that, the thresholds we used in this paper are not simply determined by some known or unknown variables only, but can be approximated by some polynomials also, which will help us to reduce the conservatism of the algorithm. Furthermore, depending on whether the state partitions and mode transitions are known or unknown as a priori, the proposed network could be divided into two categories, i.e., switched GRNs and hybrid GRNs. It is shown that, by introducing common polynomial Lyapunov functions and piecewise polynomial Lyapunov functions, conditions for stability of switched GRNs and hybrid GRNs can be obtained. Moreover, it is shown that, by solving a convex optimization problem built by using SOS techniques, stability conditions in form of LMIs for both models can be obtained. To the best of the authors'

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knowledge, up to now, little effort has been made towards such topics in GRNs, which motivates the present study.

The paper is organized as follows. Section II introduces some preliminaries about switched and hybrid GRNs and the representation of polynomials. Section III derives the stability conditions of switched GRNs and hybrid GRNs. Section IV presents an illustrative example with synthetic hybrid GRN model. Finally, Section V provides some concluding remarks.

II. PRELIMINARIES

A. Problem formulation

Notation: I_n: $n\times n$ identity matrix; 0_n: origin of $\mathbb{R}^n$; A^T: transpose of matrix A; $A>0 (A\geq 0)$: symmetric positive definite (semidefinite) matrix A; $A\otimes B$: Kronecker product of matrices A and B; diag(...): block-diagonal matrix. Matrices, if their dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

In this section, we introduce a piecewise GRN model described by differential equations as follows:

$$\begin{cases} \dfrac{dm(t)}{dt} = -Am(t) + Gdiag(\gamma(p(t)))h(p(t)) + l \\ \dfrac{dp(t)}{dt} = -Cp(t) + Dm(t) \end{cases} \tag{1}$$

where $m(t) = (m_1(t), m_2(t), ..., m_n(t))^T \in \mathbb{R}^n$ and $p(t) = (p_1(t), p_2(t), ..., p_n(t))^T \in \mathbb{R}^n$ are concentrations of mRNA and protein. $A = \text{diag}(a_1, a_2, ..., a_n) \in \mathbb{R}^{n\times n}$, $C = \text{diag}(c_1, c_2, ..., c_n) \in \mathbb{R}^{n\times n}$ are positive matrices that represent the degradation rates of mRNA and protein. $D = \text{diag}(d_1, d_2, ..., d_n) \in \mathbb{R}^{n\times n}$ is also a positive matrix that represents the translation rates. G is the coupling matrix, which defines the coupling topology, direction, and the transcriptional rate of the hybrid GRN. l is defined as a basal rate, which represents the rate of continuous supply of some chemical or processes.

In (1), $\gamma(p(t)) = (\gamma_1(p_1(t)), \gamma_2(p_2(t)), ..., \gamma_n(p_n(t)))^T \in \mathbb{R}^n$ is a function that defines the switching mechanism of the piecewise GRN. The value of $\gamma_i(p_i(t))$ depends on whether the concentration of protein p_i is above or below a threshold value $\bar{p_i}$. We have:

$$\gamma_i(p_i(t)) = \begin{cases} 1, \ if \ p_i \geq \bar{p_i} \\ 0, \ if \ p_i < \bar{p_i}, \ i\in I = \{1,2,...,N\}. \end{cases} \tag{2}$$

Remark 1: In order to reduce the conservatism, the threshold value $\bar{p_i}$ in this paper is not only simply determined by some known or unknown parameters, but can be approximated by some polynomials also, which will mimic the real situation in the gene regulation more accurately.

The function $h(p(t)) = (h_1(p_1(t)), ..., h_n(p_n(t)))^T \in \mathbb{R}^n$ represents the feedback regulation function of the piecewise GRNs, which is generally nonlinear. One special case of the regulation function is with Hill form, in such case $h(p(t))$ is given by

$$h_i(p_i(t)) = \frac{p_i(t)^H}{\beta^H + p_i(t)^H} \qquad \beta>0, \ p_i(t)>0, \quad \forall i \tag{3}$$

where H is an integer known as the Hill coefficient and the function ranges from 0 to 1 and increases as $p_i\to\infty$.

Let (m^*, p^*) be the equilibrium point of (1), i.e., a solution of the following equations:

$$\begin{cases} 0 = -Am^* + Gdiag(\gamma(p^*))h(p^*) + l \\ 0 = -Cp^* + Dm^*. \end{cases} \tag{4}$$

For convenience, let us shift the origin to the unknown equilibrium point by letting

$$\begin{cases} x(t) \triangleq (x_1(t), ..., x_n(t))^T = m(t) - m^* \\ y(t) \triangleq (y_1(t), ..., y_n(t))^T = p(t) - p^* \end{cases} \tag{5}$$

then we have:

$$\begin{cases} \dfrac{dx(t)}{dt} = -Ax(t) + Gdiag(\lambda(y(t)))r(y(t)) \\ \dfrac{dy(t)}{dt} = -Cy(t) + Dx(t) \end{cases} \tag{6}$$

where

$$\lambda_i(y_i(t)) = \begin{cases} 1, \ if \ y_i \geq \bar{y_i} \\ 0, \ if \ y_i < \bar{y_i} \end{cases}, \ \bar{y_i}(t) = \bar{p_i}(t) - p_i^* \tag{7}$$

and

$$r_i(y_i(t)) = h_i(y_i(t) + p_i^*) - h_i(p_i^*). \tag{8}$$

Since h_i is monotonically increasing and differentiable with saturation, it satisfies:

$$0 \leq \frac{h_i(s_1) - h_i(s_2)}{s_1 - s_2} \leq k_i, \ \forall s_1 \neq s_2, i = 1,2,...,n \tag{9}$$

for some k_i. From the relationship between $h(p(t))$ and $r(y(t))$, we obtain the following sector condition:

$$r_i(s)(r_i(s) - k_i s)) \leq 0 \tag{10}$$

for some $s\in\mathbb{R}$.

Let us observe that, in this paper, depending on whether the state partitions and mode transitions are unknown or known as a priori, the proposed network could be considered as a switched GRN and a hybrid GRN, respectively. And the details will be illustrated in Section III.

B. Representation of polynomials

Before proceeding, let us introduce a key technique that will be exploited in the next section. Let $s(x)$ be a polynomial in $x\in\mathbb{R}^q$ of degree $2m$. Then, $s(x)$ can be written as

$$s(x) = \triangle(S + L(\alpha), x^{\{m\}}) \tag{11}$$

where $\triangle(S + L(\alpha), x^{\{m\}})$ denotes the notation

$$\triangle(S + L(\alpha), x^{\{m\}}) = x^{\{m\}^T}(S + L(\alpha))x^{\{m\}}. \tag{12}$$

In (11), $x^{\{m\}}$ is a vector containing all monomials of degree less than or equal to m in x, S is a symmetric matrix with $s(x) = \triangle(S, x^{\{m\}})$, $L(\alpha)$ is a linear parameterization of the linear space

$$\mathfrak{L} = \{L = L^T : \triangle(L, x^{\{m\}}) = 0\} \tag{13}$$

and α is a vector of free parameters.

The representation (11) is called gram matrix method and SMR. The SMR allows one to establish whether a polynomial $s(x)$ is SOS via LMIs. Indeed, $s(x)$ is SOS if and only if there exist polynomials $s_1(x), s_2(x)$... such that

$$s(x) = \sum_i s_i(x)^2 \tag{14}$$

and this condition holds if and only if there exists α such that the following LMI holds:

$$\exists \alpha : S + L(\alpha) \geq 0. \tag{15}$$

See also [28] for further details on the SMR and on SOS polynomials.

III. STABILITY ANALYSIS

A. Switched GRNs

In switched GRNs, as mentioned in the preliminaries, the partitions of the state and transitions between modes are not characterized as a priori. In such case, the systems is under arbitrary switching, and a sufficient condition for the stability of the switched GRNs exists by introducing the common polynomial Lyapunov functions.

Then, we have:

Theorem 1: Suppose there exists a common polynomial Lyapunov functions $V(x, y)$, with $V(0, 0) = 0$, nonnegative polynomial $\delta_i(x, y, z)$, and positive polynomials $\varepsilon_1(x, y)$, $\varepsilon_{2i}(x, y, z)$, such that:

$$\begin{cases} V(x,y) - \varepsilon_1(x,y) \text{ is SOS} \\ -\dot{V}(x,y,z) + \sum_{i=1}^{n}(\delta_i(x,y,z)z_i(z_i - k_i y_i) - \varepsilon_{2i}(x,y,z)) \\ \text{is SOS}, \ \forall i \in I. \end{cases} \tag{16}$$

Then, the origin of the state space is globally asymptotically stable.

Remark 2: The new part z in $\dot{V}(x, y, z)$ is from the introducing of the sector condition (10) and which represents the nonlinear component $r(y(t))$.

Proof: According to Theorem 1, whenever the constrains in (16) hold with $\varepsilon_1 > 0$, $\varepsilon_{2i} > 0$, it follows that:

$$\begin{cases} V(x,y) - \varepsilon_1(x,y) \geq 0 \\ -\dot{V}(x,y,z) + \sum_{i=1}^{n}(\delta_i(x,y,z)z_i(z_i - k_i y_i) \\ -\varepsilon_{2i}(x,y,z)) \geq 0, \ \forall i \in I. \end{cases} \tag{17}$$

Firstly, let us consider $V(x, y) - \varepsilon_1(x, y)$. Since the polynomial $\varepsilon_1(x, y)$ is positive, we have:

$$0 < \varepsilon_1(x,y) \leq V(x,y). \tag{18}$$

Thus, the positivity of the common polynomial Lyapunov function $V(x, y)$ can be guaranteed.

Now, let us consider $-\dot{V}(x,y,z) + \sum_{i=1}^{n}(\delta_i(x,y,z)z_i(z_i - k_i y_i) - \varepsilon_{2i}(x,y,z))$. Similar to the above analysis, since $\delta_i(x, y, z)$ is a nonnegative polynomial, with a positive polynomial $\varepsilon_{2i}(x, y, z)$ and with the sector condition (10), we have:

$$\begin{aligned} 0 &> \sum_{i=1}^{n}(\delta_i(x,y,z)z_i(z_i - k_i y_i) - \varepsilon_{2i}(x,y,z)) \\ &\geq \dot{V}(x,y,z). \end{aligned} \tag{19}$$

Then, the derivative of the common polynomial Lyapunov function $\dot{V}(x, y, z)$ is guaranteed to be negative.

Consequently, the conditions of Theorem 1 hold since there exists common polynomial Lyapunov function $V(x, y)$, polynomials $\delta_i(x, y, z)$, $\varepsilon_1(x, y)$ and $\varepsilon_{2i}(x, y, z)$ fulfilling (16), $\forall i \in I$. ■

Before proceeding, let us observe that, in order to solve the conditions in Theorem 1, we can restrict our attentions to the SMR introduced in Section II-B.

Let us consider the following common polynomial Lyapunov function:

$$V(x(t), y(t)) = \Delta(P, \hat{x}^{\{m_1\}}) \tag{20}$$

where $\hat{x}^{\{m_1\}}$ is a vector containing all monomials of degree less than or equal to m_1 in x, y.

Then, let us define the derivative of the common polynomial Lyapunov function $V(x(t), y(t))$ as:

$$\dot{V}(x(t), y(t), z(t)) = \Delta(M, \tilde{x}^{\{m_2\}}) \tag{21}$$

where $\tilde{x}^{\{m_2\}}$ is a vector containing all monomials of degree less than or equal to m_2 in x, y, z.

Similarly, we have:

$$\begin{aligned} &\delta_i(x,y,z)z_i(z_i - k_i y_i) = \Psi_i(x,y,z) = \Delta(\Psi_i, \tilde{x}^{\{m_\delta\}}) \\ &\varepsilon_1(x,y) = \Delta(H_1, \hat{x}^{\{m_{e1}\}}) \\ &\varepsilon_{2i}(x,y,z) = \Delta(H_{2i}, \tilde{x}^{\{m_{e2}\}}) \end{aligned} \tag{22}$$

where the degree of the polynomial $\Psi_i(x, y, z)$ is no less than the degree of the derivative of the common polynomial Lyapunov function $\dot{V}(x, y, z)$, i.e. $m_\delta \geq m_2$. Similarly, we have $m_{e1} \geq m_1$ and $m_{e2} \geq m_2$.

Let $L_1(\alpha)$ and $L_2(\beta)$ be linear parameterizations of

$$\begin{cases} \mathfrak{L}_1 = \{L_1 = L_1^T : \Delta(L_1, \hat{x}^{\{m_1\}}) = 0\} \\ \mathfrak{L}_2 = \{L_2 = L_2^T : \Delta(L_2, \tilde{x}^{\{m_2\}}) = 0\}. \end{cases} \tag{23}$$

Then, we have:

Corollary 1: Suppose there exist symmetric matrices P, M, Ψ_i, H_1, H_{2i}, and two vectors α, β, satisfying the following LMIs:

$$\begin{cases} P - H_1 + L_1(\alpha) \geq 0 \\ -M + \sum_{i=1}^{n}(\Psi_i - H_{2i}) - L_2(\beta) \geq 0, \ \forall i \in I. \end{cases} \tag{24}$$

Then, Theorem 1 holds.

Proof: From $P - H_1 + L_1(\alpha) \geq 0$, pre- and post-multiplying by $\hat{x}^{\{m_1\}^T}$, $\hat{x}^{\{m_{e1}\}^T}$ and $\hat{x}^{\{m_1\}}$, $\hat{x}^{\{m_{e1}\}}$, respectively, one gets

$$\begin{aligned} 0 &\leq \Delta(P + L_1(\alpha), \hat{x}^{\{m_1\}}) - \Delta_1(H_1, \hat{x}^{\{m_{e1}\}}) \\ &= V(x,y) - \varepsilon_1(x,y) \end{aligned} \tag{25}$$

since

$$\Delta(L_1, \hat{x}^{\{m_1\}}) = 0. \tag{26}$$

Consider any $x, y \in \mathbb{R}$, since $\varepsilon_1(x, y)$ is positive, it implies that:

$$V(x,y) > 0. \tag{27}$$

Thus, the first condition in Theorem 1 can be satisfied.

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Now, let us consider $-M+\sum_{i=1}^{n}(\Psi_i-H_{2i})-L_2(\beta)$, pre- and post-multiplying by $\tilde{x}^{\{m_2\}^T}$, $\tilde{x}^{\{m_\delta\}^T}$, $\tilde{x}^{\{m_{e2}\}^T}$ and $\tilde{x}^{\{m_2\}}$, $\tilde{x}^{\{m_\delta\}}$, $\tilde{x}^{\{m_{e2}\}}$, respectively, one gets

$$\begin{aligned} 0 &\leq \Delta(-M-L_2(\beta),\tilde{x}^{\{m_2\}})+\Delta(\sum_{i=1}^{n}\Psi_i,\tilde{x}^{\{m_\delta\}}) \\ &\quad -\Delta(\sum_{i=1}^{n}H_{2i},\tilde{x}^{\{m_{e2}\}}) \\ &= -\dot{V}(x,y,z)+\sum_{i=1}^{n}(\Psi_i(x,y,z)-\varepsilon_{2i}(x,y,z)) \end{aligned} \tag{28}$$

since

$$\Delta(L_2,\tilde{x}^{\{m_2\}})=0. \tag{29}$$

Considering that, $\forall i\in I$, the polynomial $\Psi_i(x,y,z)$ is nonpositive and $\varepsilon_{2i}(x,y,z)$ is positive, it implies that:

$$\dot{V}(x,y,z)<0. \tag{30}$$

Thus, the second condition in Theorem 1 can be satisfied.

Consequently, the conditions of Theorem 1 hold since there exists a common polynomial Lyapunov function $V(x,y)$, polynomials $\delta_i(x,y,z)$, $\varepsilon_1(x,y)$ and $\varepsilon_{2i}(x,y,z)$ fulfilling (16), $\forall i\in I$. ■

B. Hybrid GRNs

Compared with switched GRNs, the state partitions and mode transitions in hybrid GRNs are characterized as a priori. There exists a region of the state space where a particular mode can be active corresponding to the transition law, and such state space could be defined as:

$$\Omega_i=\{x,y\in\mathbb{R}^n: g_{ik}(x,y)\geq 0,\ k=1,...,m_{\Omega_i}\} \tag{31}$$

for some polynomials $g_{ik}(x,y)$.

Let observe that, in hybrid GRNs, the state space partition must satisfy $\bigcup_{i\in I}\Omega_i=\mathbb{R}^n$, but $\text{int}(\Omega_i)\cap\text{int}(\Omega_j)$ is not necessarily empty for $i\neq j$. The switching surface or switching plane between Ω_i and Ω_j is described by:

$$S_{ij}=\Omega_i\cap\Omega_j \tag{32}$$

where the state transition between modes on S_{ij} occurs only in one direction, i.e., from j to i.

And the transition from the jth mode to the ith mode is defined by:

$$S_{ij}=\{x,y: s_{ij0}(x,y)=0, s_{ijk}(x,y)\geq 0, k=1,...,m_{S_{ij}}\} \tag{33}$$

for some polynomials $s_{ijk}(x,y)$.

Then, we have:

Theorem 2: Suppose there exists piecewise polynomial Lyapunov functions $V_i(x,y)$, with $V_i(0,0)=0$ if $(0,0)\in\Omega_i$, nonnegative polynomials $a_{ik}(x,y)$, $b_{ik}(x,y,z)$, $c_{ijk}(x,y)$, $\delta_i(x,y,z)$, and positive polynomials $\varepsilon_{1i}(x,y)$, $\varepsilon_{2i}(x,y,z)$, $\varepsilon_{3i}(x,y)$ such that:

$$\begin{cases} V_i(x,y)-\sum_{k=1}^{m_{\Omega_i}}a_{ik}(x,y)g_{ik}(x,y)-\sum_{i=1}^{n}\varepsilon_{1i}(x,y) \\ \text{is SOS},\ \forall i\in I \\ -\dot{V}_i(x,y,z)-\sum_{k=1}^{m_{\Omega_i}}b_{ik}(x,y,z)g_{ik}(x,y) \\ +\sum_{i=1}^{n}(\delta_i(x,y,z)z_i(z_i-k_iy_i)-\varepsilon_{2i}(x,y,z))\ \text{is SOS},\forall i\in I \\ -V_i(x,y)+V_j(x,y)-\sum_{k=1}^{m_{S_{ij}}}c_{ijk}(x,y)s_{ijk}(x,y) \\ -\sum_{i=1}^{n}\varepsilon_{3i}(x,y)\ \text{is SOS},\ \forall i,j. \end{cases} \tag{34}$$

Then, the origin of the state space is globally asymptotically stable.

Proof: According to Theorem 2, whenever the constrains in (34) hold with $\varepsilon_{1i},\varepsilon_{2i}$, it follows that:

$$\begin{cases} V_i(x,y)-\sum_{k=1}^{m_{\Omega_i}}a_{ik}(x,y)g_{ik}(x,y)-\sum_{i=1}^{n}\varepsilon_{1i}(x,y)\geq 0,\ \forall i\in I \\ -\dot{V}_i(x,y,z)-\sum_{k=1}^{m_{\Omega_i}}b_{ik}(x,y,z)g_{ik}(x,y) \\ +\sum_{i=1}^{n}(\delta_i(x,y,z)z_i(z_i-k_iy_i)-\varepsilon_{2i}(x,y,z))\geq 0,\ \forall i\in I \\ -V_i(x,y)+V_j(x,y)-\sum_{k=1}^{m_{S_{ij}}}c_{ijk}(x,y)s_{ijk}(x,y) \\ -\sum_{i=1}^{n}\varepsilon_{3i}(x,y)\geq 0,\ \forall i,j. \end{cases} \tag{35}$$

Firstly, let us consider $V_i(x,y)-\sum_{k=1}^{m_{\Omega_i}}a_{ik}(x,y)g_{ik}(x,y)-\sum_{i=1}^{n}\varepsilon_{1i}(x,y)$. In (31), the polynomial $g_{ik}(x,y)$ is nonnegative on Ω_i, since the polynomial $a_{ik}(x,y)$ is also nonnegative. Then, with a positive polynomial $\varepsilon_{1i}(x,y)$, we have:

$$0<\sum_{k=1}^{m_{\Omega_i}}a_{ik}(x,y)g_{ik}(x,y)+\sum_{i=1}^{n}\varepsilon_{1i}(x,y)\leq V_i(x,y). \tag{36}$$

Thus, the positivity of the piecewise polynomial Lyapunov function $V_i(x,y)$ on Ω_i can be guaranteed.

Now, let us consider $-\dot{V}_i(x,y,z)-\sum_{k=1}^{m_{\Omega_i}}b_{ik}(x,y,z)g_{ik}(x,y)+\sum_{i=1}^{n}(\delta_i(x,y,z)z_i(z_i-k_iy_i)-\varepsilon_{2i}(x,y,z))$. Similar to the above analysis, since $b_{ik}(x,y,z)$, $g_{ik}(x,y)$, $\delta_i(x,y,z)$ are all nonnegative polynomials on Ω_i, with a positive polynomial $\varepsilon_{2i}(x,y,z)$ and with the sector condition (10), we have:

$$\begin{aligned} 0 &> -\sum_{k=1}^{m_{\Omega_i}}b_{ik}(x,y,z)g_{ik}(x,y)+\sum_{i=1}^{n}(\delta_i(x,y,z)z_i(z_i-k_iy_i) \\ &\quad -\varepsilon_{2i}(x,y,z)) \\ &\geq \dot{V}_i(x,y,z). \end{aligned} \tag{37}$$

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Then, the derivative of the piecewise polynomial Lyapunov function $\dot{V}_i(x, y, z)$ is guaranteed to be negative on Ω_i.

Finally, the third condition can be proved along the similar lines as in the proofs of the second condition in Theorem 2, and it is guaranteed that $V_i(x(t), y(t)) \leq V_j(x(t), y(t))$ on S_{ij}.

Consequently, the conditions of Theorem 2 hold since there exist piecewise polynomial Lyapunov functions $V_i(x, y)$, polynomials $a_{ik}(x, y)$, $b_{ik}(x, y, z)$, $c_{ijk}(x, y)$, $\delta_i(x, y, z)$, $\varepsilon_{1i}(x, y)$, $\varepsilon_{2i}(x, y, z)$ and $\varepsilon_{3i}(x, y)$ fulfilling (34), $\forall i \in I$. ■

Remark 3: In order to be less conservative, we can increase the degree of the piecewise polynomial Lyapunov function $V_i(x, y)$. Furthermore, let us observe that, there are no requirements that the multipliers $a_{ik}(x, y)$, $b_{ik}(x, y, z)$, $c_{ijk}(x, y)$ and $\delta_i(x, y, z)$ need to be constants, they can also be polynomials of higher degree. Thus, the conditions are generally less conservative than other existing method.

Similar to Corollary 1, we have:

Corollary 2: Suppose there exist symmetric matrices P_i, M_i, Γ_{ik}, Θ_{ik}, Ψ_i, Φ_{ijk}, H_{1i}, H_{2i}, H_{3i} and three vectors α_1, α_2, β, satisfying the following LMIs:

$$\begin{cases} P_i - \sum_{k=1}^{m_{\Omega_i}} \Gamma_{ik} - \sum_{i=1}^{n} H_{1i} + L_1(\alpha_1) \geq 0, \ \forall i \in I \\ -M_i - \sum_{k=1}^{m_{\Omega_i}} \Theta_{ik} + \sum_{i=1}^{n} (\Psi_i - H_{2i}) - L_2(\beta) \geq 0, \ \forall i \in I \\ -P_i + P_j - \sum_{k=1}^{m_{S_{ij}}} \Phi_{ijk} - \sum_{i=1}^{n} H_{3i} - L_1(\alpha_2) \geq 0, \ \forall i, j \end{cases} \tag{38}$$

where

$$\begin{aligned} & a_{ik}(x, y) g_{ik}(x, y) = \Gamma_{ik}(x, y) = \Delta(\Gamma_{ik}, \hat{x}^{\{m_\Gamma\}}) \\ & b_{ik}(x, y, z) g_{ik}(x, y) = \Theta_{ik}(x, y, z) = \Delta(\Theta_{ik}, \tilde{x}^{\{m_\Theta\}}) \\ & c_{ijk}(x, y, z) s_{ijk}(x, y) = \Phi_{ijk}(x, y) = \Delta(\Phi_{ijk}, \hat{x}^{\{m_\Phi\}}) \\ & \varepsilon_{3i}(x, y) = \Delta(H_{3i}, \hat{x}^{\{m_{e3}\}}) \end{aligned} \tag{39}$$

and the degrees of the polynomials satisfy $m_\Gamma \geq m_1$, $m_\Theta \geq m_2$, $m_\Phi \geq m_1$ and $m_{e3} \geq m_1$.

Then, Theorem 2 holds.

Proof:The corollary can be proved along the similar lines as in the proofs of Corollary 1. For the length of the paper, the detailed proofs are omitted here. ■

Corollary 1 and corollary 2 provide sufficient conditions for Theorem 1 and Theorem 2 via LMI feasibility tests, respectively. Those conditions have been obtained by exploiting common polynomial Lyapunov functions and piecewise polynomial Lyapunov functions, and the SOS. The above conditions can be easily tested by using semidefinite programming techniques in Matlab.

IV. ILLUSTRATIVE EXAMPLE

Considering GRN (6) as a hybrid GRN with two state variables and two modes, we have:

$$\begin{aligned} f_1(x, y) &= \begin{bmatrix} -A_1 x(t) \\ -C_1 y(t) + D_1 x(t) \end{bmatrix}, \\ f_2(x, y) &= \begin{bmatrix} -A_2 x(t) + G_2 r(y(t)) \\ -C_2 y(t) + D_2 x(t) \end{bmatrix}, \\ \Omega_1 &= \{x, y \in \mathbb{R}^2 : \ g(x, y) < 0\}, \\ \Omega_2 &= \{x, y \in \mathbb{R}^2 : \ g(x, y) \geq 0\}. \end{aligned} \tag{40}$$

As mentioned in section II-A, the switching mechanism of the hybrid GRN (6) is determined by $\lambda(y(t))$. And the threshold value $\bar{y}_i$ in this example is approximated by a polynomial $t(x, y)$, particularly we choose $t(x, y) = y_1^2$, and we have:

$$g(x, y) = y_1 - y_1^2. \tag{41}$$

In (40), the regulation function $r_i(y(t))$ is with Hill form, and the Hill coefficient equals to 2, i.e., $y_i^2/(1 + y_i^2)$. It is easy to know that $k_i < 0.65$, and let us choose $k = diag(0.65, 0.65)$ in order to fulfill the sector condition (10).

The network parameters are selected as:

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}, C_1 = \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}, D_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.6 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0.6 & 0 \\ 0 & 1 \end{bmatrix}, C_2 = \begin{bmatrix} 1.3 & 0 \\ 0 & 0.1 \end{bmatrix},$$
$$D_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 1.1 \end{bmatrix}, G_2 = \begin{bmatrix} 0 & -0.1 \\ 0.3 & 0 \end{bmatrix}.$$

By solving the conditions in Theorem 2 by using semidefinite programming techniques in Matlab, we can obtain a feasible solution with the following obtained matrix variables.

$$P_1 = \begin{bmatrix} 1.8030 & 0.0856 \\ 0.0856 & 1.3346 \end{bmatrix},$$
$$P_2 = \begin{bmatrix} 1.1146 & 0.0735 \\ 0.0735 & 0.7206 \end{bmatrix}.$$

This shows that the synthetic hybrid GRN is globally asymptotically stable under the above conditions.

V. CONCLUSIONS

In this paper, we addressed the problem of establishing globally asymptotically stability of piecewise GRNs. Depending on whether the state partitions and mode transitions are known or unknown as a priori, the proposed network could be divided into two categories, i.e., switched GRNs and hybrid GRNs. Specifically, based on the approaches of common polynomial Lyapunov functions and piecewise polynomial Lyapunov functions, two conditions are established to ensure the globally asymptotically stability for switched and hybrid GRNs, respectively. Then, by using SOS techniques, stability conditions in form of LMIs for both systems can be obtained. An example with synthetic hybrid GRN model has been used to illustrate the use of the proposed methodology.

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