New Analytical Error Rate Bounds for GSM PHY Layer in CCI Limited and Rayleigh Faded Channels

Ahmed M. Alaa and Hazim Tawfik

Abstract− **The Quality of Service (QoS) of a GSM system is quantified in terms of Bit Error Rate (BER) and Frame Erasure Rate (FER) observed by the user. The problem of obtaining analytical expressions for BER and FER in a fading channel with multiple cochannel interferers (CCI) is an extremely complex mathematical problem. The reason for this complexity is that the involvement of several GSM PHY layer modules is mandatory. Besides, one needs to characterize the transmission environment from inception by obtaining the statistical properties of faded cochannel interferers. Thus, error rate metrics are usually obtained by simulating the GSM physical layer rather than treating the problem analytically. A reliable interface between system and link level models can be obtained by evaluating the BER and FER in terms of the Signal-to-Interference Ratio (SIR) analytically, instead of the pre-defined statistical mapping data usually used in literature. In this work, bounds on the uplink BER and FER are obtained for the GSM physical layer assuming a CCI limited system where both the desired and interference signals are subjected to Rayleigh fading.**

Index terms; Cochannel Interference (CCI); Convolutional Coding; Frame Erasure Rate (FER); GMSK; GSM

I. INTRODUCTION

THE problem of The problem of designing a GSM network that satisfies the Quality of Service (QoS) requirements usually requires the inclusion of both system and link level models. The system level model is used to obtain statistics for the Signal-to-Interference Ratio (SIR), while the link level model is used to map these SIR values to corresponding BER and FER values. The QoS requirements are defined in terms of the FER. Mapping system level to link level data requires the usage of statistical field measurements or simulations for the complete GSM receiver chain. However, these statistical data differ for different system loads, number of hopping frequencies and number of interferers. Besides, Monte carlo simulations are usually time consuming. Thus, it is more reliable to obtain analytical expressions for BER and FER that are functions of all system parameters. The problem of obtaining the error rates analytically for a faded and CCI limited channel with the whole GSM physical layer included was seldom encountered in literature. The reason for this is the great complexity of the problem; in order to obtain the exact probability of error for the system's link level, all GSM physical layer modules that impact the error rate must be included. In [1], a bound was developed on the error rate observed at the output of the GSM convolutional decoder, but without considering CCI statistics or GMSK bit error rate. The statistical characterization of the CCI limited and faded channel differs from the conventional noise limited (AWGN) faded channel. Thus, the proposed analysis must start from inception as one can't directly make use of the GMSK error probability expressions based on Gaussian noise found in literature. Previous research on GSM spectral capacity evaluation [2], capacity gain due random Frequency Hopping [3] and system-link level interfacing [4] [5] fully relied on statistical mapping data obtained from either field measurements or physical layer simulations. In this work, novel analytical bounds for the BER and FER are pursued for the sake of generic system-link level interfacing. The rest of the paper is organized as follows: section II presents the system under study, objectives and assumptions, in addition to the physical layer model. Section III includes a derivation of the approximate BER for the GMSK modulation. Section IV shows the upper bound of the GSM convolutional encoder error probability.

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II. SYSTEM MODEL

We consider a cellular system the first tier of interferers. The hexagonal cell radius in the cellular system is *R*. The distance between the home cell and the reuse cells is *D*. Without loss of generality, it is assumed that the Mobile Stations (MS) have fixed locations; the home user is located at the cell border at a distance *R* from the Base station (BTS). The interferers are located at the centers of co-cells. Assume that the number of interferers is n and the maximum number of interferers is N = 6. The physical layer of the GSM system consists of several processing and coding modules. The output of the speech encoder is classified into three classes of bits [6]. The class IA bits are 50 bits and are the most significant bits and are protected via CRC parity bits.

We track the error rate on the full rate speech channel (TCH). Figure 1 depicts the physical layer chain considered in the analysis.

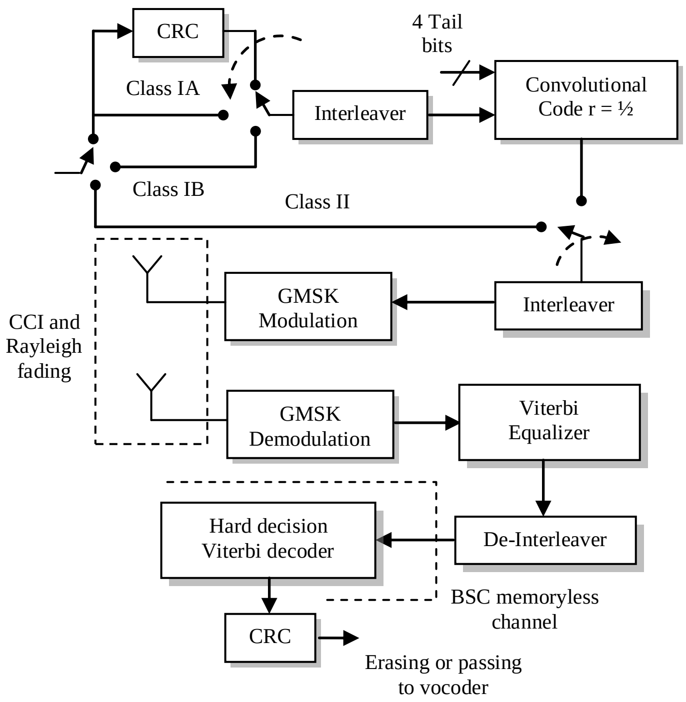

Figure 1. GSM PHY layer model

II. GMSK Bit Error Rate

The first module with the lowest level of abstraction in the physical layer of the GSM system is the GMSK modem. It is required to obtain the probability of error of the modem in a CCI limited channel with Rayleigh faded signals. The probability of error in Minimum Shift Keying (MSK) behaves exactly as conventional Binary Phase Shift Keying (BPSK) [7]. For a CCI limited channel, one cannot employ conventional error probability expressions in the analysis as the encountered additive channel is non-Gaussian. Hence, it is mandatory to start the analysis from inception by characterizing the additive interference statistical properties. The Signal-to-Interference Ratio (SIR) is given by:

$$\frac{S}{I} = \frac{|h|^2}{\sum_i |h_i|^2} \tag{1}$$

where |h| is the channel gain amplitude multiplied by the signal/interferer power, this amplitude follows a Rayleigh distribution with a parameter σ = mean power of the received signal. The mean power of the received signal is proportional to the path loss; $\bar{S}$ α $R^{-\gamma}$ and $\bar{I}$ α $nD^{-\gamma}$ where γ is the *path loss exponent*. The distribution of |h| is given by:

$$p_h(h) = \frac{h}{\sigma^2} e^{-\frac{h^2}{2\sigma^2}} \;,\; h \geq 0 \tag{2}$$

The mean power received is given by $P_{mean} = 2\sigma^2$, by applying transformation to the random variable |h| to h^2, the channel gain follows a *Chi-squared* random variable with two degrees of freedom :

$$p_{h^2}(x = h^2) = \frac{1}{P_{mean}} e^{\frac{-x}{P_{mean}}}, x \geq 0 \tag{3}$$

The PDF of the received power for the signal S and each of the *n* interferers (denoted by *J*):

$$P_s(S) = \frac{1}{R^{-\gamma}} e^{\frac{-S}{R^{-\gamma}}}, S \geq 0 \tag{4}$$

$$P_J(J) = \frac{1}{D^{-\gamma}} e^{\frac{-J}{D^{-\gamma}}}, J \geq 0 \tag{5}$$

The PDF of the summation of *n* interferers that are Chi-squared distributed is obtained by convolving *n* Chi-squared PDFs:

$$I = \sum_{i=1}^{n} J_i \tag{6}$$

$$p_I(I) = p_{J_1}(J_1) \otimes p_{J_2}(J_2) \otimes ... p_{J_n}(J_n) \tag{7}$$

The convolution problem presented in (7) is very tedious. By using the *central limit theorem for causal functions* [8], the summation yields a *Gamma distributed* random variable. The PDF of the equivalent CCI additive signal power is given by:

$$p_I(I) = p_I(\sum_{i=1}^{n} d_i) \sim \Gamma(n, 2\sigma^2) \sim \Gamma(n, D^{-\gamma}) \tag{8}$$

In order to calculate the error probability of the GMSK modem, one can utilize the constellation diagram of the FSK (Frequency Shift Keying) with a Gamma distributed additive interference power. The probability of error is calculated by integrating over the square-root Gamma distributed additive interference amplitude (as shown in figure 2), and then the BER is obtained by averaging the error probability over the PDF of the Chi-squared faded signal power. By applying square-root transformation for the CCI signal, one can show that the PDF of the additive interference is given by:

$$I \sim \Gamma(n, D^{-\gamma}) = \frac{1}{D^{-n\gamma}} \frac{1}{\Gamma(n)} I^{n-1} e^{\frac{-I}{D^{-\gamma}}} \tag{9}$$

$$p_y(y = \sqrt{I}) = \frac{2}{D^{-n\gamma}\Gamma(n)} y^{2n-1} e^{\frac{-y^2}{D^{-\gamma}}} \tag{10}$$

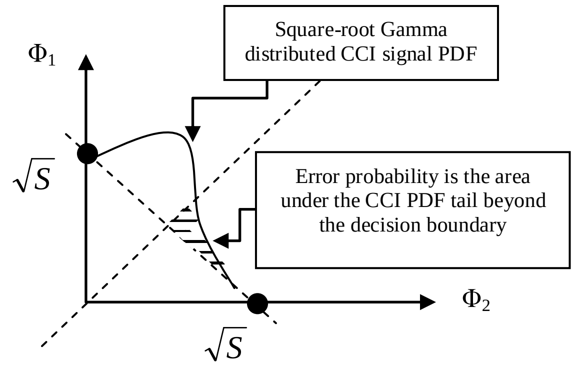

Figure 2. The MSK constellatio diagram (Φ_1 and Φ_2 are the orthonormal basis functions)

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Based on the constellation depicted in figure 3 and by translating the axes to obtain an equivalent BPSK constellation, it can be easily shown that the probability of error will be given by:

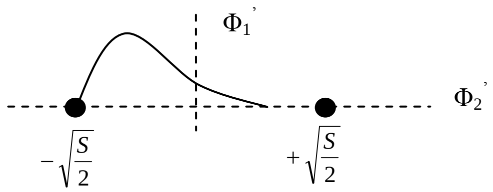

Figure 3. Equivalent BPSK constellations

$$P_e = \int_{\sqrt{\frac{S}{2}}}^{\infty} \frac{2}{D^{-n\gamma}\Gamma(n)} y^{2n-1} e^{\frac{-y^2}{D^{-\gamma}}} dy = \frac{\Gamma(n, \frac{n}{2} \cdot \frac{S}{I})}{\Gamma(n)} \tag{11}$$

This novel error probability expression considers only the faded CCI effect, but doesn't consider the Rayleigh fading of the user's signal. The channel gain follows a Chi-squared PDF:

$$P_\psi(\psi) = \frac{1}{SIR} e^{\frac{-\psi}{SIR}} \tag{12}$$

Where SIR = S/I and the BER is given by:

$$P_e = \int P_{e|\psi}(SIR \mid \psi).P(\psi).d\psi$$
$$= (1 - (\frac{\frac{n}{2} SIR}{\frac{n}{2} SIR + 1})^n) \tag{13}$$

Unlike the MSK modem, GMSK is subjected to pulse shaping before the FM modulation process. Thus, residual ISI (Inter Symbol Interference) is experienced at the receiver side causing SIR degradation. Let the degraded SIR be equivalent to λ.SIR, where $0 < \lambda < 1$ is an empirical factor that is dependent on the BT (Bandwidth-symbol duration product). The BT product is equal to 0.3 in GSM, and λ is usually found to be 0.8 [8]. Note that the Signal-to-Interference Ratio included in equation (13) is proportional to the ratio $(D/R)^{\gamma}$. The ratio Q = (D/R) is called the *quality reuse ratio*, and is considered as an important system parameter that controls the amount of CCI and thus affects the system's QoS.. As the quality reuse ratio decreases, the variance of the CCI signal increases and the probability of error increases. Notice that by plugging n = 0 in equation (13), one obtains $P_e = 0$ (no errors encountered when there are no active interferers). The expression obtained in (13) is a function of the number of cochannel interferers *n*. This number ranges between n = 0 to N = 6 for the first tier of co-cells only. The number of interferers at a given instant is decided by the system load. Another factor that affects the number of co-cell interferers is random *Frequency Hopping*. For each burst, every interferer is assigned a specific hop frequency. Thus, the probability of collision between the home user and co-cell interferers is reduced by a factor that is dependent on the number of hopping frequencies. The *Probability Mass Function (PMF)* of the number of co-cell interferers can be modelled by a binomial distribution as:

$$P_n(n) = \binom{N}{n} \cdot \zeta^n \cdot (1-\zeta)^{N-n} \tag{14}$$

where ζ is the probability that a certain channel is occupied (traffic/channel). This factor is denoted by the *system load*. The factor ζ incorporates many network parameters and this leads to linking the error probability with system configuration parameters like: system bandwidth B_{sys} (in MHz), GSM channel bandwidth B_{ch} (in MHz), traffic ρ (Erlangs), Reuse figure K_f and number of sectors per site N_{sec}. The factor ζ is given by:

$$\zeta = \frac{\rho \cdot K_f \cdot B_{ch} \cdot N_{\sec}}{U_{TDMA} \cdot B_{sys}} \tag{15}$$

The *PMF* of the number of interferers as function of system configuration parameters is given by:

$$P_n(n) = \tag{16}$$

$$\binom{N}{n} \cdot (\frac{\rho \cdot K_f \cdot B_{ch} \cdot N_{\sec}}{B_{sys}})^n \cdot (1 - \frac{\rho \cdot K_f \cdot B_{ch} \cdot N_{\sec}}{B_{sys}})^{N-n}$$

The averaged overall error probability for GMSK is given by:

$$P_b = \sum_{n=0}^{N} P_e(SIR \mid n) \cdot P_n(n) \tag{17}$$

The probability of error expression is developed in (17) as a function of all system configuration parameters. The *interference diversity* and *averaging* effects resulting from frequency hopping [3] are accounted for by considering all bits fed to the Viterbi decoder to be independent, and thus, burst errors are not considered in our analysis (this is justified by the presence of an interleaver). Besides, the error probability for individual class IA bits will be considered independent while calculating the rate of frame erasures. The overall Bit Error Rate (BER) of the GMSK is plotted versus SIR in figure 4. We assume a GSM system with system load of 25% and first tier interferers.

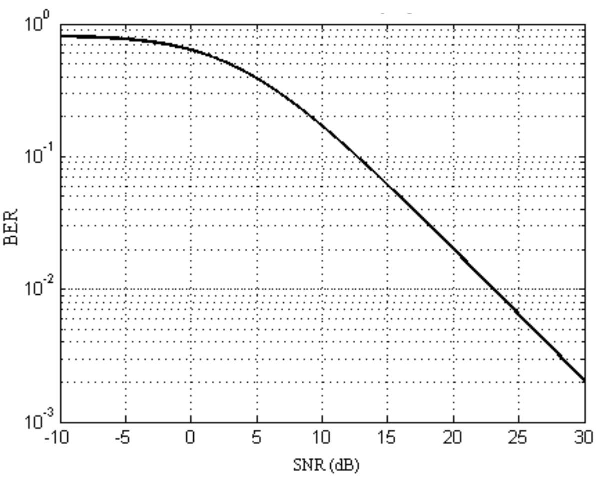

Figure 4. The BER for GMSK in CCI limited and Rayleigh faded channel

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IV. Post-decoding Bit Error Rate

The GSM full rate speech channel (TCH) consists of 260 bits that are protected via block and convolutional coding. The class IA bits are the most important 50 bits in a TCH frame and are protected by 3 CRC parity bits and rate r = ½ convolutional code. The class IB bits are 132 bits and are only applied to convolutional coding. The class II bits are 78 un-coded bits. The usage of bit interleaving, frequency hopping and the assumption of flat fading channel allow the assumption of a memoryless Binary Symmetric Channel (BSC). The BSC is characterized by a *crossover probability*, which represents the probability that the receiver receives an erroneous bit. The analysis approach for the channel decoder is to set the BSC crossover probability as the error probability for the GMSK modulation, and then study the performance of the GSM conventional Viterbi decoder. As the aim of this work is to obtain an upper bound on the error rate performance, it is assumed that the receiver utilizes a hard decision Viterbi decoder (using soft decision decoder would improve the performance). The GSM convolutional encoder is a rate r = ½ encoder with constraint length of 5. The *channel transition figure* of the memoryless BSC is drawn in figure 5. The error correction capability of convolutional codes depends on the distance between codeword sequences. Determining the code's free distance (hamming distance) d_H is done by calculating the distance between the minimum weight and the all zero codewords.

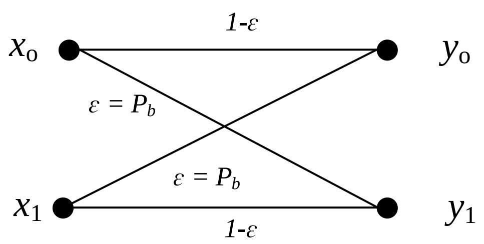

Figure. 5 Channel transition figure of the BSC perceived by Viterbi decoder. Crossover probability is calculated in (17).

Assuming an all zero codeword is transmitted, the error probability is bounded by the probability that the hamming distance between the received vector $\underline{Y}$ and the error pattern $\underline{E}$ is less than the distance between $\underline{Y}$ and the all-zero word, summed over all possible error patterns.

$$\begin{aligned} P_e &\le P\{\bigcup_E (d_H(\underline{Y},E) \le d_H(\underline{Y},0)) \mid \underline{0}\ was\ sent\} \\ &\overset{Union Bound}{\le} \sum_E P\{d_H(\underline{Y},E) \le d_H(\underline{Y},0) \mid \underline{0}\ was\ sent\} \\ &\le \frac{1}{k}\cdot \sum_{w=d_H}^{\infty} A_w \cdot P_w(\varepsilon) \end{aligned} \tag{18}$$

where k is the number of input bits to the encoder (185 in GSM) and $\{A_w\}$ is the weight distribution of a convolutional code, which represents the number of detours from the all-zero path with code sequence weight of w [9]. The crossover probability ε is the probability of error for the GMSK modulator calculated in (17). The quantity $P_w(\varepsilon)$ is the probability of error between two codewords that are w distance apart. An error between two binary codewords that are of distance d apart occurs if more than d/2 of the bits composing the hamming distance (the bits where the two codewords differ) are erroneous. The number of erroneous bits is randomly distributed with a binomial distribution; the probability of more than d/2 erroneous bits out of d bits is a summation over the binomial distribution from d/2 to d.

$$P_d(\varepsilon) = \begin{cases} \displaystyle\sum_{e=(d+1)/2}^{d} \binom{d}{e}\cdot \varepsilon^e \cdot (1-\varepsilon)^{d-e} \\ , d\ is\ odd \\ \displaystyle\frac{1}{2}\cdot\binom{d}{d/2}\cdot \varepsilon^{d/2}\cdot(1-\varepsilon)^{d/2} \\ \displaystyle +\sum_{e=\frac{d}{2}+1}^{d} \binom{d}{e}\cdot\varepsilon^e\cdot(1-\varepsilon)^{d-e},\ d\ is\ even \end{cases} \tag{19}$$

This piecewise definition can be reduced into a unified expression by neglecting the first term in the expression of even values of d [8]. The probability of error between two codewords of distance d apart reduces to:

$$P_d(\varepsilon) \cong \sum_{e=\left\lceil \frac{d}{2} \right\rceil}^{d} \binom{d}{e}\cdot\varepsilon^e\cdot(1-\varepsilon)^{d-e} \tag{20}$$

where $\lceil\cdot\rceil$ is the ceiling operator. In [11], it is stated that for the GSM convolutional encoder, the hamming distance d_H is 7 and the weight distribution belongs to the set {4, 12, 20 ...}. The overall GSM system BER would incorporate the GMSK probability of error with the error probability bound for the rate ½ convolutional encoder. The resulting expression includes all system configuration parameters in addition to the coder's free distance and weight distribution. An expression for the coded GSM BER in a faded and CCI limited channel is expressed in (21). A comparison between the pre-decoding and post-decoding probability of error is depicted in figure 6. The results closely match the post and pre decoding error rate bounds obtained via simulation in [11].

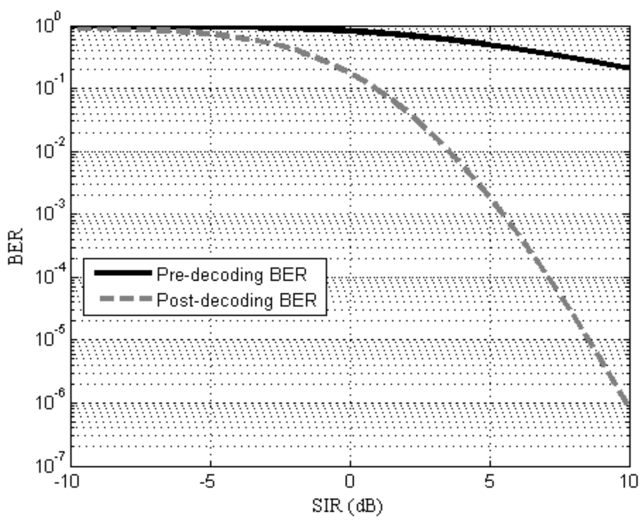

Figure. 6 The post-decoding BER and pre-decoding BER bounds

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$$P_e < \frac{1}{k}\cdot\sum_{w=d_H}^{d_H+2} A_w \cdot \sum_{e=\left\lceil \frac{w}{2}\right\rceil}^{w} \binom{w}{e}\cdot\left(\sum_{n=0}^{N}\left(1-\left(\frac{\frac{n}{2}\cdot\lambda\cdot SIR}{\frac{n}{2}\cdot\lambda\cdot SIR+1}\right)^{n}\right)\cdot\binom{N}{n}\cdot\zeta^{n}\cdot(1-\zeta)^{N-n}\right)^{e}\cdot \left(1-\left(\sum_{n=0}^{N}\left(1-\left(\frac{\frac{n}{2}\cdot\lambda\cdot SIR}{\frac{n}{2}\cdot\lambda\cdot SIR+1}\right)^{n}\right)\cdot\binom{N}{n}\cdot\zeta^{n}\cdot(1-\zeta)^{N-n}\right)\right)^{w-e} \tag{21}$$

V. Frame Erasure Rate Calculation

A speech frame is erased if the 3-bit CRC parity detects an error in the significant class bits. A frame is erased if the CRC error detector signals a bit error. Assuming that all bit errors in the class Ia bits are detected. This gives an upper bound on the FER. The FER can be calculated as the probability that any of the 50 class Ia bits are erroneous. This is equivalent to the complement of the probability that all class Ia bits are correct. Equation (22) represents the FER as a function of all system configuration parameters (I_a denotes the number of class IA bits). The FER is plotted for different values of the system loads in figure 7.

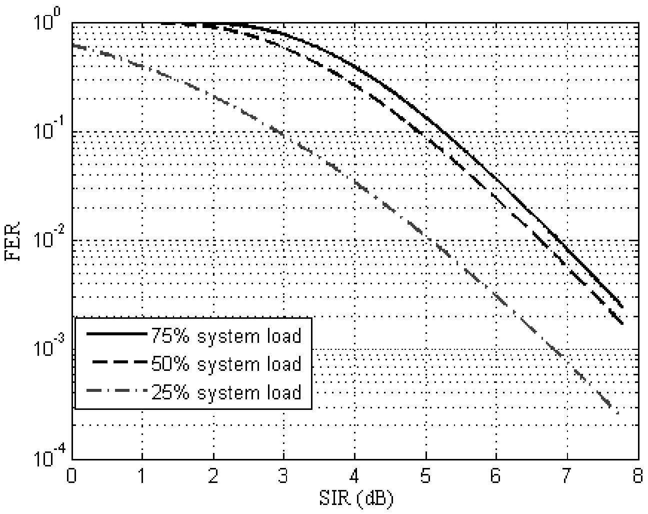

Figure. 7 The tight FER bound for GSM system versus SIR

The accuracy of the proposed bound was examined by comparison with simulation results. In [12], the GSM link level was simulated to obtain the FER performance versus SIR in a TU3 channel using 8 hopping frequencies. It is shown in figure 8 that the proposed bound acts as an upper bound for SIR < 5.5 dB. As the SIR increases, the system becomes limited by the inter-symbol interference (ISI) resulting from the urban channel profile and an offset of 1 dB is observed in the FER curve. The analytical approach can be enhanced further by modelling ISI as another additive random process. However, this would introduce significant complexity to the problem.

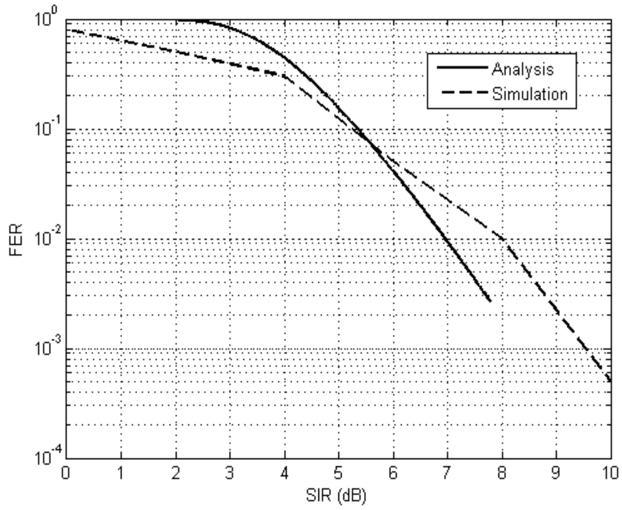

Figure. 8 Comparison between simulation and analytical results

VI. Conclusion

New analytical expressions for evaluating upper bounds for BER and FER of GSM cellular system are developed in this work. These expressions can provide means for system-to-link level mapping for a GSM network in a computationally efficient manner instead of the time consuming link level simulators. The developed expressions link the QoS metrics (BER and FER) with all system configuration parameters (system bandwidth, reuse figure, sectors per site, offered traffic...). Hence, one can use such expressions to optimize system parameters while satisfying the GSM standard QoS requirements. All GSM PHY layer modules that impact the error rate are considered in the analysis. A Cochannel Interference (CCI) limited and Rayleigh channel is

$$P_{FER} < 1-(1-P_e)^{I_a}$$
$$=1-\left(1-\frac{1}{k}\cdot\sum_{w=d_H}^{d_H+2} A_w \cdot \sum_{e=\left\lceil \frac{w}{2}\right\rceil}^{w} \binom{w}{e}\cdot\left(\sum_{n=0}^{N}\left(1-\left(\frac{\frac{n}{2}\cdot\lambda\cdot SIR}{\frac{n}{2}\cdot\lambda\cdot SIR+1}\right)^{n}\right)\cdot\binom{N}{n}\cdot\zeta^{n}\cdot(1-\zeta)^{N-n}\right)^{e}\cdot \left(1-\left(\sum_{n=0}^{N}\left(1-\left(\frac{\frac{n}{2}\cdot\lambda\cdot SIR}{\frac{n}{2}\cdot\lambda\cdot SIR+1}\right)^{n}\right)\cdot\binom{N}{n}\cdot\zeta^{n}\cdot(1-\zeta)^{N-n}\right)\right)^{w-e}\right)^{I_a} \tag{22}$$

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considered for evaluating the raw BER of the GSM GMSK modulation. The calculated BER is then used as a crossover probability for a Binary Symmetric Channel (BSC) that is perceived by the convolutional code. The channel is considered memoryless due to interleaving and frequency hopping effects. An upper bound for the error probability of a hard decision Viterbi decoder is derived. This post-decoding error probability represents the GSM system BER that can be easily converted into an equivalent Frame Erasure Rate (FER). Plots for BER and FER are shown and the impact of system load on the error rate bound is emphasized.

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