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An Arithmetic Approach to the General Two Water Jugs Problem

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***Abstract*—The water jugs problem is a well-known problem in recreational mathematics, problem-solving, artificial intelligence, computer programming and cognitive psychology. The methods of solutions are usually based on heuristics or search methods such as breadth first search (BFS) or depth first search (DFS), which could be time and memory consuming sometimes. In this paper, we present an arithmetic approach to solve this problem, which is simple and suitable for manual calculation or programming language implementation. Analysis of the solution steps involved and some illustrative examples are provided.**

***Index Terms*—water jugs problem, arithmetic approach, problem-solving, artificial intelligence.**

I. Introduction

THE water jugs problem is a well-known problem in problem-solving [1], geometry [2], recreational mathematics [3], discrete mathematics [4], computer programming [5], cognitive psychology [6, 9 10] and artificial intelligence [11], etc. The problem says:

> *"You are at the side of a river. You have a 3 liter jug and a 5 liter jug. The jugs do not have markings to allow measuring smaller quantities. How can you use the jugs to measure 4 liters of water? "*

There are various methods to solve this problem, including the working backwards approach [1], the billiards approach [2, 3], the diagraph approach [4], the search approach (such as BFS or DFS) [5, 11] and the use of heuristics [6, 8, 9, 10]. However, the drawbacks of using these methods could be time and memory consuming sometimes. In this paper, we present a simple arithmetic approach to solve the problem that was introduced by the author in [12]. A novel feature of this approach is that one can deduce the total amount of water (say V) in the jugs at each pouring step by simple addition or subtraction only and the actual pouring sequence can be easily determined by looking at the computed value of V. Due to its simplicity, it is very suitable for manual calculation or implementation in computer languages for automation of the pouring steps. Also, the upper bound on the number of pouring steps involved is a linear function of the capacities of the given water jugs and there is no additional memory cost required for doing searching and branching like some common search methods when performing the pouring steps.

The whole paper is organized as follows. In section 2, we will present the arithmetic approach for solving the general two water jugs problem and describe the mathematical background behind. In the third section, we will illustrate how to use the new approach by some examples. In the fourth section, we will discuss how to obtain an upper bound on the number of pouring steps involved. Then, we will provide some concluding remarks in the final section.

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II. An Arithmetic Approach to the Water Jugs Problem

An arithmetic approach to the general two water jugs problem was introduced in [12], which can be applied to solve the problem below:

> *"Let m, n, d be positive integers. You are at the side of a river. You have a m-liter jug and a n-liter jug, where 0<m<n. The jugs do not have markings to allow measuring smaller quantities. How can you use the jugs to measure d (<n) liters of water? "*

This problem can be modeled by means of a Diophantine equation, namely $mx+ny=d$, whose solvability is determined by the following theorem [7].

Theorem 2.1. The Diophantine equation $mx+ny=d$ is solvable if and only if gcd(m, n) divides d.

For example, the water jugs problem mentioned in the introduction section is solvable since gcd(3, 5) divides 4. However, if the jugs are replaced by a 3-litre jug and a 9-litre jug, then it will be insolvable because gcd(3, 9) cannot divide 4. Now, let us assume that $mx+ny=d$ is solvable in the discussions below. Depending on which jug is chosen to be filled first, there are two possible solutions for the two water jugs problems, say M_1 and M_2, which can be determined by the integer sequences obtained by applying the algorithms below:

Algorithm 2.1.
Input: The integers m, n and d, where $0<m<n$ and $d<n$.
Output: An integer sequence corresponding to a feasible solution called M_1 by filling the m-litre jug first.
Procedure:
Step 1. Initialize the sequence by a dummy variable $k = 0$.
Step 2. If $k \neq d$, then repeat adding m to k and assign the result to k until $k = d$ or $k > n$.

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Step 3. If $k > n$, then subtract n from k and assign the result to k.
Step 4. If $k = d$, then stop. Otherwise, repeat the steps from Step 2 to Step 4.

In this algorithm, the number of additions (say x_1) and subtractions (say y_1) involved will provide a solution to the $mx+ny=d$, namely $x = x_1$, $y = -y_1$. The actual pouring sequence can be determined easily by looking at the numbers appeared in the integer sequence obtained.

Algorithm 2.2.
Input: The integers m, n and d, where $0<m<n$ and $d<n$.
Output: An integer sequence corresponding to a feasible solution called M_2 by filling the n-litre jug first.
Procedure:
Step 1. Initialize the sequence by a dummy variable $k = 0$.
Step 2. If $k \neq d$, then add n to k and assign the result to k.
Step 3. If $k > d$, then repeat subtracting m from k and assign the result to k until $k = d$ or $k < m$.
Step 4. If $k = d$, then stop. Otherwise, repeat the steps from Step 2 to Step 4.

In this algorithm, the number of subtractions (say x_2) and additions (say y_2) involved will provide a solution to the $mx+ny=d$, namely $x = -x_2$, $y = y_2$. Again, the actual pouring sequence can be determined easily by looking at the numbers appeared in the integer sequence obtained.

III. EXAMPLES

We now illustrate how to use the arithmetic approach to solve the two water jugs problem below.

Example 3.1. There are a 3-litre jug and a 5-litre jug. We want to use them to measure 4 liters of water, as described in the introduction before. Using the above notations, we have $m=3$, $n=5$, $d=4$ and the associated Diophantine equation is $3x+5y=4$. Applying Algorithm 2.1, we can obtain an integer sequence for M_1:

$$\boxed{0} \xrightarrow[+3]{} \boxed{3} \xrightarrow[+3]{} \boxed{6} \xrightarrow[-5]{} \boxed{1} \xrightarrow[+3]{} \boxed{4}$$

The number of additions and subtractions involved are 3 and 1 respectively, so $x = 3$, $y = -1$ is a solution to the equation $3x+5y=4$. Since the integers inside the boxes in the sequence refer to the total amount of water in the jugs at different stages, we can work out the corresponding pouring steps very easily. If we use a coordinate (x, y) to represent the amounts of water inside the 3-litre jug and the 5-litre jug in each pouring step, then the successive pouring steps for M_1 are as follows:

$(0,0) \rightarrow (3,0) \rightarrow (0,3) \rightarrow (3,3) \rightarrow (1,5) \rightarrow (1,0) \rightarrow (0,1) \rightarrow (3,1) \rightarrow (0, 4)$

Thus, the total number of pouring steps involved in M_1 is 8.

Similarly, we can obtain an integer sequence for M_2 by applying Algorithm 2.2, namely:

$$\boxed{0} \xrightarrow[+5]{} \boxed{5} \xrightarrow[-3]{} \boxed{2} \xrightarrow[+5]{} \boxed{7} \xrightarrow[-3]{} \boxed{4}$$

The number of additions and subtractions involved are 2 and 2 respectively, so $x = -2$, $y = 2$ is a solution to the equation $3x+5y=4$. The corresponding pouring steps for M_2 are:

$(0,0) \rightarrow (0,5) \rightarrow (3,2) \rightarrow (0,2) \rightarrow (2,0) \rightarrow (2,5) \rightarrow (3,4)$

Thus, the total number of pouring steps involved in M_2 is 6. By comparing the number of steps in M_1 and M_2, we know M_2 provides a more optimal solution to this particular water jug problem.

Example 3.2. There are a 3-litre jug and a 7-litre jug. We want to use them to measure 5-litre of water. So, $m=3$, $n=7$, $d=5$ and $3x+7y=5$ is the associated Diophantine equation. Applying Algorithm 2.1, we can obtain an integer sequence for M_1:

$$\boxed{0} \xrightarrow[+3]{} \boxed{3} \xrightarrow[+3]{} \boxed{6} \xrightarrow[+3]{} \boxed{9} \xrightarrow[-7]{} \boxed{2} \xrightarrow[+3]{} \boxed{5}$$

The number of additions and subtractions involved are 3 and 1 respectively, so $x = 4$, $y = -1$ is a solution to the equation $3x+7y=5$. The corresponding pouring steps for M_1 are:

$(0,0) \rightarrow (3,0) \rightarrow (0,3) \rightarrow (3,3) \rightarrow (0,6) \rightarrow (3,6) \rightarrow (2,7) \rightarrow (2,0) \rightarrow (0, 2) \rightarrow (3,2) \rightarrow (0,5)$

Thus, the total number of pouring steps involved in M_1 is 10.

Similarly, we can obtain an integer sequence for M_2 by applying Algorithm 2.2, namely:

$$\boxed{0} \xrightarrow[+7]{} \boxed{7} \xrightarrow[-3]{} \boxed{4} \xrightarrow[-3]{} \boxed{1} \xrightarrow[+7]{} \boxed{8} \xrightarrow[-3]{} \boxed{5}$$

The number of additions and subtractions involved are 2 and 3 respectively, so $x = -3$, $y = 2$ is a solution to the equation $3x+7y=5$. The corresponding pouring steps for M_2 are:

$(0,0) \rightarrow (0,7) \rightarrow (3,4) \rightarrow (0,4) \rightarrow (3,1) \rightarrow (0,1) \rightarrow (1,0) \rightarrow (1,7) \rightarrow (3, 5)$

Thus, the total number of pouring steps involved in M_2 is 8. By comparing the number of steps in M_1 and M_2, we know M_2 provides a more optimal solution to this particular water jug problem.

IV. AN UPPER BOUND ON THE NUMBER OF POURING STEPS

Assume the Diophantine equation $mx+ny=d$ is solvable. By using linear congruence, the smallest positive integral solution, say (x', y'), can be found by solving the equations:

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$$mx \equiv d \pmod{n}; \quad ny \equiv d \pmod{m}.$$

Hence, we have:

$$|x'| \le n-1; \quad |y'| \le m-1.$$

Let N be the number of pouring steps involved in the arithmetic approach. Since each number in the integer sequence M_1 or M_2 represents the total amount of water in the two jugs at different step and there are at most two pouring steps associated with such a number (see the examples above), so we have:

$$N \le 2(|x'| + |y'|) \le 2(m+n-2).$$

In general, the value of N may not be the same for M_1 and M_2. Also, it is strictly less than 2(m+n-2) in most cases, as shown in the examples above.

V. Concluding Remarks

A simple arithmetic approach for solving the general two water jugs problem is presented in this paper. It is suitable for hand calculations or implementation in common computer languages. An upper bound on the total number of pouring steps involved can be easily obtained by substituting the values of m, n into the expression 2(m+n-2). Moreover, the integer sequences for M_1 or M_2 can be computed easily by using simple additions and subtractions only. There is no additional memory cost required for doing searching and branching like some common search methods when performing the pouring steps. Due to its simplicity and novelty, this approach is suitable for introduction to students or researchers involved in studying or doing researches in the areas of recreational mathematics, computer programming, problem-solving, cognitive psychology, artificial intelligence or discrete mathematics. Further extension of this approach to handle a more general k (>2) water jugs problem will be a meaningful research problem to pursue.

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