Proceedings of the World Congress on Engineering 2013 Vol III,
WCE 2013, July 3 - 5, 2013, London, U.K.

Free Vibration Of Antisymmetric Angle-Ply Laminated Annular Circular Plate

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Abstract— **Free vibration of laminated antisymmetric angle-ply annular circular plates are studied with inclusion of first order shear deformation theory using a spline function approximation by applying a point collocation method. The equations of motion of the plates are derived using first order shear deformation theory. The vibration of two- and four-layered plates are analysed, made up of two types of materials and two types of boundary conditions. A generalized eigenvalue problem is obtained and solved numerically for obtaining the required eigenfrequency parameters and associate eigenvectors are spline coefficients. The stability of the annular circular plate is analyzed with respect to the radii ratio, circumferential node number, different materials, number of laminates, ply orientations under different boundary conditions for two and four layered plates.**

Index Terms— **angle-ply, free vibration, shear deformation, splines**

I. INTRODUCTION

Composite structural elements play a significant role in several fields including aerotechnology, automobiles and ship building because of their more desirable temperature resistant, low cost, light weight, damping and shock absorbing characteristics than those of homogeneous ones. In composite plates the influence of shear deformation becomes more significant as the plate thickness increases and hence theories incorporating this aspect are highly desired, along with suitable numerical techniques. Unlike the isotropic materials, the properties of composite materials can be tailored to have very high strength and yet be very light. An adjustment in the natural frequency of any composite structure can be made by using the different materials, ply angles, changing the thickness and adjusting the boundary conditions. So far some significant theories and methods of analysis have been developed and applied for stress and vibration studies of composite elements.

The classical laminate theory which is based on Kirchhoff and Love, neglects shear deformation. First order shear deformation theory (FSDT) was developed as an extension of the theory of Mindlin for laminated isotropic plates in which transverse shear and rotary inertia effects are included. This theory is called YNS theory and this has been generalized to laminated anisotropic plates by [1]. A closed form solution [2] and FEM solution [3] was used to analyze the free vibration of simply supported rectangular plates of anti-symmetric angle-ply laminates. Recently [4] analyzed annular Mindlin plate using the singular convolution method without including the shear deformation theory.

In this paper a semi-numerical technique of applying the point-collocation method with spline function approximation is used for analyzing the free vibration of anti-symmetric angle-ply plates under shear deformation theory. This theory is dependent on a choice of shear correction factors and number of procedures were developed to find the values of shear correction factors [5]-[8]. In the study of [9] a meshless method of discretization is used in the third order theory of Reddy on composite laminated plate, along with the multiquadric radial function method which allows an accurate prediction of field variables. A layerwise theory of composite and sandwich laminated plates using polyharmonic splines is analysed [10].

This paper deals with the free vibration of antisymmetric angle-ply laminated composite annular circular plates including shear deformation using the method of collocation with splines. Same, as well as different types of materials used in different layers. The problem is formulated using YNS theory from which, a system of coupled differential equations are obtained for a set of assumed displacement and rotational functions which are functions of a space co-ordinate. A spline technique is used, preferring over a number of other methods available for such problems, like those of Galerkin, Runge-Kutta, Frobenius, Chebyshev, Differential quadrature and Rayleigh-Ritz methods. The choice of the spline method is due to the possibility that a chain of lower order approximations, as used here, which can yield greater accuracy than a global high order approximation [11]. The spline collocation method was tested successfully over a two point boundary value problem with a cubic spline [11] and also recently spline approximation was applied successfully by [12] –[14].

In this work three displacement functions and two rotational functions are approximated by splines, which are cubic, in a system of coupled equations. The convergence characteristics are brought out. These splines are simple and clear for analytical process and have significant computational advantage. Assuming the solution in a separable form , a system of coupled differential equations in displacement and rotational functions, is obtained and these functions are approximated by Bickley-type splines of order three. Collocation with these splines yield a set of field equations which, along with the equations of boundary

Manuscript received March 06, 2013; revised March 12, 2013. This work was supported by UTM-MOHE Research fund under the Vote No. 0714J, Research Management Centre (RMC), Universiti Teknologi Malaysia, Johor Bahru, Johr, Malayisa.

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conditions, reduce to a system of homogeneous simultaneous algebraic equations on the assumed spline coefficients. The resulting generalized eigenvalue problem is solved for a frequency parameter, using eigensolution techniques, to obtain as many frequencies as required, starting from the least. From the eigenvectors the spline coefficients are computed from which the mode shapes and shear rotations can be constructed.

Parametric studies are made considering the variation of frequency parameters with respect to the radii ratio, circumferential node number, different sequence and number of lay-ups. The effect of two and four layers using two different materials and boundary conditions on the frequency parameter are analysed. Significant mode shapes are obtained. Numerical results are presented in terms of graphs and tables and discussed.

II. Formulation of the Problem

Consider a composite laminated annular circular plate with an arbitrary number of layers, which are perfectly bonded together. Consider $r_a = a$ is the inner radius and $r_b = b$ is the outer radius of the annular circular plate and $\ell = b - a$ be the width of the annular circular plate. The curvilinear coordinate system (r, θ, z) is fixed at its reference surface, which is taken to be its middle surface.

According to [15], the displacement components are assumed to be

$$\begin{aligned} u(r,\theta,z,t) &= u_0(r,\theta,t) + z\psi_r(r,\theta,t) \\ v(r,\theta,z,t) &= v_0(r,\theta,t) + z\psi_\theta(r,\theta,t) \\ w(r,\theta,z,t) &= w_0(r,\theta,t) \end{aligned} \tag{1}$$

where u, v, w are the displacement components in the r, θ and z directions respectively, u_0, v_0 and w_0 are the in-plane displacements of the middle plane and ψ_r and ψ_θ are the shear rotations of any point on the middle surface of the plate.

The equation of stress-resultants and displacements are in the form.

$$\begin{pmatrix} N_r \\ N_\theta \\ N_{r\theta} \\ M_r \\ M_\theta \\ M_{r\theta} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \gamma_{r\theta} \\ \kappa_r \\ \kappa_\theta \\ \kappa_{r\theta} \end{pmatrix}$$

$$\begin{pmatrix} Q_\theta \\ Q_r \end{pmatrix} = K \begin{pmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{pmatrix} \begin{pmatrix} \gamma_{\theta z} \\ \gamma_{rz} \end{pmatrix} \tag{2}$$

where

$$\varepsilon_r = \frac{\partial u_0}{\partial r} + z\frac{\partial \psi_r}{\partial r}, \varepsilon_\theta = \frac{1}{r}\frac{\partial v_0}{\partial \theta} + \frac{u_0}{r} + z\left(\frac{1}{r}\frac{\partial \psi_\theta}{\partial \theta} + \frac{\psi_r}{r}\right)$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u_0}{\partial \theta} + \frac{\partial v_0}{\partial r} + \frac{v_0}{r} + z\left(\frac{1}{r}\frac{\partial \psi_r}{\partial \theta} + \frac{\partial \psi_\theta}{\partial r} - \frac{\psi_\theta}{r}\right)$$

$$\gamma_{rz} = \psi_r + \frac{\partial w}{\partial r}, \gamma_{\theta z} = \psi_\theta + \frac{1}{r}\frac{\partial w}{\partial \theta},$$

$$\kappa_r = \frac{\partial \psi_r}{\partial \theta}, \quad \kappa_\theta = \frac{1}{r}\frac{\partial \psi_\theta}{\partial \theta} + \frac{\psi_r}{r},$$

$$\kappa_{r\theta} = \frac{\partial \psi_\theta}{\partial r} + \frac{1}{r}\frac{\partial \psi_r}{\partial \theta} - \frac{\psi_\theta}{r}$$

and

$$A_{ij} = \sum_k Q_{ij}^{(k)}(z_k - z_{k-1}), B_{ij} = \frac{1}{2}\sum_k Q_{ij}^{(k)}(z_k^2 - z_{k-1}^2),$$

$$D_{ij} = \frac{1}{3}\sum_k Q_{ij}^{(k)}(z^3{}_k - z^3{}_{k-1}) \quad (i, j = 1, 2, 6)$$

$$A_{ij} = K\sum Q_{ij}^{(k)}(z_k - z_{k-1}) \quad (i, j = 4, 5) \tag{3}$$

where K is the shear correction coefficient and z_{k-1} and z_k are the boundaries of the k-th layer and $\bar{Q}_{ij}^{(k)}$ are defined in [12] . The value of K for a general laminate depends on lamina properties and lamination scheme and may be calculated by various static and dynamic methods [5].

The coefficients A_{16}, A_{26}, A_{45}, B_{11}, B_{12}, B_{22}, B_{66}, D_{16} and D_{26} are identically zero for antisymmetric angle-ply laminates [16].

The governing equations of motion in u, v, w, ψ_r and ψ_θ are obtained.

The displacement components u_0, v_0, w and shear rotations ψ_r and ψ_θ are assumed in the form

$$\begin{aligned} u_0(r,\theta,t) &= U(r)e^{n\theta}e^{i\omega t} \\ v_0(r,\theta,t) &= V(r)e^{n\theta}e^{i\omega t} \\ w(r,\theta,t) &= W(r)e^{n\theta}e^{i\omega t} \\ \psi_\theta(r,\theta,t) &= \Psi_\theta(r)e^{n\theta}e^{i\omega t}, \\ \psi_r(r,\theta,t) &= \Psi_r(r)e^{n\theta}e^{i\omega t} \end{aligned} \tag{4}$$

where r and θ are polar coordinates describes the radial and circumferential direction, ω is the angular frequency of vibration, t is the time and n is the circumferential node number. Substituting (4) into the governing equations of motion. The equations are obtained in terms of displacement functions and rotational functions.

Introducing the non-dimensional parameters as:

$$R = \frac{r-a}{l}, \; a \le r \le b \text{ and } \quad R \in [0,1] \; \lambda = \omega l\sqrt{\frac{I_1}{A_{11}}} \quad ,$$

$$\beta = \frac{a}{b}\ ,\ \gamma = \frac{h}{r_a}\ ,\ \delta_k = \frac{h_k}{h} \qquad (5)$$

Here h_k is the thickness of the k-th layer, h is the total thickness of the plate, r_a is the radius of inner circular plate and A_{11} is a standard extensional rigidity co-efficient.

The new set of differential equations is obtained by substituting the non-dimensional co-efficients and the equations consists of the second order derivatives in $U(R)$, $V(R)$, $W(R)$, $\Psi_r(R)$, $\Psi_\theta(R)$.

III. PMETHOD OF SOLUTION

The displacement and rotational functions are approximated as

$$\begin{aligned}
U^*(R) &= \sum_{i=0}^{2} a_i R^i + \sum_{j=0}^{N-1} b_j (R-R_j)^3\ H(R-R_j) \\
V^*(R) &= \sum_{i=0}^{2} c_i R^i + \sum_{j=0}^{N-1} d_j (R-R_j)^3\ H(R-R_j) \\
W^*(R) &= \sum_{i=0}^{2} e_i R^i + \sum_{j=0}^{N-1} f_j (R-R_j)^3\ H(R-R_j) \\
\Psi_r^*(R) &= \sum_{i=0}^{2} g_i R^i + \sum_{j=0}^{N-1} p_j (R-R_j)^3\ H(R-R_j) \\
\Psi_\theta^*(R) &= \sum_{i=0}^{2} l_i R^i + \sum_{j=0}^{N-1} q_j (R-R_j)^3\ H(R-R_j)
\end{aligned} \qquad (6)$$

in which $H(R-R_j)$ is the Heaviside function and $a_i, c_i, e_i, g_i, l_i, b_j, d_j, f_j, p_j$ and q_j are unknown coefficients (*i*.e., spline coefficients).

These functions have very attractive characteristics compared to many others from the point of convergence, accuracy and elegance of usage analytically and efficiency for computational work.

Let us assume that the interval $R\,\varepsilon\,[\,0,\ 1\,]$ is divided into N equal sub-intervals. The knots are at $R = R_s = \frac{s}{N};\ s = 0,1,....,N$. Imposing the condition that the final differential equations satisfies (6) at each $R = R_s$, we have the system of $(5N+5)$ homogeneous equations with $(5N+15)$ spline coefficients.

The two types of boundary conditions are considered

(1) (C - C): Clamped clamped

(2) (S - S): Simply supported

Each of these cases gives 10 more equations, thus making a total of $(5N+15)$ equations, in the same number of unknowns. The resulting field and boundary condition equations may be written in the form

$$[\ M\ ]\{\ q\ \} = \lambda^2\ [\ P\ ]\{\ q\ \} \qquad (7)$$

where $[M]$ and $[P]$ are square matrices, $\{q\}$ is a column matrix. This is treated as a generalized eigenvalue problem in the eigen parameter λ and the eigenvector $\{q\}$ whose elements are the spline coefficients.

IV. RESULTS AND DISCUSSION

In this work free vibration of antisymmetric angle-ply annular circular plates of variable thickness under shear deformation theory is studied. The spline method is used to approximate the displacement functions for analyzing the vibration behavior of the layered circular plates with different boundary conditions and different types of materials. The convergence and comparative study have been done to check the validity of spline function. Numerical computations were carried out for studying the results of convergence. The convergence study of the frequency parameter is carried out to fix the value of N (the number of knots) with respect to the thickness to radius ratio, radii ratio, number of layers and circumferential node number. The program performed for N=2 onwards and finally it is fixed as N=14 would be enough to achieve percentage of the next value of N is 0.35%.

TABLE I
VARIATION OF FREQUENCY PARAMETER $\lambda_m\,(m=1,2,3)$ WITH n UNDER C-C BOUNDARY CONDITIONS

	$30^0/-30^0$ KGE/KGE			$30^0/-30^0/30^0/-30^0$ KGE/AGE/AGE/KGE		
n	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
1	0.21746	0.576408	1.077354	0.26977	0.689585	1.246933
2	0.210654	0.568357	1.069747	0.262512	0.068057	1.238558
3	0.199002	0.554733	1.05955	0.250266	0.665304	1.224478
4	0.181856	0.535185	1.038802	0.23275	0.643386	1.204502
5	0.157666	0.509096	1.015023	0.209391	0.61415	1.178352
6	0.121575	0.47519	0.985173	0.178911	0.576486	1.145653
7	0.028287	0.429255	0.938123	0.147571	0.52824	1.105899
8	0.242279	0.320388	0.90784	0.117512	0.47825	1.065899
9	0.136142	0.354368	0.857672	0.078745	0.39067	1.003088

Table I depicts the variation of frequency parameter $\lambda_m\,(m=1,2,3)$ with reference to the circumferential node number n for 2 and 4 layered antisymmetric angle-ply plates made of Kevlar-49/epoxy (KGE) and Graphite Epoxy (AS4/3501-6) (AGE) materials with a clamped-clamped (C-C) boundary conditions arranging the layers in the order of $30^0/\text{-}30^0$ (KGE/KGE) and $30^0/\text{-}30^0/30^0/\text{-}30^0$ (KGE/AGE/AGE/KGE). The value of frequency parameter is higher for 4 layered as compared to 2 layered plates. In the case of two layered plates the value of frequencyparemeter $\lambda_m\,(m=1,2,3)$ decreases with increasing the circumferential node number n. Only the λ_1 and λ_2 have zigzag condition for n=8 and 9. This may be due to the node number and material oder. In case of four layered plates as the circumferential node number n increases the value of the frequency parameter decreases for λ_1, λ_2 and λ_3.

TABLE II
VARIATION OF FREQUENCY PARAMETER $\lambda_m\ (m=1,2,3)$ WITH n UNDER S-S-BOUNDARY CONDITIONS

	$30^0/-30^0$ KGE/KGE			$30^0/-30^0/30^0/-30^0$ KGE/AGE/AGE/KGE		
n	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
1	0.15288	0.357923	0.905084	0.141874	0.492137	1.05064
2	0.065779	0.335471	0.859984	0.106979	0.476782	1.037467
3	0.061698	0.303261	0.825659	0.070009	0.449997	1.015701
4	0.011714	0.25438	0.788791	0.064529	0.410025	0.980985
5	0.072015	0.183504	0.741186	0.125138	0.352578	0.94361
6	0.054362	0.115487	0.685125	0.174141	0.268357	0.89079
7	0.09475	0.205712	0.615027	0.231094	0.127951	0.823825
8	0.106427	0.28824	0.525976	0.151555	0.324356	0.739627
9	0.101456	0.361542	0.408694	0.202906	0.438821	0.632355

Table II depicts the variation of frequency parameter $\lambda_m\ (m=1,2,3)$ with reference to the circumferential node number n for two- and four-layered antisymmetric angle-ply plates made of Kevlar-49/epoxy (KGE) and Graphite Epoxy (AS4/3501-6) (AGE) materials under simply-supported (S-S) boundary conditions arranging layers in the order of $30^0/-30^0$ (KGE/KGE) and $30^0/-30^0/30^0/-30^0$ (KGE/AGE/AGE/KGE). The value of frequency parameter is higher for four-layered as compared to two-layered plates except for n=1.As the circumferential node number n increases for two- and four-layered plate the value of the frequency parameter λ_1 varies randomly.

For two- layered plate the value of frequency parameter decreases for λ_2 up to n=6 and starts increasing for n>6 In the case of four-layered plate the value of the frequency parameter decreases for λ_2 up to $n=7$ then it starts increasing for n>7. As the circumferential node number n increases for two- and four-layered plate the value of frequency parameter decrease for λ_3.

Figure 1 depicts the variation of angular frequency ω_m with respect to radii ratio β by fixing the circumferential node number n=2 and ratios of thickness to radius of the inner circle $\gamma=0.05$ for two-layered antisymmetric angle-ply plates under C-C boundary conditions. The plies are oriented for Fig. 1 (a), (b) and (c) as 30^0/-30 45^0/-45^0 and 60^0/-60^0 respectively and arranged materials in the order of KGE-KGE in all cases. It is seen in Fig. 1 (a) that as the radii ratio β increases there is a slight increase in angular frequency up to β< 0.5 but there is a significant increase in angular frequency for $\beta\geq$ 0.5. The angular ω_m is higher for higher modes. The same trend can be seen in Fig. 1 (b) and (c).

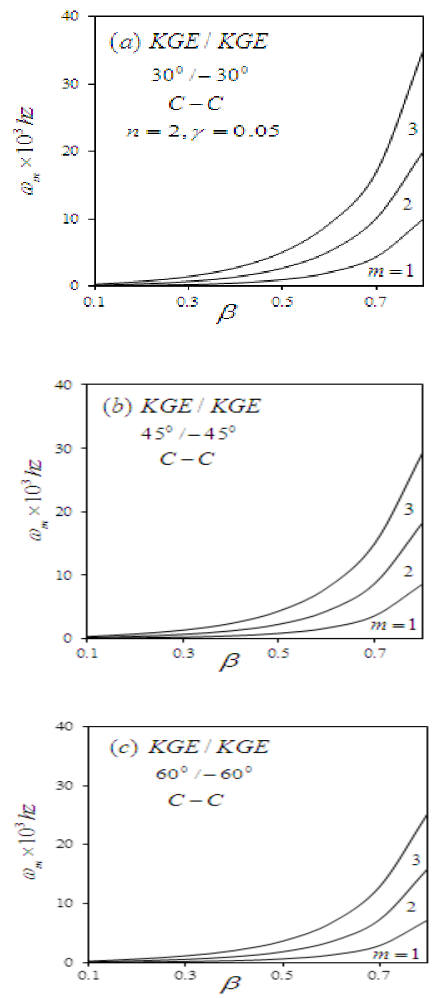

Fig. 1 ((a)-(c)). Effect of radii ratio on the angular frequency of two-layered plate under C-C boundary conditions with n=2

Figure 2 depicts the variation of angular frequency ω_m with respect to radii ratio β by fixing the circumferential node number n=2 and ratios of thickness to radius of the inner circle $\gamma=0.05$ for four-layered antisymmetric angle-ply plates under C-C boundary conditions. The plies are oriented for Fig. 2 (a), (b), (c) as $30^0/-30^0/30^0/-30^0$, $45^0/-45^0/45^0/-45^0$ and $60^0/-60^0/60^0/-60^0$ respectively in the order of materials KGE/AGE/AGE/KGE. It is seen in Fig. 2 (a) that as the radii ratio β increases there is a slight increase in angular frequency ω_m up to $\beta<0.5$ and there is a significant increase in angular frequency ω_m for $\beta \geq 0.5, 0.7, 0.8$. The angular ω_m is higher for higher modes. The same trend can be seen in Figs. 2 (b) and (c).

Figure 3 depicts the variation of angular frequency ω_m with respect to radii ratio β by fixing the circumferential node number n=2 and ratios of thickness to radius of the inner circle γ=0.05 for two-layered antisymmetric angle-ply plates under S-S boundary conditions. The plies are oriented for Fig. 3 (a), (b) and (c) as $30^0/-30^0$ (KGE/KGE), $45^0/-45^0$ (KGE/KGE) and $60^0/-60^0$ (KGE/KGE) respectively. It is seen in Fig. 3 (a) that as the radii ratio β increases there is a slight increase in angular frequency up to $\beta<0.5$ but there is a significant increase in angular frequency for $\beta \geq 0.5, 0.7, 0.8$.. The angular ω_m is higher for higher modes. The same trend can be seen in Fig. 3 (b) and (c).

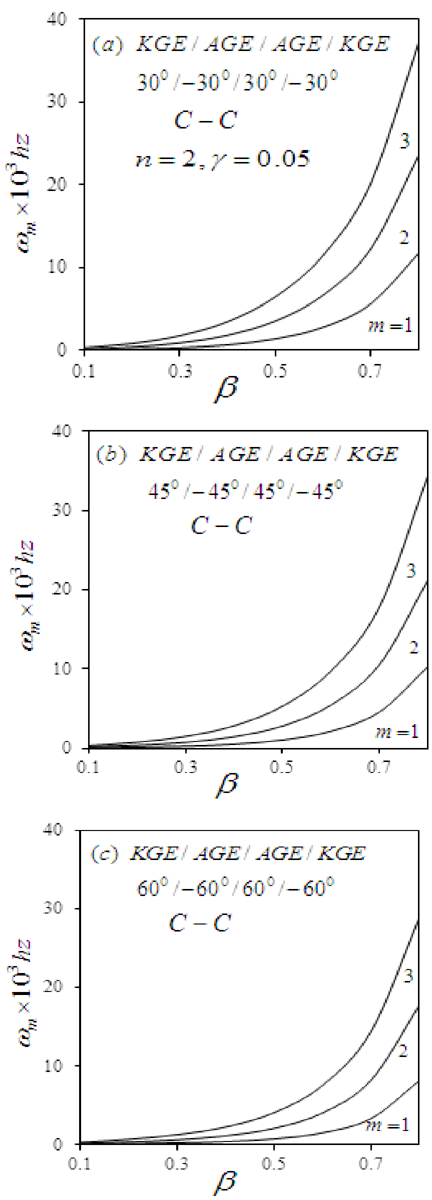

Fig. 2 ((a)-(c)). Effect of radii ratio on the angular frequency of four-layered plate under C-C boundary conditions with n=2

with

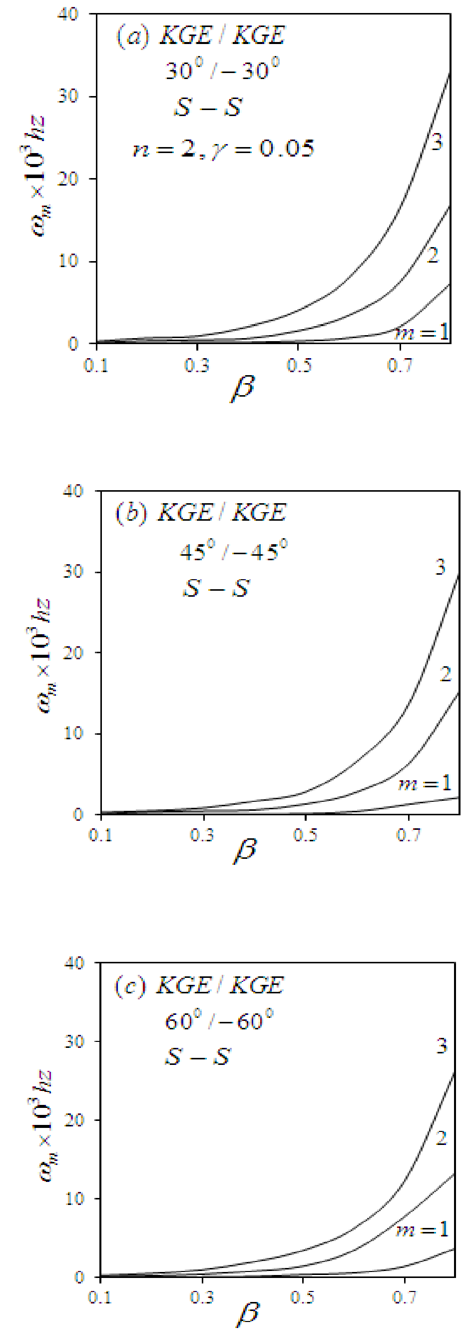

Fig. 3 ((a)-(c)). Effect of radii ratio on the angular frequency of two-layered plate under S-S boundary conditions with n=2

Figure 4 depicts the variation of angular frequency ω_m with respect to radii ratio β by fixing the circumferential node number n=2 and ratios of thickness to radius of the inner circle γ=0.05 for four-layered antisymmetric angle-ply plates under S-S boundary conditions. The plies are oriented for Fig. 4(a), (b), (c) as $30^0/\text{-}30^0/30^0/\text{-}30^0$ (45^0/-

$45^0/\text{-}45^0\ 45^0$ and $60^0/\text{-}60^0/60^0/\text{-}60^0$ in the order of KGE/ AGE/AGE/KGE respectively. It is seen in Fig. 4 (a) that as the radii ratio β increases there is a slight increase in angular frequency up to β=0.5 but there is a significant increase in angular frequency for $\beta = 0.6, 0.7, 0.8$. The same trend can be seen in Fig. 4 (b) and (c).

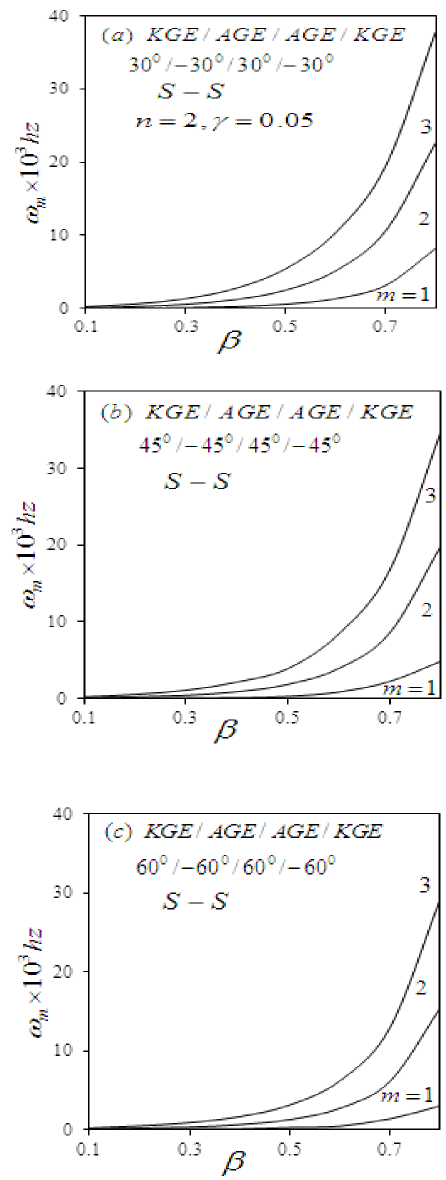

Fig. 4 ((a)-(c)). Effect of radii ratio on the angular frequency of four-layered plate under S-S boundary conditions with n=2

V. CONCLUSION

Free vibration of antisymmetric angle-ply laminated annular circular plates is studied using spline function approximation under shear deformation theory. The inclusion of shear deformation theory significantly lowers the values of the frequency parameters when we compared to the values predicted by the classical plate theory. The results presented in this paper may be fruitful for designers to choose the materials, ply-angle and circumferential node number for making the circular plate structures according to their needs in designing appropriately.

ACKNOWLEDGMENT

The author thankfully acknowledges the financial support from the UTM - MOHE Research Grant Vote Nos. 05J17 and 07J49, Research Management Centre(RMC), Universiti Teknologi Malaysia, Malaysia for completion of this research work and presenting in WCE 20013 conference, London, UK.

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ISBN: 978-988-19252-9-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)