Fourier Approximation of Functions Conjugate to the Functions Belonging to Weighted Lipschitz Class

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***Abstract*—The study of error estimates of periodic functions in $L^p(p \geq 1)$-spaces through Fourier series, although is an old problem and known as *Fourier approximation* in the existing literature, has been of a growing interests over the last four decades. The most common methods used for the determination of the degree of approximation of periodic functions are based on the minimization of the L^p-norm of $f(x) - T_n(x)$, where $T_n(x)$ is a trigonometric polynomial of degree n, and called the approximant of f. The degree of approximation of f, so obtained depends heavily on p. In this paper, we obtain the degree of approximation of $\tilde{f}$, conjugate to the function f belonging to weighted Lipschitz class $W(L^p, \xi(t))$ by a trigonometric polynomial generated by the product matrix means of the conjugate Fourier series of f. The degree of approximation obtained in our theorems of this paper is sharper than others and free from p. Some corollaries have also been deduced from our theorems.**

***Index Terms*—Fourier approximation, $W(L^p, \xi(t))$-class, $C^1.T$ means, periodic functions, $b_{n,n-k} \geq 0$.**

I. INTRODUCTION

FOR a 2π-periodic function $f \in L^p := L^p[0, 2\pi]$, $p \geq 1$, integrable in the sense of Lebesgue, let

$$s_n(f;x) := \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx), \ \ n \in \mathbb{N}$$
$$\text{and } s_0(f;x) = \frac{a_0}{2}, \quad (1)$$

denote the $(n+1)^{\text{th}}$ partial sums, called trigonometric polynomials of degree (or order) n, of the Fourier series of f. The conjugate series of the Fourier series of f is defined by $\sum_{k=1}^{\infty}(a_k \sin kx - b_k \cos kx)$ and its n^{th} partial sum is defined as

$$\tilde{s}_n(f;x) := \sum_{k=1}^{n}(a_k \sin kx - b_k \cos kx), \quad n \in \mathbb{N}$$
$$\text{and } \tilde{s}_0(f;x) = 0. \quad (2)$$

The conjugate of f denoted by $\tilde{f}$ is defined as

$$\tilde{f}(x) = -\frac{1}{2\pi} \lim_{\epsilon \to 0} \int_{\epsilon}^{\pi} \psi_x(t) \cot(t/2) dt, \quad (3)$$

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where $\psi_x(t) = f(x+t) - f(x-t)$ [1, p.131].
The L^p-norm of $f \in L^p[0, 2\pi]$ is defined by

$$\|f\|_p = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \mid f(x) \mid^p dx \right\}^{1/p} \quad (1 \leq p < \infty) \text{ and}$$
$$\|f\|_\infty = \sup_{x \in [0,2\pi]} \mid f(x) \mid .$$

We determine the degree of approximation (error estimates) $E_n(f)$ of $f \in L^p$-space by n^{th} degree trigonometric polynomials $T_n(x)$ given by

$$E_n(f) = \min_{T_n} \| f(x) - T_n(x) \|_p .$$

The $T_n(x)$ is called Fourier approximant of f, and this method of approximation is called Fourier approximation. In this paper , we consider the following function classes:

$$Lip\alpha := \{f : [0, 2\pi] \to R : |f(x+t) - f(x)| = O(t^\alpha)\},$$

$$Lip(\alpha, p) := \{f \in L^p[0, 2\pi] : \|f(x+t) - f(x)\|_p = O(t^\alpha)\},$$

$Lip(\xi(t), p) := \{f \in L^p[0, 2\pi] : \|f(x+t) - f(x)\|_p = O(\xi(t))\}$, $W(L^p, \xi(t)) := \{f \in L^p[0, 2\pi] : \|(f(x+t) - f(x)) \sin^\beta(x/2)\|_p = O(\xi(t))\}$, where $p \geq 1$, $0 < \alpha \leq 1$, $\beta \geq 0$, $t > 0$ and $\xi(t)$ is a positive increasing function of t [2, 3].
It is important to note that the increasing function $\xi(t)$ in the definition of $W(L^p, \xi(t))$-class is not the same as in the definition of $Lip(\xi(t), p)$-class. The $\xi(t)$ in $Lip(\xi(t), p)$-class depends on t only, whereas in $W(L^p, \xi(t))$-class it depends on t and β [3]. In particular, if we take $\xi(t) = t^\beta \psi(t)$ for $\beta \geq 0$ and some positive increasing function $\psi(t)$, then $W(L^p, \xi(t))$-class defined above reduces to $W'(L^p, \psi(t))$-class defined by Khan [3]. We also note that

$$Lip\alpha \subseteq Lip(\alpha, p) \subseteq Lip(\xi(t), p) \subseteq W(L^p, \xi(t))[2, 4].$$

Let $T \equiv (a_{n,k})$ be a lower triangular matrix with non-negative entries such that $a_{n,-1} = 0, A_{n,k} = \sum_{r=k}^{n} a_{n,r}$ and $A_{n,0} = 1, n \in \mathbb{N}_0$. The sequence-to-sequence transformation

$$\tilde{t}_n(f;x) := \sum_{k=0}^{n} a_{n,k} \tilde{s}_k(f;x), \ \ n \in \mathbb{N}_0,$$

defines the matrix means of $\{\tilde{s}_n(f;x)\}$. The conjugate Fourier series of the function f is said to be T-summable to s, if $\tilde{t}_n(f;x) \to s$ as $n \to \infty$.
By superimposing C^1-summability (Cesàro summability of order 1) upon T-summability, we get the $C^1.T$-summability.

Thus the $C^1.T$ means of $\{\tilde{s}_n(f;x)\}$ denoted by $\tilde{t}_n^{C^1.T}(f;x)$ are given by

$$\tilde{t}_n^{C^1.T}(f;x) := (n+1)^{-1}\sum_{r=0}^{n}\left(\sum_{k=0}^{r} a_{r,k}\tilde{s}_k(f;x)\right), n\in\mathbb{N}_0. \quad (4)$$

If $\tilde{t}_n^{C^1.T}(f;x)\to s_1$ as $n\to\infty$, then the conjugate Fourier series of f is said to be $C^1.T$ - summable to the sum s_1. The regularity of methods C^1 and T implies regularity of method $C^1.T$.

We also write

$$(C^1.T)_n(t) = \frac{1}{2\pi(n+1)}\sum_{r=0}^{n}\sum_{k=0}^{r} a_{r,r-k}\frac{\cos(r-k+1/2)t}{\sin(t/2)},$$

$b_{n,n-k} = \Delta_n a_{n,n-k} = a_{n,n-k} - a_{n+1,n+1-k}$ and $\tau = [1/t]$, the integral part of $1/t$.

In the last four decades, many researchers have been approximated the function $\tilde{f}$, conjugate of a function f belonging to $Lip\alpha$, $Lip(\alpha,p)$, $Lip(\xi(t),p)$ and $W(L^p,\xi(t))$-classes with $p\geq 1$, by different summability means of the conjugate Fourier series of f and obtained the error of approximation $E_n(\tilde{f})$, which depends heavily on p [2, 5–7]. A detailed review of the previous work done in this direction can be seen in our recent paper [2], in which authors have determined the degree of approximation of $\tilde{f}$, conjugate of $f\in W(L^p,\xi(t))$ by the Hausdorff means of the conjugate Fourier series of f and improved previous results in the light of Kranz et al. [8], Łenski and Szal [9] and Mishra et al. [4]. The degree of approximation of $\tilde{f}$ obtained in [2] is of order $(n+1)^{\beta+1/p}\xi(1/(n+1))$, which clearly depends on p, and leads to an open question whether this error of approximation can be made independent of p.

II. Main Results

The importance of Fourier approximation discussed in [5] and the observation mentioned above motivate us to study further the degree of approximation of $\tilde{f}$. In this paper, we obtain the degree of approximation of conjugate of functions belonging to the Lipschitz class $W(L^p,\xi(t))$ $(p\geq 1)$, by a general summability method, i.e., $C^1.T$ means of their conjugate Fourier series. This work is an attempt to make degree of approximation free from p. More precisely, we prove the following.

Theorem 1. *Let $T\equiv(a_{n,k})$ be a lower triangular regular matrix with non-negative and non-decreasing (with respect to k) entries which satisfy*

$$b_{n,n-k}\geq 0 \ for\ 0\leq k\leq n. \quad (5)$$

Then the degree of approximation of $\tilde{f}$, conjugate of a 2π-periodic function f belonging to the weighted Lipschitz class $W(L^p,\xi(t))$, with $p>1$ and $0\leq\beta<1/p$ by $C^1.T$ means of its conjugate Fourier series is given by

$$\| \tilde{t}_n^{C^1.T}(f;x)-\tilde{f}(x)\|_p = O\left((n+1)^{\beta}\xi(1/(n+1))\right), \quad (6)$$

provided a positive increasing function $\xi(t)$ satisfies the following conditions:

$$\xi(t)/t^{\beta+1-\sigma} \ \ \textit{is non-decreasing}, \quad (7)$$

$$\left\{\int_0^{\pi/(n+1)}\left(\frac{t^{-\sigma}\mid\psi_x(t)\mid\sin^{\beta}(t/2)}{\xi(t)}\right)^p dt\right\}^{1/p} = O((n+1)^{\sigma-1/p}), \quad (8)$$

for $\beta<\sigma<1/p$,

$$\xi(t)/t \ \ \textit{is non-increasing}, \quad (9)$$

$$\left\{\int_{\pi/(n+1)}^{\pi}\left(\frac{t^{-\delta}\mid\psi_x(t)\mid}{\xi(t)}\right)^p dt\right\}^{1/p} = O((n+1)^{\delta-1/p}), \quad (10)$$

where δ is an arbitrary number such that $1/p<\delta<\beta+1/p$ and $p^{-1}+q^{-1}=1$. The conditions (8) and (10) hold uniformly in x.

The conditions (8) and (10) can be verified by using the fact that $\psi_x(t)\in W(L_p,\xi(t))$ and $\psi_x(t)/\xi(t)$ is a bounded function. The condition (10) above is improved version of condition (14) of [2].

Note 1: Condition (9) implies that $\xi(\pi/(n+1))/(\pi/(n+1))\leq\xi(1/(n+1))/(1/(n+1))$, i.e., $(n+1)/\pi\xi(\pi/(n+1))\leq(n+1)\xi(1/(n+1))$.

Lemmas

We need the following lemmas for the proof of our theorem.

Lemma 1. *For $0<t\leq\pi/(n+1)$, $(C^1.T)_n(t)=O(1/t)$.*

Proof. Using $|\cos t|\leq 1$ and $\sin(t/2)\geq t/\pi$ for $0<t\leq\pi/(n+1)$, we have

$$\begin{aligned}
&|(C^1.T)_n(t)|\\
=\ & (2\pi(n+1))^{-1}\times\\
&\left|\sum_{r=0}^{n}\sum_{k=0}^{r}a_{r,r-k}\,(\cos(r-k+1/2)t)/(\sin t/2)\right|\\
\leq\ & (2\pi(n+1))^{-1}\times\\
&\sum_{r=0}^{n}\sum_{k=0}^{r}a_{r,r-k}\,|(\cos(r-k+1/2)t)/(\sin t/2)|\\
\leq\ & (2\pi(n+1))^{-1}\sum_{r=0}^{n}\sum_{k=0}^{r}a_{r,r-k}\ 1/(t/\pi)\\
=\ & O((n+1)t)^{-1}\sum_{r=0}^{n}\left(\sum_{k=0}^{r}a_{r,r-k}\right)=O(1/t).
\end{aligned}$$

Lemma 2. *If $\{a_{n,k}\}$ is non-negative and non-decreasing (with respect to k) sequence satisfying (5), then*

$$|(C^1.T)_n(t)| = O\left(t^{-2}/(n+1)\right),\ \textit{for}\ \pi/(n+1)<t\leq\pi.$$

Proof. Using $\sin(t/2)\geq t/\pi$, for $\pi/(n+1)<t\leq\pi$, we

have

$$\begin{aligned} &|(C^1.T)_n(t)| \\ &= (2\pi(n+1))^{-1} \times \\ &\quad \left|\sum_{r=0}^{n}\sum_{k=0}^{r} a_{r,r-k}(\cos(r-k+1/2)t)/\sin(t/2)\right| \\ &= O(t(n+1))^{-1}\left|Re\sum_{r=0}^{n}\sum_{k=0}^{r} a_{r,r-k}e^{i(r-k+1/2)t}\right| \\ &= O(t(n+1))^{-1}\left|\sum_{r=0}^{n}\sum_{k=0}^{r} a_{r,r-k}e^{i(r-k)t}\right|. \end{aligned}$$

Following [10, pp. 445-446], we have

$$\begin{aligned} &|\sum_{r=0}^{n}\sum_{k=0}^{r} a_{r,r-k}e^{i(r-k)t}| \\ &\leq \left|\sum_{r=0}^{\tau}\sum_{k=0}^{r} a_{r,r-k}e^{i(r-k)t}\right| + \left|\sum_{r=\tau+1}^{n}\sum_{k=0}^{\tau} a_{r,r-k}e^{i(r-k)t}\right| \\ &\quad + \left|\sum_{r=\tau+1}^{n}\sum_{k=\tau+1}^{r} a_{r,r-k}e^{i(r-k)t}\right| \\ &\leq K_1 + K_2 + K_3, say. \end{aligned}$$

Now

$$\begin{aligned} K_1 &\leq \sum_{r=0}^{\tau}\sum_{k=0}^{r} a_{r,r-k}\left|e^{i(r-k)t}\right| \\ &\leq \sum_{r=0}^{\tau} A_{r,0} = (\tau+1) = O(t^{-1}). \end{aligned}$$

Using Abel's transformation after changing the order of summation in K_2, we have

$$\begin{aligned} K_2 &= \left|\sum_{k=0}^{\tau}\sum_{r=\tau+1}^{n} a_{r,r-k}e^{i(r-k)t}\right| \\ &= \left|\sum_{k=0}^{\tau}\left[\sum_{r=\tau+1}^{n-1}\left\{b_{r,r-k}\sum_{v=0}^{r} e^{i(v-k)t}\right\}\right.\right. \\ &\quad \left.\left. + a_{n,n-k}\sum_{v=0}^{n} e^{i(v-k)t} - a_{\tau+1,\tau+1-k}\sum_{v=0}^{\tau} e^{i(v-k)t}\right]\right| \\ &= O(t^{-1})\sum_{k=0}^{\tau}\left(\sum_{r=\tau+1}^{n-1} b_{r,r-k} + a_{n,n-k} + a_{\tau+1,\tau+1-k}\right) \\ &= O(t^{-1})\sum_{k=0}^{\tau}\left(\sum_{r=\tau+1}^{n-1}(a_{r,r-k} - a_{r+1,r+1-k})\right. \\ &\quad \left. + a_{n,n-k} + a_{\tau+1,\tau+1-k}\right) \\ &= O(t^{-1})\sum_{k=0}^{\tau}(2a_{\tau+1,\tau+1-k} + a_{n,n-k} + a_{n+1,n+1-k}) \\ &= O(t^{-1})\sum_{k=0}^{\tau}(a_{\tau,\tau-k} + a_{n,n-k}) \\ &= O(t^{-1})\left(\sum_{k=0}^{\tau} a_{\tau,\tau-k} + \sum_{k=0}^{n} a_{n,n-k}\right) \\ &= O(t^{-1})(A_{\tau,0} + A_{n,0}) = O(t^{-1}), \end{aligned}$$

in view of $b_{r,r-k} = a_{r,r-k} - a_{r+1,r+1-k} \geq 0$ for $0 \leq k \leq r$.

Again using Abel's transformation in K_3, we have

$$\begin{aligned} K_3 &= \left|\sum_{r=\tau+1}^{n}\left[\sum_{k=\tau+1}^{r-1}\left\{\Delta_k a_{r,r-k}\sum_{v=0}^{k} e^{i(r-v)t}\right\}\right.\right. \\ &\quad \left.\left. + a_{r,0}\sum_{v=0}^{r} e^{i(r-v)t} - a_{r,r-\tau-1}\sum_{v=0}^{\tau} e^{i(r-v)t}\right]\right| \\ &= O(t^{-1})\sum_{r=\tau+1}^{n}\left[\sum_{k=\tau+1}^{r-1}\left|a_{r,r-k} - a_{r,r-k-1}\right|\right. \\ &\quad \left. + a_{r,0} + a_{r,r-\tau-1}\right] \\ &= O(t^{-1})\sum_{r=\tau+1}^{n}(a_{r,r-\tau-1}) \\ &= O(t^{-1})\left[a_{\tau+1,0} + a_{\tau+2,1} + a_{\tau+3,2} + ... + a_{n,n-\tau-1}\right] \\ &= O(t^{-1})\left[a_{\tau+1,0} + a_{\tau+1,1} + a_{\tau+1,2} + ... + a_{\tau+1,n-\tau-1}\right] \\ &= O(t^{-1})\sum_{r=0}^{n-\tau-1} a_{\tau+1,r} = O(t^{-1})(1) = O(t^{-1}), \end{aligned}$$

in view of $b_{r,r-k} = a_{r,r-k} - a_{r+1,r+1-k} \geq 0$ for $0 \leq k \leq r$ and $a_{n,k} \leq a_{n,k+1}$.
Collecting K_1, K_2 and K_3, we get

$$|(C^1.T)_n(t)| = O\left(t^{-2}/(n+1)\right).$$

Proof of Theorem 1. The integral representation of $\tilde{s}_n(f;x)$ defined in (2) is given by

$$\tilde{s}_n(f;x) = -\frac{1}{\pi}\int_0^{\pi}\psi_x(t)\left\{\frac{\cos(t/2) - \cos(n+1/2)t}{2\sin(t/2)}\right\}dt$$

and,

$$\tilde{s}_n(f;x) - \tilde{f}(x) = \frac{1}{2\pi}\int_0^{\pi}\psi_x(t)\frac{\cos(n+1/2)t}{\sin(t/2)}dt.$$

Therefore,

$$\begin{aligned} &\tilde{t}_n^{C^1.T}(f;x) - \tilde{f}(x) \\ &= \frac{1}{n+1}\sum_{r=0}^{n}\sum_{k=0}^{r} a_{r,k}\left[\tilde{s}_k(f;x) - \tilde{f}(x)\right] \\ &= \int_0^{\pi}\psi_x(t)(2\pi(n+1))^{-1} \times \\ &\quad \sum_{r=0}^{n}\sum_{k=0}^{r} a_{r,r-k}\frac{\cos(r-k+1/2)t}{\sin(t/2)}dt \\ &= \int_0^{\pi/(n+1)}\psi_x(t)(C^1.T)_n(t)dt \\ &\quad + \int_{\pi/(n+1)}^{\pi}\psi_x(t)(C^1.T)_n(t)dt \\ &= I_1 + I_2, \text{ say.} \end{aligned} \tag{11}$$

Using Hölder's inequality, conditions (7), (8), $\sin(t/2) \geq t/\pi$, Lemma 1, the mean value theorem for integrals and $p^{-1} + q^{-1} = 1$, we have

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$$
\begin{aligned}
|I_1| &= \lim_{\epsilon\to 0}\int_{\epsilon}^{\pi/(n+1)} \left|\left[t^{-\sigma}(\psi_x(t)\sin^{\beta}(t/2)/\xi(t)) \times \right.\right.\\
&\quad \left.\left.(\xi(t)(C^1.T)_n(t)/t^{-\sigma}\sin^{\beta}(t/2))\right]\right| dt\\
&\le \left[\int_0^{\pi/(n+1)} \left(t^{-\sigma}|\psi_x(t)|\sin^{\beta}(t/2)/\xi(t)\right)^p dt\right]^{1/p} \times\\
&\left[\lim_{\epsilon\to 0}\int_{\epsilon}^{\pi/(n+1)} \left(\xi(t)|(C^1.T)_n(t)|/t^{-\sigma}\sin^{\beta}(t/2)\right)^q dt\right]^{1/q}\\
&= O((n+1)^{\sigma-1/p}) \times\\
&\quad \left[\lim_{\epsilon\to 0}\int_{\epsilon}^{\pi/(n+1)} \left|\xi(t)/(t^{1-\sigma}.\sin^{\beta}(t/2))\right|^q dt\right]^{1/q}\\
&= O((n+1)^{\sigma-1/p}) \times\\
&\quad \left[\lim_{\epsilon\to 0}\int_{\epsilon}^{\pi/(n+1)} \left|\xi(t)/(t^{\beta+1-\sigma})\right|^q dt\right]^{1/q}\\
&= O((n+1)^{\sigma-1/p})(n+1)^{\beta+1-1/q-\sigma}\xi(\pi/(n+1))\\
&= O((n+1)^{\beta}\ \xi(\pi/(n+1))). \qquad (12)
\end{aligned}
$$

Again using Lemma 2, Hölder's inequality and $(\sin(t/2))^{-1} \le \pi/t$ for $0 < t \le \pi$, we have

$$
\begin{aligned}
|I_2| &= \left[\int_{\pi/(n+1)}^{\pi} |\psi_x(t)| \left[O\left(t^{-2}/(n+1)\right)\right] dt\right]\\
&= O\left[\int_{\pi/(n+1)}^{\pi} t^{-2}|\psi_x(t)|/(n+1)dt\right]\\
&= O\left(\int_{\pi/(n+1)}^{\pi} \frac{t^{-\delta}|\psi_x(t)|\sin^{\beta}(t/2)}{(n+1)\xi(t)}\frac{t^{-1}\xi(t)}{t^{-\delta}t\sin^{\beta}(t/2)}dt\right)\\
&= O\left\{\frac{1}{n+1}\int_{\pi/(n+1)}^{\pi}\left(\frac{t^{-\delta}|\psi_x(t)|}{\xi(t)}\right)^p dt\right\}^{1/p}\\
&\quad \times\left\{\int_{\pi/(n+1)}^{\pi}\left(\frac{t^{-1}\xi(t)}{t^{-\delta+\beta+1}}\right)^q dt\right\}^{1/q}\\
&= O\left[(n+1)^{\delta-1-1/p}\xi\left(\frac{\pi}{n+1}\right)\left(\frac{n+1}{\pi}\right)\right.\\
&\quad \left.\left(\int_{\pi/(n+1)}^{\pi} t^{-(\beta+1-\delta)q}dt\right)^{1/q}\right]\\
&= O\left[(n+1)^{\delta-1/p}\xi(\pi/(n+1))(n+1)^{\beta+1-\delta-1/q}\right]\\
&= O[(n+1)^{\beta}\xi(\pi/(n+1))], \qquad (13)
\end{aligned}
$$

in view of (9), (10), the mean value theorem for integrals, $1/p < \delta < \beta + 1/p$ and $p^{-1} + q^{-1} = 1$.
Collecting (11)-(13), we get

$$
|\ \tilde{t}_n^{C^1.T}(f;x) - \tilde{f}(x)\ | = O[(n+1)^{\beta}\xi(\pi/(n+1))]. \qquad (14)
$$

Fianally from (14), we easily get

$$
\|\ \tilde{t}_n^{C^1.T}(f;x) - \tilde{f}(x)\ \|_p = O\left((n+1)^{\beta}\xi(1/(n+1))\right), \qquad (15)
$$

in view of Note 1. This completes the proof of Theorem 1.

As mentioned in Remark 2 of [2], the above proof is not valid for $p = 1$. Therefore, for $p = 1$, we have the following theorem:

Theorem 2. *Let $T \equiv (a_{n,k})$ be the same as in Theorem 1. Then the degree of approximation of $\tilde{f}$, conjugate of a 2π-periodic function f belonging to the weighted Lipschitz class $W(L^1, \xi(t))$, with $0 \le \beta < 1$ by $C^1.T$ means of its conjugate Fourier series is given by*

$$
\|\ \tilde{t}_n^{C^1.T}(f;x) - \tilde{f}(x)\ \|_1 = O\left((n+1)^{\beta}\xi(1/(n+1))\right), \qquad (16)
$$

provided a positive increasing function $\xi(t)$ satisfies conditions (7) to (10) of Theorem 1 for $p = 1$, $\beta < \sigma < 1$ and $1 < \delta < \beta + 1$.

Proof of Theorem 2. Following the proof of Theorem 1, for $p = 1, i.e., q = \infty$, we have

$$
\begin{aligned}
I_1 &= \int_0^{\pi/(n+1)}\left(\frac{t^{-\sigma}|\psi_x(t)|\sin^{\beta}(t/2)}{\xi(t)}\right)dt \times\\
&\quad ess \sup_{0<t\le\pi/(n+1)}\left|\frac{\xi(t)|(C^1.T)_n(t)|}{t^{-\sigma}\sin^{\beta}(t/2)}\right|\\
&= \int_0^{\pi/(n+1)}\left(\frac{t^{-\sigma}|\psi_x(t)|\sin^{\beta}(t/2)}{\xi(t)}\right)dt \times\\
&\quad ess \sup_{0<t\le\pi/(n+1)}\left|\frac{\xi(t)}{t^{-\sigma+1}\sin^{\beta}(t/2)}\right|\\
&= O((n+1)^{\sigma-1})\ ess \sup_{0<t\le\pi/(n+1)}\left|\frac{\xi(t)}{t^{\beta-\sigma+1}}\right|\\
&= O((n+1)^{\sigma-1})\left\{\frac{\xi(\pi/(n+1))}{(\pi/(n+1))^{\beta-\sigma+1}}\right\}\\
&= O((n+1)^{\beta}\ \xi(\pi/(n+1))). \qquad (17)
\end{aligned}
$$

in view of conditions (7) and (8) for $p = 1$.

$$
\begin{aligned}
I_2 &= O\left\{\frac{1}{n+1}\int_{\pi/(n+1)}^{\pi}\frac{t^{-\delta}|\psi_x(t)|\sin^{\beta}(t/2)}{\xi(t)}dt\right\} \times\\
&\quad ess \sup_{\pi/(n+1)\le t\le\pi}\left|\frac{\xi(t)}{t^{-\delta+\beta+2}}\right|\\
&= O\left[(n+1)^{\delta-2}\xi\left(\frac{\pi}{n+1}\right)\left(\frac{(n+1)^{2+\beta-\delta}}{\pi^{2+\beta-\delta}}\right)\right]\\
&= O[(n+1)^{\beta}\xi(\pi/(n+1))], \qquad (18)
\end{aligned}
$$

in view of (9), i.e., decreasing nature of $\xi(t)/t^{-\delta+\beta+2}$ and (10). Collecting (17) and (18), we get

$$
|\ \tilde{t}_n^{C^1.T}(f;x) - \tilde{f}(x)\ | = O[(n+1)^{\beta}\xi(\pi/(n+1))]. \qquad (19)
$$

Finally from (19), we easily get

$$
\|\ \tilde{t}_n^{C^1.T}(f;x) - \tilde{f}(x)\ \|_1 = O\left((n+1)^{\beta}\xi(1/(n+1))\right), \qquad (20)
$$

in view of Note 1. This completes the proof of Theorem 2.

III. Corollaries

The following corollaries can be derived from our theorems:

1. If $\beta = 0$, then for $f \in Lip(\xi(t), p)$ with $p \ge 1$,

$$
\|\tilde{t}_n^{C^1.T}(f;x) - \tilde{f}(x)\|_p = O\left(\xi(1/(n+1))\right).
$$

2. If $\beta = 0$, $\xi(t) = t^{\alpha}(0 < \alpha \le 1)$, then for $f \in Lip(\alpha, p)(0 < \alpha \le 1)$,

$$
\|\tilde{t}_n^{C^1.T}(f;x) - \tilde{f}(x)\|_p = O\left((n+1)^{-\alpha}\right).
$$

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3. If $p \to \infty$ in Corollary 2, then for $f \in Lip\alpha(0 < \alpha < 1)$,

$$\|\tilde{t}_n^{C^1.T}(f;x) - \tilde{f}(x)\|_\infty = O((n+1)^{-\alpha}).$$

For $\alpha = 1$, we can write an independent proof to obtain

$$\|\tilde{t}_n^{C^1.T}(f;x) - \tilde{f}(x)\|_\infty = O(log(n+1)/(n+1)).$$

If we replace matrix T by Nörlund matrix (N_p), i.e., $a_{n,k} = p_{n-k}/P_n$ for $0 \le k \le n$ and $a_{n,k} = 0$ for $k > n$, where $P_n = \sum_{k=0}^{n} p_k \to \infty$ as $n \to \infty$, then we get $C^1.N_p$ analogues of our theorems and corollaries of this paper.

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