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The Convergence Iterative Scheme for Quasi-variational Problems and Fixed Point Problems

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***Abstract*—In this paper, we introduce an iterative scheme for finding a common element of the set solutions of quasi-variational inclusion problems, fixed point problems, and generalized equilibrium problems in Hilbert spaces. Under suitable conditions, some strong convergence theorem for a sequence of nonexpansive mappings be proved. The results presented in this paper improve and extend the corresponding results announced by many others.**

***Index Terms*—Fixed point, quasi-variational inclusion, generalized equilibrium problems, minimization problems**

I. Introduction

THIS paper we always assume that H is a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Let be a nonlinear mapping and let F be a bifunction of $C\times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The generalized equilibrium problem is to find a point $x\in C$ such that

$$F(x,y)+\langle B(x),y-x\rangle\geq 0,\forall y\in C. \quad (1.1)$$

The set of solutions of (1.1) is denoted by GEP (see in [3]). If $B=0,$ then (1.1) reduces to the equilibrium problem: to find $x\in C$ such that

$$F(x,y)\geq 0,\forall y\in C. \quad (1.2)$$

Let $A:H\to H$ be a single-valued nonlinear mapping and $M:H\to 2^{H}$ be a set-valued mapping. The quasi-variational inclusion problem (see in [9]), is to find $x\in H$ such that

$$f\in A(x)+M(x). \quad (1.3)$$

The set of solutions of (1.3) is denoted by $VI(H,A,M).$ A special case of the problem (1.3) is to find an element $x\in H$ such that

$$\theta\in A(x)+M(x), \quad (1.4)$$

where θ is the zero vector in $H.$ If $M=\partial_{\delta_C}$ and $\delta_C:H\to[0,+\infty)$ is the indicator function of $C,$ that is

$$\delta_C(x)=\begin{cases}0, & x\in C\\ -\infty, & x\notin C.\end{cases} \quad (1.5)$$

Then the quasi-variational inclusion problem (1.4) is equivalent the classical variational inequality problem, denoted by $VI(C,A),$ to find $x\in H$ such that

$$\langle A(x),v-x\rangle\geq 0,\forall v\in C. \quad (1.6)$$

It is known that (1.4) provides a convenient framework for the unified study of optimal solutions in many optimization related areas including optimal control, equilibria and variational inequalities (see [1] and the references therein).

Let $S:H\to H$ be a nonlinear mapping. The mapping S is said to be contractive with coefficient $k\in(0,1)$ if

$$\|Sx-Sy\|\leq\alpha\|x-y\|,\forall x,y\in H. \quad (1.7)$$

The mapping S is said to be nonexpansive if

$$\|Sx-Sy\|\leq\|x-y\|,\forall x,y\in H. \quad (1.8)$$

The fixed point set of S is denoted by F(S). For finding a common element of the set of fixed points of a nonexpansive mapping and of the set solutions to variational inequality (1.6), Iiduka and Takahashi [6], introduced the following iterative scheme. Starting with $x_1=x\in C$ and define a sequence $\{x_n\}$ by

$$x_{n+1}=\alpha_n x+(1-\alpha_n)SP_C(x_n-\lambda_n Ax_n), \quad (1.9)$$

for all $n\in N,$ where $\{\alpha_n\}$ be a sequence in [0,1) and $\{\lambda_n\}$ be a sequence in [a, b]. They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{\lambda_n\}$, the sequence $\{x_n\}$ converges strongly to $P_{F(S)\cap VI(C,A)}x.$

Recently, Zhang et al. [14] introduced an iterative method for nonexpansive mapping and equilibrium problem (1.2) in a Hilbert space H:

$$x_t=SP_C\left((1-t)J_{M,\lambda}(I-\lambda A)T_{\mu}(I-\mu B)\right)x_t,t\in(0,1). \quad (1.10)$$

Under suitable conditions, they proved that the sequence $\{x_n\}$ generated by (1.10) converges strongly to the fixed point which is the unique solution of the quadratic minimization problem:

$$\|x^*\|^2=\min_{x\in F(S)\cap VI(H,A,M)\cap GEP}\|x\|^2.$$

Motivated and inspired by Iiduka and Takahashi [6], Zhang et al. [14], Zhang et al. [13], Khongtham and Plubtieng [8], Plubtieng and Punpaeng [10], Noor and Noor [9], and Tan [7], we introduce an iterative scheme for finding a common element quasi-variational inclusion problems, fixed point problems, and generalized equilibrium problems in Hilbert spaces. We will present in the section III.

II. Preliminaries

Let C be a nonempty closed convex subset of $H.$ It is well known that

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$$\|\gamma x+(1-\gamma)y\|^2=\gamma\|x\|^2+(1-\gamma)\|y\|^2-\gamma(1-\gamma)\|x-y\|^2, \quad (2.1)$$

for all $x,y\in H$ and $\gamma\in[0,1]$. For any $x\in H$, there exists a unique nearest point in C, denote by $P_C x$ such that $\|x-P_C x\|\le\|x-y\|$ for all $y\in C$. Such a mapping P_C is called the metric projection from H into C. We know that P_C is nonexpansive mapping, $P_C x\in C$ and

$$\langle x-P_C x, P_C x-y\rangle\ge 0, \forall x\in H, y\in C. \quad (2.2)$$

Recalled that a mapping $A:H\to H$ is called $\alpha-$ inverse strongly monotone (see [6],[4]), if there exists a positive α such that

$$\langle Ax-Ay, x-y\rangle\ge\alpha\|Ax-Ay\|^2, \forall x,y\in H. \quad (2.3)$$

It is well known that A is an $(1/\alpha)-$ Lipschitz continuous and monotone mapping. Moreover, $I-\lambda A$ is a nonexpansive mapping, if $0<\lambda\le 2\alpha$ and I is the identity mapping on H (see [13]).

Recalled that a set-valued mapping $M:H\to 2^H$ is called monotone if for all $x,y\in H, f\in Mx$, and $g\in My$ imply $\langle x-y, f-g\rangle\ge 0$. A monotone mapping $M:H\to 2^H$ is maximal if and only if for $(x,f)\in H\times H$, $\langle x-y, f-g\rangle\ge 0$, for every $(y,g)\in G(M)$ implies $f\in Mx$. The single- valued mapping $J_{M,\lambda}:H\to H$ defined by

$$J_{M,\lambda}(x)=(I+\lambda M)^{-1}(x), \forall x\in H \quad (2.4)$$

is called the resolvent operator associated with M, where λ is any positive number and I is the identity mapping. We know that the resolvent operator $J_{M,\lambda}$ associated with M, is a nonexpansive for all $\lambda>0$, that is,

$$\|J_{M,\lambda}(x)-J_{M,\lambda}(y)\|\le\|x-y\|, \forall x,y\in H, \forall\lambda>0, \quad (2.5)$$

(see [14]).

In addition, the resolvent operator $J_{M,\lambda}$ is $1-$inverse strongly monotone, that is, for all $x,y\in H$,

$$\|J_{M,\lambda}(x)-J_{M,\lambda}(y)\|^2\le\langle x-y, J_{M,\lambda}(x)-J_{M,\lambda}(y)\rangle, \quad (2.6)$$

(see [14]).

The following lemmas are useful in our proof.

Lemma 2.1 (see [11]). Let sequence $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X. Let $\{\beta_n\}$ be a sequence in [0, 1] with $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$. Suppose that $x_{n+1}=(1-\beta_n)y_n+\beta_n x_n, \forall n\ge 0$, and $\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\le 0$. Then, $\lim_{n\to\infty}\|y_n-x_n\|=0$.

Lemma 2.2 (see [2]). Let C be a nonempty closed subset of a Banach space and let $\{S_n\}$ be a sequence of mappings of C into itself. Suppose that

$$\sum_{n=1}^{\infty}\sup\{\|S_{n+1}z-S_n z\|: z\in C\}<\infty.$$

Then, for each $x\in C$, $\{S_n y\}$ converges strongly to some point of C. Let S be a mapping from C into itself defined by $Sy=\lim_{n\to\infty}S_n y, \forall y\in C$.
Then, $\lim_{n\to\infty}\sup\{\|S_n z-Sz\|: z\in C\}=0$.

We assume that the bifunction $F:C\times C\to H$ satisfies the following conditions:
(A1) $F(x,x)=0$ for all $x\in C$,
(A2) F is monotone, that is, $F(x,y)+F(y,x)\le 0, \forall x,y\in C$;
(A3) for each $x,y,z\in C$,

$$\lim_{t\to 0}F(tz+(1-t)x,y)\le F(x,y);$$

(A4) for each $x\in C, y\mapsto F(x,y)$ is convex and lower semi-continuous.

Lemma 2.3 (see [5]). Let H be a real Hilbert space, C be a nonempty closed convex subset of H, and $F:C\times C\to H$ be a bifunction satisfying the conditions (A1) –(A4). Let $\tau>0$ and $x\in H$. Then, there exists a point $z\in C$ such that

$$F(z,y)+(1/\tau)\langle y-z, z-x\rangle\ge 0, \forall y\in C.$$

Define a mapping $T_\tau:H\to C$ by

$$T_\tau=\{z\in C: F(z,y)+(1/\tau)\langle y-z, z-x\rangle\ge 0, \forall y\in C\}, \quad (2.7)$$

for all $z\in H$. Then the following hold:
(i) T_τ is single-valued and firmly nonexpansive, that is, for any $x,y\in H$,

$$\|T_\tau x-T_\tau y\|^2\le\langle T_\tau x-T_\tau y, x-y\rangle; \quad (2.8)$$

(ii) EP(F) is closed and convex and $EP(F)=F(T_\tau)$.

Lemma 2.4 (i) (see [13]) $u\in H$ is a solution of variational inclusion (1.4) if and only if

$$u=J_{M,\lambda}(u-\lambda Au), \forall\lambda>0. \quad (2.9)$$

that is,

$$VI(H,A,M)=F(J_{M,\lambda}(u-\lambda Au)), \forall\lambda>0. \quad (2.10)$$

(ii) (see [14]) $u\in C$ is a solution of generalized equilibrium problem (1.6) if and only if

$$u=T_\tau(u-\tau Bu), \forall\tau>0, \quad (2.11)$$

that is,

$$GEP=F(T_\tau(I-\tau B)), \forall\tau>0. \quad (2.12)$$

(iii) (see [14]) Let $A:H\to H$ is an $\alpha-$inverse strongly monotone mapping and $B:C\to H$ is a $\delta-$ inverse strongly monotone mapping. If $\lambda\in(0,2\alpha]$ and $\tau\in(0,2\delta]$, then VI(H, A, M) is a closed convex subset in H and GEP is a closed convex subset in C.

Lemma 2.5 (see [12]). Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1}\le(1-\alpha_n)a_n+\delta_n, n\ge 0,$$

where $\{\alpha_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that:
(i) $\sum_{n=1}^{\infty}\alpha_n=\infty$;
(ii) $\limsup_{n\to\infty}(\delta_n/\alpha_n)\le 0$ or $\sum_{n=1}^{\infty}|\delta_n|<\infty$.
Then $\lim_{n\to\infty}a_n=0$.

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III. Main Result

In this section, we prove the strong convergence theorem for solving a common element of the set solutions of quasi-variational inclusion problems, fixed point problems, and generalized equilibrium problems in a real Hilbert spaces.

Theorem 3.1 Let H be a real Hilbert space, let F be a bifunction from $C\times C$ into H satisfying the conditions (A1)-(A4) and let $\{S_n\}$ is a sequence of nonexpansive mappings on C. Let $A:H\to H$ is an $\alpha-$ inverse strongly monotone mapping and $B:C\to H$ is a $\delta-$ inverse strongly monotone mapping. Let $M:H\to 2^H$ is maximal monotone mapping such that

$$\Omega:=\bigcap_{n=1}^{\infty}F(S_n)\bigcap VI(H,A,M)\bigcap GEP\neq\varnothing.$$

Let f be a contraction of H into itself with a constant $k\in(0,1)$. Let $x_1\in H$ and

$$\begin{aligned} u_n &= T_\tau\left(I-\tau B\right)x_n,\\ y_n &= J_{M,\lambda}\left(I-\lambda A\right)u_n, \forall n\geq 0, \qquad (3.1)\\ x_{n+1} &= \alpha_n f(x_n)+\beta_n x_n+\gamma_n S_n P_C\left(\left(1-t_n\right)y_n\right), \end{aligned}$$

for all $n\in N$, the mapping $T_\tau:H\to C$ is defined as (2.7) in Lemma 2.3, $\lambda\in(0,2\alpha]$, $\tau\in(0,2\delta]$, and $\{t_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are four sequences in [0,1) satisfy

(i) $\alpha_n+\beta_n+\gamma_n=1$;

(ii) $\lim_{n\to\infty}\alpha_n=0, \sum_{n=1}^{\infty}\alpha_n=\infty$;

(iii) $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$;

(iv) $\lim_{n\to\infty}t_n=0, \sum_{n=1}^{\infty}t_n=\infty$.

Suppose that $\sum_{n=1}^{\infty}\sup\left\{\|S_{n+1}z-S_nz\|:z\in C\right\}<\infty$ for any bounded subset C of H. Let S is a mapping from C into itself defined by $Sx=\lim_{n\to\infty}S_nx, \forall x\in C$ and suppose that $F(S)=\bigcap_{n=1}^{\infty}F(S_n)$. Then, $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ converge strongly to $q\in\Omega$, which is the unique solution of the quadratic minimization problem:

$$\|q\|^2=\min_{x\in\Omega}\|x\|^2. \qquad (3.2)$$

Proof. Put $Q=P_{F(S)\cap VI(H,A,M)\cap GEP}$. It easy to see that Qf is a contraction. By Banach contraction principle, there exists $z_0\in F(S)\bigcap VI(H,A,M)\bigcap GEP$ such that

$$z_0=Qf(z_0)=P_{F(S)\cap VI(H,A,M)\cap GEP}f(z_0).$$

Otherwise, we see that $I-\lambda A$, $I-\tau B$, T_τ, and $J_{M,\lambda}$ are nonexpansive. First, we will show that $\{x_n\}$ is bounded. Put $p\in\Omega$. We observed that

$$\begin{aligned}\|u_n-p\|^2 &= \|T_\tau(I-\tau B)x_n-T_\tau(I-\tau B)p\|^2\\ &\leq\|x_n-p\|^2+\tau(\tau-2\beta)\|Bx_n-Bp\|^2 \qquad (3.3)\end{aligned}$$

and

$$\begin{aligned}\|y_n-p\|^2 &= \|J_{M,\lambda}(I-\lambda A)u_n-J_{M,\lambda}(I-\lambda A)p\|^2\\ &\leq\|x_n-p\|^2+\lambda(\lambda-2\alpha)\|Au_n-Ap\|^2\\ &\quad+\tau(\tau-2\beta)\|Bx_n-Bp\|^2. \qquad (3.4)\end{aligned}$$

Using (3.3) and (3.4), we have that

$$\|y_n-p\|\leq\|u_n-p\|\leq\|x_n-p\|. \qquad (3.5)$$

From (3.1) and (3.5), we calculated that

$$\begin{aligned}\|x_{n+1}-p\| &= \left\|\alpha_n f(x_n)+\beta_n x_n+\gamma_n S_n P_C\left(\left(1-t_n\right)y_n\right)-p\right\|\\ &\leq\alpha_n\|f(x_n)-p\|+\beta_n\|x_n-p\|+\gamma_n\left(\left(1-t_n\right)\|x_n-p\|+t_n\|p\|\right)\\ &\leq\alpha_n\|f(p)-p\|+\left(1-\alpha_n(1-k)\right)\|x_n-p\|+\gamma_n t_n\|p\|. \qquad (3.6)\end{aligned}$$

Using (3.6) and by inductions, we get that

$$\|x_n-p\|\leq\max\left\{\|x_1-p\|,\left(1/\left(1-k\right)\right)\|f(p)-p\|,\|p\|\right\}, \forall n\geq 1.$$

This implies that $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{u_n\}$, $\{Au_n\}$, $\{Bx_n\}$, $\{f(x_n)\}$, and $\left\{S_nP_C\left(\left(1-t_n\right)y_n\right)\right\}$. Put $v_n=P_C\left(\left(1-t_n\right)y_n\right)$ and $z_n=S_nv_n$. Next, we show that $\|x_{n+1}-x_n\|\to 0$, as $n\to\infty$. Let $x_{n+1}=\left(1-\beta_n\right)e_n+\beta_nx_n$. We note that $e_n=\left[(x_{n+1}-\beta_nx_n)/(1-\beta_n)\right]$. Then, we have that

$$\begin{aligned}\|e_{n+1}-e_n\|-\|x_{n+1}-x_n\| &\leq\tfrac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})\|+\tfrac{\alpha_n}{1-\beta_n}\|f(x_n)\|\\ &\quad+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\left[\left(1-t_{n+1}\right)\|x_{n+1}-x_n\|+\|t_{n+1}-t_n\|\|e_n\|\right]\\ &\quad+\frac{\gamma_n}{1-\beta_n}\left[\sup\left\{\|S_{n+1}z-S_nz\|:z\in C\right\}\right]-\|x_{n+1}-x_n\|. \qquad (3.7)\end{aligned}$$

From (3.7) and the conditions (i)-(iv), we have that $\limsup_{n\to\infty}\left(\|e_{n+1}-e_n\|-\|x_{n+1}-x_n\|\right)\leq 0$. By Lemma 2.1 and Lemma 2.2, we have $\lim_{n\to\infty}\|e_n-x_n\|=0$. Consequently,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}\left(1-\beta_n\right)\|e_n-x_n\|=0$$

and so are $\lim_{n\to\infty}\|v_{n+1}-v_n\|=0$, $\lim_{n\to\infty}\|u_{n+1}-u_n\|=0$, and $\lim_{n\to\infty}\|y_{n+1}-y_n\|=0$. Since

$$x_{n+1}-x_n=\alpha_nf(x_n)+\beta_nx_n+\gamma_nS_nv_n-x_n, \qquad (3.8)$$

it follows by (ii) and $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, that $\lim_{n\to\infty}\|Sv_n-x_n\|=0$. And we also get that $\lim_{n\to\infty}\|x_n-u_n\|=0$, $\lim_{n\to\infty}\|u_n-y_n\|=0$, and $\lim_{n\to\infty}\|x_n-y_n\|=0$. Moreover, we have that $\lim_{n\to\infty}\|x_{n+1}-Sx_{n+1}\|=0$. By the same argument as in the proof Theorem 3.1(pp. 13-14) of [7], we conclude that $\lim_{n\to\infty}\|x_n-q\|=0$, where $q\in\Omega$. Finally, we show that $\lim_{n\to\infty}\|x_n-q\|=0$, where q is the unique solution of the quadratic minimization problem (3.2). For any $r\in\Omega$ and $\lim_{n\to\infty}\|Sv_n-x_n\|=0$, we get that

$$\begin{aligned}\limsup_{n\to\infty}\left\langle f(r)-r,x_n-r\right\rangle &= \limsup_{n\to\infty}\left\langle f(r)-r,Sv_n-r\right\rangle\\ &= \lim_{i\to\infty}\left\langle f(r)-r,Sv_{n_i}-r\right\rangle\\ &= \left\langle f(r)-r,q-r\right\rangle\leq 0. \qquad (3.9)\end{aligned}$$

And

$$\begin{aligned}\|z_n-r\|^2 &= \|S_nv_n-r\|^2\\ &= \left\|S_nP_C\left(\left(1-t_n\right)y_n\right)-S_nP_Cr\right\|^2\end{aligned}$$

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$$\leq \|y_n - r\|^2 - 2t_n \langle y_n, y_n - r\rangle + t_n{}^2 \|y_n\|^2$$
$$\leq (1-2t_n)\|x_n - r\|^2 + 2t_n \langle r, r - y_n\rangle + t_n{}^2 \|y_n\|^2.$$

This implies that

$$\|x_{n+1} - r\|^2 = \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n v_n - r\|^2$$
$$\leq \|\beta_n (x_n - r) + \gamma_n (S_n v_n - r)\|^2$$
$$+2\alpha_n k \|x_n - r\|\|x_{n+1} - r\|\|x_{n+1} - r\|$$
$$+2\alpha_n \langle f(r) - r, x_{n+1} - r\rangle$$
$$\leq \beta_n \|x_n - r\|^2 + \gamma_n ((1-2t_n)\|x_n - r\|^2$$
$$+2t_n \langle r, r - y_n\rangle + t_n{}^2 \|y_n\|^2)$$
$$+2\alpha_n k \|x_n - r\|\|x_{n+1} - r\|$$
$$+2\alpha_n \langle f(r) - r, x_{n+1} - r\rangle. \quad (3.10)$$

Put $r = q$ in (3.10). Then using (3.9), (3.10), and Lemma 2.5 we have that $\lim_{n\to\infty} \|x_n - q\| = 0$, where q is the unique solution of the quadratic minimization problem: $\|q\|^2 = \min_{x\in\Omega} \|x\|^2$. This complete the proof.

In Theorem 3.1, if $S = S_n, \forall n \geq 1$, then, we have the following corollary.

Corollary 3.2 Let H be a real Hilbert space, let F be a bifunction from $C\times C$ into H satisfying the conditions (A1)-(A4) and let S be a nonexpansive mapping on H. Let $A: H \to H$ be an $\alpha-$ inverse strongly monotone mapping and $B: C \to H$ be a $\delta-$ inverse strongly monotone mapping. Let $M: H \to 2^H$ be a maximal monotone mapping such that $\Omega_1 := F(S)\bigcap VI(H, A, M)\bigcap GEP \neq \emptyset$. Let f be a contraction of H into itself with a constant $k \in (0,1)$. Let $x_1 \in H$ and

$$u_n = T_\tau (I - \tau B) x_n,$$
$$y_n = J_{M,\lambda} (I - \lambda A) u_n, \forall n \geq 0, \quad (3.11)$$
$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n SP_C ((1-t_n) y_n),$$

for all $n \in N$, the mapping $T_\tau : H \to C$ is defined as (2.7) in Lemma 2.3, $\lambda \in (0, 2\alpha]$, $\tau \in (0, 2\delta]$, and $\{t_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are four sequences in [0,1) satisfy

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iv) $\lim_{n\to\infty} t_n = 0, \sum_{n=1}^{\infty} t_n = \infty$.

Then, $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ converge strongly to $q \in \Omega_1$, which is the unique solution of the quadratic minimization problem: $\|q\|^2 = \min_{x\in\Omega_1} \|x\|^2$.

In Theorem 3.1, if $S = S_n, \forall n \geq 1$, $\beta_n = 0, \forall n \geq 1$, $f(x_n) = x_n, \forall n \geq 1$, then we have the following corollary.

Corollary 3.3 Let H be a real Hilbert space, let F be a bifunction from $C\times C$ into H satisfying the conditions (A1)-(A4) and let S be a nonexpansive mapping on H. Let $A: H \to H$ be an $\alpha-$ inverse strongly monotone mapping and $B: C \to H$ be a $\delta-$ inverse strongly monotone mapping. Let $M: H \to 2^H$ be a maximal monotone mapping such that $\Omega_2 := F(S)\bigcap VI(H, A, M)\bigcap GEP \neq \emptyset$. Let $x_1 \in H$ and

$$u_n = T_\tau (I - \tau B) x_n,$$
$$y_n = J_{M,\lambda} (I - \lambda A) u_n, \forall n \geq 0, \quad (3.12)$$
$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) SP_C ((1-t_n) y_n),$$

for all $n \in N$, the mapping $T_\tau : H \to C$ is defined as (2.7) in Lemma 2.3, $\lambda \in (0, 2\alpha]$, $\tau \in (0, 2\delta]$, and $\{\alpha_n\}$ and $\{t_n\}$ are sequences in [0,1) satisfy

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\lim_{n\to\infty} t_n = 0, \sum_{n=1}^{\infty} t_n = \infty$.

Then, $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ converge strongly to $q \in \Omega_2$, which is the unique solution of the quadratic minimization problem: $\|q\|^2 = \min_{x\in\Omega_2} \|x\|^2$.

IV. CONCLUSION

The convergence theorems shown that the iterative sequence converges to the unique solution of the quadratic minimization problem: $\|q\|^2 = \min_{x\in F(S)\cap VI(H,A,M)\cap GEP} \|x\|^2$.

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