The Convergence Iterative Scheme for Quasi-variational Problems and Fixed Point Problems

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Abstract—In this paper, we introduce an iterative scheme for finding a common element of the set solutions of quasi-variational inclusion problems, fixed point problems, and generalized equilibrium problems in Hilbert spaces. Under suitable conditions, some strong convergence theorem for a sequence of nonexpansive mappings be proved. The results presented in this paper improve and extend the corresponding results announced by many others.

Index Terms—Fixed point, quasi-variational inclusion, generalized equilibrium problems, minimization problems

I. INTRODUCTION

THIS paper we always assume that $H$ is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Let be a nonlinear mapping and let $F$ be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The generalized equilibrium problem is to find a point $x \in C$ such that

$$ F(x, y) + \langle B(x), y - x \rangle \geq 0, \forall y \in C. \quad (1.1) $$

The set of solutions of (1.1) is denoted by $\text{GEP}(F, B, C)$. If $B = 0$, then (1.1) reduces to the equilibrium problem: to find $x \in C$ such that

$$ F(x, y) \geq 0, \forall y \in C. \quad (1.2) $$

Let $A : H \to H$ be a single-valued nonlinear mapping and $M : H \to 2^H$ be a set-valued mapping. The quasi-variational inclusion problem (see in [3]), is to find $x \in H$ such that

$$ f \in A(x) + M(x). \quad (1.3) $$

The set of solutions of (1.3) is denoted by $\text{VI}(H, A, M)$. A special case of the problem (1.3) is to find an element $x \in H$ such that

$$ \theta \in A(x) + M(x), \quad (1.4) $$

where $\theta$ is the zero vector in $H$. If $M = \delta_C$ and $\delta_C : H \to [0, +\infty)$ is the indicator function of $C$, that is

$$ \delta_C(x) = \begin{cases} 0, & x \in C \\ +\infty, & x \not\in C \end{cases}. \quad (1.5) $$

Then the quasi-variational inclusion problem (1.4) is equivalent the classical variational inequality problem, denoted by $\text{VI}(C, A)$, to find $x \in H$ such that

$$ \langle A(x), v - x \rangle \geq 0, \forall v \in C. \quad (1.6) $$

It is known that (1.4) provides a convenient framework for the unified study of optimal solutions in many optimization related areas including optimal control, equilibria and variational inequalities (see [1] and the references therein).

Let $S : H \to H$ be a nonlinear mapping. The mapping $S$ is said to be contractive with coefficient $k \in (0, 1)$ if

$$ \|sx - sy\| \leq k\|x - y\|, \forall x, y \in H. \quad (1.7) $$

The mapping $S$ is said to be nonexpansive if

$$ \|sx - sy\| \leq \|x - y\|, \forall x, y \in H. \quad (1.8) $$

The fixed point set of $S$ is denoted by $\text{F}(S)$. For finding a common element of the set of fixed points of a nonexpansive mapping and of the set solutions to variational inequality (1.6), Iiduka and Takahashi [6], introduced the following iterative scheme. Starting with $x_1 = x \in C$ and define a sequence $\{x_n\}$ by

$$ x_{n+1} = \alpha_n x + (1 - \alpha_n)SP(x_n - \lambda_n Ax_n), \quad (1.9) $$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ be a sequence in $[0, 1]$ and $\{\lambda_n\}$ be a sequence in $[a, b]$. They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{\lambda_n\}$, the sequence $\{x_n\}$ converges strongly to $\text{P}_{\text{F}(S) \cap \text{VI}(C, A)} \cdot x$.

Recently, Zhang et al. [14] introduced an iterative method for nonexpansive mapping and equilibrium problem (1.2) in a Hilbert space $H$:

$$ x_t = SP \left( (1-t)J_{M, \lambda} \left( 1-\lambda A \right) T_t \left( 1-\mu B \right) \right) x_1, t \in (0, 1). \quad (1.10) $$

Under suitable conditions, they proved that the sequence $\{x_n\}$ generated by (1.10) converges strongly to the fixed point which is the unique solution of the quadratic minimization problem:

$$ \|x^*\|^2 = \min_{x \in F(S) \cap \text{VI}(H, A, M) \cap \text{GEP}(F, B, C)} \|x\|^2. $$

Motivated and inspired by Iiduka and Takahashi [6], Zhang et al. [14], Zhang et al. [13], Khongtham and Plubtieng [8], Plubtieng and Punpaeng [10], Noor and Noor [9], and Tan [7], we introduce an iterative scheme for finding a common element quasi-variational inclusion problems, fixed point problems, and generalized equilibrium problems in Hilbert spaces. We will present in the section III.

II. PRELIMINARIES

Let $C$ be a nonempty closed convex subset of $H$. It is well known that
\[ \|y + (1 - \gamma)z\| = \|y\| + (1 - \gamma)\|z - y\|, \tag{2.1} \]
for all \( x, y \in H \) and \( \gamma \in \{0, 1\} \). For any \( x \in H \), there exists a unique nearest point in \( C \), denoted by \( P_C x \) such that \( \|x - P_C x\| = \|x - y\| \) for all \( y \in C \). Such a mapping \( P_C \) is called the metric projection from \( H \) into \( C \). We know that \( P_C \) is nonexpansive mapping, \( P_C x \in C \) and
\[ \langle x - P_C x, P_C x - y \rangle \geq 0, \forall x \in H, y \in C. \tag{2.2} \]

Recalled that a mapping \( A: H \rightarrow H \) is called \( \alpha \)-inverse strongly monotone (see [6],[4]), if there exists a positive \( \alpha \) such that
\[ \langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \forall x, y \in H. \tag{2.3} \]

It is well known that \( A \) is an \( (1/\alpha) \)-Lipschitz continuous and monotone mapping. Moreover, \( I - \lambda A \) is a nonexpansive mapping, if \( 0 < \lambda \leq 2 \alpha \) and \( I \) is the identity mapping on \( H \) (see [13]).

Recalled that a set-valued mapping \( M: H \rightarrow 2^H \) is called monotone if for all \( x, y \in H, f \in Mx, \) and \( g \in My \) imply
\[ \langle x - y, f - g \rangle \geq 0. \]
A monotone mapping \( M: H \rightarrow 2^H \) is maximal if and only if for \( (x, f) \in H \times H, \langle x - y, f - g \rangle \geq 0, \)
for every \( (y, g) \in G(M) \) implies \( f \in Mx \). The single-valued mapping \( J_{M,\lambda}: H \rightarrow H \) defined by
\[ J_{M,\lambda}(x) = (I + \lambda M)^{-1}(x), \forall x \in H \tag{2.4} \]
is called the resolvent operator associated with \( M \), where \( \lambda \) is any positive number and 1 is the identity mapping. We know that the resolvent operator \( J_{M,\lambda} \) associated with \( M \), is a nonexpansive for all \( \lambda > 0 \), that is,
\[ \|J_{M,\lambda}(x) - J_{M,\lambda}(y)\| \leq \|x - y\|, \forall x, y \in H, \forall \lambda > 0, \tag{2.5} \]
(see [14]).
In addition, the resolvent operator \( J_{M,\lambda} \) is 1-inversion strongly monotone, that is, for all \( x, y \in H, \)
\[ \|J_{M,\lambda}(x) - J_{M,\lambda}(y)\|^2 \leq \langle x - y, J_{M,\lambda}(x) - J_{M,\lambda}(y) \rangle. \tag{2.6} \]
(see [14]).

The following lemmas are useful in our proof.

Lemma 2.1 (see [11]). Let sequence \( \{x_n\} \) and \( \{y_n\} \) be bounded sequences in a Banach space \( X \). Let \( \{\beta_n\} \) be a sequence in \( [0, 1] \) with \( 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1. \) Suppose that \( x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n, \forall n \geq 0, \) and \( \limsup_{n \to \infty} \left\| y_{n+1} - y_n \right\| \leq \left\| x_{n+1} - x_n \right\|. \) Then,
\[ \lim_{n \to \infty} \left\| y_{n+1} - x_n \right\| = 0. \]

Lemma 2.2 (see [2]). Let \( C \) be a nonempty closed subset of a Banach space and let \( \{S_n\} \) be a sequence of mappings of \( C \) into itself. Suppose that
\[ \sum_{n=1}^\infty \sup_{z \in C} \left\| S_{n+1}z - S_nz \right\| < \infty, \]
Then, for each \( x \in C, \) \( \{S_ny\} \) converges strongly to some point of \( C \). Let \( S \) be a mapping from \( C \) into itself defined by \( S = \lim_{n \to \infty} S_n \), \( \forall y \in C. \)
Then, \( \lim_{n \to \infty} \sup_{z \in C} \left\| S_nz - S_nz \right\| = 0. \)

We assume that the bifunction \( F: C \times C \rightarrow H \) satisfies the following conditions:
(A1) \( F(x, x) = 0 \) for all \( x \in C \),
(A2) \( F \) is monotone, that is, \( F(x, y) + F(y, x) \leq 0, \forall x, y \in C; \)
(A3) for each \( x, y, z \in C, \)
\[ \lim_{\tau \to 0^+} F(x + (1 - \tau)y, x) \leq F(x, y); \]
(A4) for each \( x, y \in C, y \mapsto F(x, y) \) is convex and lower semi-continuous.

Lemma 2.3 (see [5]). Let \( H \) be a real Hilbert space, \( C \) be a nonempty closed convex subset of \( H \), and \( F: C \times C \rightarrow H \) be a bifunction satisfying the conditions (A1) – (A4). Let \( \tau > 0 \) and \( x \in H \). Then, there exists a point \( z \in C \) such that
\[ F(z, y) + \left(1/\tau\right)\|y - z, x - x\| \leq 0, \forall y \in C. \tag{2.7} \]

Define a mapping \( T_\tau: H \rightarrow C \) by
\[ T_\tau = \left\{ z \in C : F(z, y) + \left(1/\tau\right)\|y - z, x - x\| \leq 0, \forall y \in C \right\}. \tag{2.7} \]
(i) \( T_\tau \) is single-valued and firmly nonexpansive, that is, for any \( x, y \in H, \)
\[ \|T_\tau x - T_\tau y\|^2 \leq \langle T_\tau x - T_\tau y, x - y \rangle; \tag{2.8} \]
(ii) \( EP(F) \) is closed and convex and \( EP(F) = F(T_\tau). \)

Lemma 2.4 (i) (see [13]) \( u \in H \) is a solution of variational inclusion (1.4) if and only if
\[ u = J_{M,\lambda}(u - \lambda Au), \forall \lambda > 0. \tag{2.9} \]
that is,
\[ VI(H, A, M) = F(J_{M,\lambda}(u - \lambda Au)), \forall \lambda > 0. \tag{2.9} \]

(ii) (see [14]) \( u \in C \) is a solution of generalized equilibrium problem (1.6) if and only if
\[ u = T_\tau(u - \tau Bu), \forall \tau > 0, \tag{2.11} \]
that is,
\[ GEP = F(T_\tau(1 - \tau B)), \forall \tau > 0. \tag{2.12} \]
(iii) (see [14]) Let \( A: H \rightarrow H \) is an \( \alpha \)-inverse strongly monotone mapping and \( B: C \rightarrow H \) is a \( \delta \)-inverse strongly monotone mapping. If \( \lambda \in (0, 2\alpha] \) and \( \tau \in (0, 2\delta] \), then
\[ VI(H, A, M) \] is a closed convex subset in \( H \) and \( GEP \) is a closed convex subset in \( C \).

Lemma 2.5 (see [12]). Assume \( \{\alpha_n\} \) is a sequence of nonnegative real numbers such that
\[ \alpha_n = (1 - \alpha_n)\alpha_n + \delta_n, n \geq 0, \]
where \( \{\alpha_n\} \) is a sequence in \( (0, 1) \) and \( \{\delta_n\} \) is a sequence in \( \mathbb{R} \) such that:
(i) \( \sum_{n=1}^\infty \alpha_n = \infty; \)
(ii) \( \limsup_{n \to \infty} \left(\delta_n / \alpha_n\right) \leq 0 \) or \( \sum_{n=1}^\infty \delta_n < \infty. \)
Then \( \lim_{n \to \infty} \alpha_n = 0. \)
Let $\Omega := \bigcap_{n=1}^{\infty} F(S_n) \cap VI(H,A,M) \cap GEP \neq \emptyset$.

Let $f$ be a contraction of $H$ into itself with a constant $k \in (0,1)$. Let $x_1 \in H$ and
\begin{equation}
\begin{aligned}
& u_n = T_n (1 - \tau B) x_n, \\
& y_n = J_{M,\lambda} (1- \lambda A) u_n, \forall n \geq 0,
\end{aligned}
\end{equation}
for all $n \in \mathbb{N}$, the mapping $T_n : H \to C$ is defined as (2.7) in Lemma 2.3. Let $\lambda \in (0,2\alpha]$, $\tau \in (0,2\delta)$, and $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are four sequences in $(0,1)$ satisfying
\begin{enumerate}[(i)]
\item $\alpha_n + \beta_n + \gamma_n = 1$;
\item $\lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;
\item $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$;
\item $\lim_{n \to \infty} \gamma_n = 0, \sum_{n=1}^{\infty} \gamma_n = \infty$.
\end{enumerate}

Suppose that $\sum_{n=1}^{\infty} \sup \{ \|S_n z - S_n z\| : z \in C \} < \infty$ for any bounded subset $C$ of $H$. Let $S$ be a mapping from $C$ into itself defined by $S x = \lim_{n \to \infty} S_n x, \forall x \in C$ and suppose that $F(S) = \bigcap_{n=1}^{\infty} F(S_n)$. Then, $\{x_n\}, \{y_n\}$, and $\{u_n\}$ converge strongly to $x \in \Omega$, which is the unique solution of the quadratic minimization problem:
\begin{equation}
\|x\|^2 = \min_{x \in \Omega} \|x\|^2. \tag{3.2}
\end{equation}

Proof. Put $Q = P_{\bigcap_{n=1}^{\infty} F(S_n) \cap VI(H,A,M) \cap GEP}$. It is easy to see that $Qf$ is a contraction. By Banach contraction principle, there exists $z_0 \in F(S) \cap VI(H,A,M) \cap GEP$ such that $z_0 = Qf(z_0) = P_{\bigcap_{n=1}^{\infty} F(S_n) \cap VI(H,A,M) \cap GEP}(z_0)$. Otherwise, we see that $1 - \lambda A, 1 - \tau B, T_n$, and $J_{M,\lambda}$ are nonexpansive. First, we will show that $\{x_n\}$ is bounded. Put $p \in \Omega$. We observed that
\begin{equation}
\begin{aligned}
\|u_n - p\|^2 & = \|T_n (1 - \tau B) x_n - T_n (1 - \tau B) p\|^2 \\
& \leq \|x_n - p\|^2 + \tau (\tau - 2\beta) \|B x_n - B p\|^2, \tag{3.3}
\end{aligned}
\end{equation}
and
\begin{equation}
\begin{aligned}
\|y_n - p\|^2 & = \|J_{M,\lambda} (1 - \lambda A) u_n - J_{M,\lambda} (1 - \lambda A) p\|^2 \\
& \leq \|x_n - p\|^2 + \lambda (\lambda - 2\alpha) \|A u_n - A p\|^2 \\
& \quad + \tau (\tau - 2\beta) \|B x_n - B p\|^2. \tag{3.4}
\end{aligned}
\end{equation}

Using (3.3) and (3.4), we have that
\begin{equation}
\begin{aligned}
\big| \|u_n - p\|^2 - \|y_n - p\|^2 \big| & \leq \|x_n - p\|^2.
\end{aligned}
\end{equation}
From (3.1) and (3.5), we calculated that
\begin{equation}
\begin{aligned}
\|x_{n+1} - p\|^2 & = \|J_n f(x_n) + \beta_n x_n + \gamma_n S_n P_C \left((1 - t_n) y_n - p\right) - p\| \\
& \leq \alpha_n \|f(x_n) - p\| + \beta_n \|x_n - p\| + \gamma_n \|\left((1 - t_n) y_n - p\right) + t_n \|p\|\| \\
& \leq \alpha_n \|f(p) - p\| + \beta_n \|x_n - p\| + \gamma_n \|\left((1 - t_n) y_n - p\right) + t_n \|p\|. \tag{3.6}
\end{aligned}
\end{equation}

Using (3.6) and by inductions, we get that
\begin{equation}
\|x_n - p\|^2 \leq \max \{\|x_1 - p\|^2, \frac{1}{(1 - k)}(f(p) - p), \|p\|\}, \forall n \geq 1.
\end{equation}

This implies that $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{u_n\}$, $\{A u_n\}$, $\{B x_n\}$, $\{f(x_n)\}$, and $\{S_n P_C ((1 - t_n) y_n)\}$.

Put $v_n = P_C \left((1 - t_n) y_n\right)$ and $z_n = S_n v_n$. Next, we show that $x_{n+1} = (1 - \beta_n) e_n + \beta_n x_n$. We note that $e_n = \left((x_{n+1} - \beta_n x_n)/(1 - \beta_n)\right)$. Then, we have that
\begin{equation}
\begin{aligned}
\|e_n - e_n\| & \leq \|x_{n+1} - x_n\| \leq \alpha_n \|f(x_n)\| + \gamma_n \|\left((1 - t_n) y_n - p\right) + t_n \|p\| \\
& \quad + \frac{\gamma_n}{1 - \beta_n} \left(\|x_n - p\|^2 - \|y_n - p\|^2\right) \quad \text{(3.7)}
\end{aligned}
\end{equation}

From (3.7) and the conditions (i)-(iv), we have that $\limsup_{n \to \infty} \|e_n - e_n\| - \|x_{n+1} - x_n\| \leq 0$. By Lemma 2.1 and Lemma 2.2, we have $\lim_{n \to \infty} (e_n - x_n) = 0$. Consequently,
\begin{equation}
\begin{aligned}
\lim_{n \to \infty} (x_{n+1} - x_n) = \lim_{n \to \infty} (1 - \beta_n) (e_n - x_n) = 0,
\end{aligned}
\end{equation}
and so are $\lim_{n \to \infty} (y_{n+1} - y_n) = 0, \lim_{n \to \infty} (u_{n+1} - u_n) = 0$, and $\lim_{n \to \infty} (y_n - y_n) = 0$. Since
\begin{equation}
\begin{aligned}
x_{n+1} - x_n = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n v_n - x_n,
\end{aligned}
\end{equation}

it follows by (ii) and $\lim_{n \to \infty} (x_{n+1} - x_n) = 0$, that $\lim_{n \to \infty} (S_n v_n - x_n) = 0$ and $\lim_{n \to \infty} (y_n - y_n) = 0$. And we also get that $\lim_{n \to \infty} (u_n - u_n) = 0, \lim_{n \to \infty} (y_n - y_n) = 0, and \lim_{n \to \infty} (x_n - y_n) = 0$. Moreover, we have that $\lim_{n \to \infty} (S_n v_n - x_n) = 0$. By the same argument as in the proof Theorem 3.1 (pp. 13-14) of [7], we conclude that $\lim_{n \to \infty} (x_n - q) = 0$, where $q \in \Omega$. Finally, we show that $\lim_{n \to \infty} (x_n - q) = 0$, where $q$ is the unique solution of the quadratic minimization problem (3.2). For any $r \in \Omega$ and $\limsup_{n \to \infty} (f(r) - r, x_n - r) = \lim_{n \to \infty} (f(r) - r, S_n v_n - r)$
\begin{equation}
\begin{aligned}
= \lim_{n \to \infty} (f(r) - r, S_n v_n - r) = \langle f(r) - r, q - r\rangle \leq 0.
\end{aligned}
\end{equation}

And
\begin{equation}
\begin{aligned}
\|z_n - r\|^2 = \|S_n v_n - r\|^2 = \|S_n P_C ((1 - t_n) y_n) - S_n P_C r\|^2
\end{aligned}
\end{equation}
\[
\begin{align*}
\leq &\|y_n - r\|^2 - 2t_n \langle y_n, y_n - r \rangle + t_n^2 \|y_n\|^2 \\
\leq &\left(1 - 2t_n\right)\|x_n - r\|^2 + 2t_n \langle r, r - y_n \rangle + t_n^2 \|y_n\|^2.
\end{align*}
\]

This implies that
\[
\|x_{n+1} - r\|^2 = \|\alpha f(x_n) + \beta_n x_n + \gamma_n S_n y_n - r\|^2 \\
\leq &\|\alpha f(x_n) + \beta_n x_n + \gamma_n y_n - S_n y_n - r\|^2 \\
&\leq 2\alpha_n \|x_n - r\|^2 + 2\alpha_n \|r - y_n\|^2 + 2\alpha_n \langle r, x_n - r \rangle + 2\alpha_n \langle r, x_n - r \rangle \\
&\leq \beta_n \|x_n - r\|^2 + \gamma_n \left(1 - 2t_n\right)\|x_n - r\|^2 \\
&\leq 2t_n \langle r, r - y_n \rangle + t_n^2 \|y_n\|^2 \\
&\leq 2\alpha_n \|x_n - r\|^2 + 2\alpha_n \|x_n - r\|^2 + 2\alpha_n \langle r, x_n - r \rangle.
\]

(3.10)

Put \(r = q\) in (3.10). Then using (3.9), (3.10), and Lemma 2.5 we have that \(\lim_{n \to \infty} \|x_n - q\| = 0\), where \(q\) is the unique solution of the quadratic minimization problem:
\[
\|q\|^2 = \min_{x \in \Omega} \|x\|^2.
\]

This complete the proof.

In Theorem 3.1, if \(S = S_n, \forall n \geq 1\), then we have the following corollary.

**Corollary 3.2** Let \(H\) be a real Hilbert space, let \(F\) be a bifunction from \(C \times C\) into \(H\) satisfying the conditions (A1)-(A4) and let \(S\) be a nonexpansive mapping on \(H\). Let \(A: H \to H\) be an \(\alpha\)-inverse strongly monotone mapping and \(B: C \to H\) be a \(\delta\)-inverse strongly monotone mapping. Let \(M: H \to 2^H\) be a maximal monotone mapping such that \(\Omega := F(S) \cap \bigcap \text{VI}(H, A, M) \cap \text{GEP} \neq \emptyset\). Let \(x\) be a contraction of \(H\) into itself with a constant \(k \in (0,1)\). Let \(x_1 \in H\) and
\[
u_n = T_r \left(1 - \tau B\right)x_n,
\]
\[
y_n = J_{M, \lambda} \left(1 - \|\lambda A\|\right)\nu_n, \forall n \geq 0,
\]
\[
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n y_n - r,
\]
for all \(n \in \mathbb{N}\), the mapping \(T_r : H \to C\) is defined as (2.7) in Lemma 2.3, \(\lambda \in (0, 2\alpha]\), \(\tau \in (0, 28]\), and \(\{t_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}\) are sequences in \([0,1)\) satisfy

(i) \(\alpha_n + \beta_n + \gamma_n = 1\);

(ii) \(\lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty; \)

(iii) \(0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1; \)

(iv) \(\lim_{n \to \infty} t_n = 0, \sum_{n=1}^{\infty} t_n = \infty. \)

Then, \(\{x_n\}, \{y_n\}\), and \(\{\nu_n\}\) converge strongly to \(q \in \Omega\), which is the unique solution of the quadratic minimization problem:
\[
\|q\|^2 = \min_{x \in \Omega} \|x\|^2.
\]

In Theorem 3.1, if \(S = S_n, \forall n \geq 1, \beta_n = 0, \forall n \geq 1, f(x_n) = x_n, \forall n \geq 1\), then we have the following corollary.

**Corollary 3.3** Let \(H\) be a real Hilbert space, let \(F\) be a bifunction from \(C \times C\) into \(H\) satisfying the conditions (A1)-(A4) and let \(S\) be a nonexpansive mapping on \(H\). Let \(A: H \to H\) be an \(\alpha\)-inverse strongly monotone mapping and \(B: C \to H\) be a \(\delta\)-inverse strongly monotone mapping. Let \(M: H \to 2^H\) be a maximal monotone mapping such that \(\Omega := F(S) \cap \bigcap \text{VI}(H, A, M) \cap \text{GEP} \neq \emptyset\). Let \(x \in H\) and
\[
u_n = T_r \left(1 - \tau B\right)x_n,
\]
\[
y_n = J_{M, \lambda} \left(1 - \|\lambda A\|\right)\nu_n, \forall n \geq 0,
\]
\[
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n y_n - r,
\]
for all \(n \in \mathbb{N}\), the mapping \(T_r : H \to C\) is defined as (2.7) in Lemma 2.3, \(\lambda \in (0, 2\alpha]\), \(\tau \in (0, 28]\), and \(\{t_n\}, \{\alpha_n\}, \{\beta_n\}\) are sequences in \([0,1)\) satisfy

(i) \(\lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty; \)

(ii) \(\lim_{n \to \infty} t_n = 0, \sum_{n=1}^{\infty} t_n = \infty. \)

Then, \(\{x_n\}, \{y_n\}\), and \(\{\nu_n\}\) converge strongly to \(q \in \Omega\), which is the unique solution of the quadratic minimization problem:
\[
\|q\|^2 = \min_{x \in \Omega} \|x\|^2.
\]

IV. CONCLUSION

The convergence theorems shown that the iterative sequence converges to the unique solution of the quadratic minimization problem:
\[
\|q\|^2 = \min_{x \in F(S) \cap \text{VI}(H, A, M) \cap \text{GEP}} \|x\|^2.
\]

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REFERENCES


