L_1-induced Performance Analysis and Sparse Controller Synthesis for Interval Positive Systems

Xiaoming Chen, James Lam, Ping Li, and Zhan Shu

***Abstract*—This paper is concerned with the design of L_1-induced sparse controller for continuous-time positive systems with interval uncertainties. A necessary and sufficient condition for stability and L_1-induced performance of positive linear systems is proposed in terms of linear inequalities. Based on this, conditions for the existence of robust state-feedback controllers are established. Moreover, the total number of all the nonzero elements of the controller gain is to be minimized, while satisfying a guaranteed level of L_1-induced performance. Then, we propose an ℓ_1-minimization problem to relax the ℓ_0 objective function for optimization and an iterative convex optimization approach is developed to solve the conditions. Finally, an illustrative example is provided to show the effectiveness and applicability of the theoretical results.**

***Index Terms*—Interval uncertainties, Linear Lyapunov functions, L_1-induced performance, Positive systems, Sparse control.**

I. Introduction

In many practical systems, there is such a kind of systems whose state variables are confined to be positive. Such systems are frequently encountered in various fields, for instance, ecology [1], industrial engineering [2]. These systems belong to the class of positive systems, whose state variables and outputs take only nonnegative values for nonnegative inputs and initial conditions. Positive systems possess many special characteristics, mainly due to the fact that the states of positive systems are confined within a cone located in the positive orthant rather than in the whole space. The special characteristics brings about many new issues, which cannot be solved in general by using well-established methods for general linear systems. Therefore, the study on positive system theory has drawn the attention of many researchers all over the world in recent years.

After a system-theoretic approach to positive systems was proposed in [3], a large number of theoretical contributions have appeared in the literature [4], [5], [6], [7], [8]. Among these research results, a positive state-space representation of a given transfer function has been characterized in [9]. Necessary and sufficient conditions for positive realizability by means of convex analysis have been derived in [10]. Reachability and controllability for positive systems have been investigated thoroughly in [11] and [12]. The synthesis problem of state-feedback controllers guaranteeing the closed-loop system to be positive and asymptotically stable has been investigated by the LMI approach and the linear programming approach in [13] and [14], respectively. Stability theory for nonnegative and compartmental dynamic systems with time delay has been investigated in [15], [16], [17]. Some results on 2-D positive systems can be found in [7], [18]. As for the results on model reduction problem for positive systems, we refer readers to [19], [20].

Although numerous results have been developed, it is noted that most of the above mentioned research works are based on quadratic Lyapunov functions. The results are often formulated under the linear matrix inequality (LMI) framework [21]. In recent years, some new results based on linear Lyapunov functions have emerged [14], [22], [23], [24]. The motivation is that the state of positive systems is nonnegative, making a linear Lyapunov function a valid candidate. However, the above-mentioned works mainly focused on the stability analysis problem and less efforts have been made in designing the controllers for positive systems. In addition, some frequently used costs such as H_∞ norm are based on the L_2 signal space [25] and these costs are not very natural to describe some of the features of practical physical systems. By contrast, 1 norm provides a more useful description for positive systems, for instance, if the values represent the amount of material or the number of animal in a species. On the other hand, finding sparse vectors is important in many applications such as in parameter estimation or identification, signal processing or model reduction [26], [27]. A vector or signal is said to be sparse, if most of its entries are zero. The ℓ_0-norm quantifies sparsity by counting the number of nonzero entries in a vector or signal. However, finding sparse vectors is difficult because minimizing the ℓ_0-norm is a nonconvex problem. In compressive sensing, sparse signals are reconstructed by replacing the ℓ_0-minimization with an ℓ_1-minimization [27], [28]. However, there are few results on sparse controller synthesis for positive systems, especially with linear Lyapunov functions. Moreover, it is worth noting that the system parameters are usually assumed to be exactly known in the literature [29], [30]. Actually, practical systems are often affected by environmental changes, variations, perturbations or disturbances, and consequently it is inevitable that uncertainties enter the system parameters [31]. Owing to the complexity caused by parameter uncertainties, the synthesis problems for uncertain positive systems have not been fully investigated. Motivated by the aforementioned discussions, in this paper, we consider the sparse state-feedback stabilization problem for continuous-time interval positive systems under L_1-induced performance.

The remaining parts of this article are organized as follows. In Section II, preliminaries are presented and the L_1-induced performance is introduced for continuous-time positive systems. In Section III, the exact value of L_1-induced norm is computed and a characterization is developed under which the positive linear system is asymptotically stable and satisfies the performance. In Section IV, the sparse controller

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design problem is formulated by minimizing the ℓ_0-norm of the controller gain K and the ℓ_0-optimization problem is relaxed by a ℓ_1-minimization problem. Moreover, the resulting problem can be tackled by finding a solution of iterative convex optimization problems. An examples is provided in Section V to show the effectiveness and applicability of the theoretical results. Conclusions are given in Section VI.

II. Problem Formulation

In this section, we introduce notations and several results concerning continuous-time linear positive systems.

Let $\mathbb{R}$ be the set of real numbers; $\mathbb{R}^n$ denotes the n-column real vectors; $\mathbb{R}^{n\times m}$ is the set of all real matrices of dimension $n\times m$. Let $\bar{\mathbb{R}}^n_+$ denote the nonnegative orthants of $\mathbb{R}^n$; that is, if $x\in\mathbb{R}^n$, then $x\in\bar{\mathbb{R}}^n_+$ is equivalent to $x\geq\geq 0$. $\mathbb{N}$ is the set of natural numbers. For a matrix $A\in\mathbb{R}^{m\times n}$, a_{ij} denotes the element located at the ith row and the jth column. $A\geq\geq 0$ (respectively, $A>>0$) means that for all i and j, $a_{ij}\geq 0$ (respectively, $a_{ij}>0$). The notation $A\geq\geq B$ (respectively, $A>>B$) means that the matrix $A-B\geq\geq 0$ (respectively, $A-B>>0$). The matrix $A\in\mathbb{R}^{n\times n}$ is called Metzler, if all its off-diagonal elements are positive, that is, $\forall(i,j),\ i\neq j,\ a_{ij}\geq 0$. For matrices $A,\underline{A},\overline{A}\in\mathbb{R}^{n\times m}$, the notation $A\in[\underline{A},\overline{A}]$ means that $\underline{A}\leq\leq A\leq\leq\overline{A}$. The symbol $\mathrm{col}_i(A)$ denotes the i_{th} column of matrix A. The superscript "T" denotes matrix transpose. $\|\cdot\|$ represents the Euclidean norm for vectors. The 1-norm of a vector $x(t)=(x_1(t),x_2(t),\ldots,x_n(t))$ is defined as $\|x(t)\|_1\triangleq\sum_{i=1}^{n}|x_i(t)|$ and the induced 1-norm of a matrix $Q\triangleq[q_{ij}]\in\mathbb{R}^{m\times n}$ is denoted by $\|Q\|_1\triangleq\max_{1\leq j\leq n}(\sum_{i=1}^{m}|q_{ij}|)$. The L_1-norm of x is defined as $\|x\|_{L_1}\triangleq\int_0^\infty\|x(t)\|_1dt$. The all-ones vector in $\mathbb{R}^n$ is denoted by $\mathbf{1}_n$. Given a matrix $A=[a_1,\ldots,a_n]$ with a_i being its ith column, we define a vector $\mathrm{vec}(A)\triangleq[a_1^T\ldots a_n^T]^T$. Matrices, if their dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

Consider an interval system:

$$\left\{\begin{array}{rcl}\dot{x}(t)&=&Ax(t)+B_ww(t),\\ y(t)&=&Cx(t)+D_ww(t),\end{array}\right.\tag{1}$$

where $x(t)\in\mathbb{R}^n, w(t)\in\mathbb{R}^m$ and $y(t)\in\mathbb{R}^r$ are the system state, input and output, respectively.

In this paper, the system matrices A, B_w, C and D_w are not precisely known, but belong to the following admissible uncertainty domain:

$$A\in[\underline{A},\overline{A}],\ B_w\in[\underline{B}_w,\overline{B}_w],\ C\in[\underline{C},\overline{C}],\ D_w\in[\underline{D}_w,\overline{D}_w].\tag{2}$$

We have the following definitions throughout the paper.

Definition 1: For a vector $x=[x_1,\ldots,x_n]^T\in\mathbb{R}^n$, it is called sparse if its ℓ_0-norm is small compared to the dimension of the vector, where ℓ_0-norm of x is defined as

$$\|x\|_0=\sum_{i=1}^{n}|\mathrm{sign}(x_i)|.$$

Definition 2: System (1) is said to be a continuous-time positive linear system if for all $x(0)\geq\geq 0$ and $w(t)\geq\geq 0$, we have $x(t)\geq\geq 0$ and $y(t)\geq\geq 0$ for $t>0$.

Definition 3: System (1) is said to be positive and robustly stable if it is positive and asymptotically stable over all admissible uncertainty domain in (2).

In the following, we introduce some useful results which will be used in the sequel.

Lemma 1 ([32]): The system in (1) is a continuous-time positive linear system if and only if

$$A \text{ is Metzler},\ B_w\geq\geq 0,\ C\geq\geq 0,\ D_w\geq\geq 0.$$

Proposition 1 ([22]): The positive linear system given by (1) is asymptotically stable if and only if there exists a vector $p\geq\geq 0$ (or $p>>0$) satisfying

$$p^TA<<0.\tag{3}$$

Lemma 2 ([27]): The convex envelope of the function $f=\|x\|_0=\sum_{i=1}^{n}|\mathrm{sign}(x_i)|$ on $X=\{x\in\mathbb{R}^n|\|x\|_\infty\leq 1\}$ is $f_{env}(x)=\|x\|_1=\sum_{i=1}^{n}|x_i(t)|$.

Now, we are in a position to give the definition of L_1-induced norm. For a stable positive linear system given in (1), its L_1-induced norm is defined as

$$\|\Im\|_{(L_1,L_1)}\triangleq\sup_{w\neq 0,\ w\in L_1}\frac{\|y\|_{L_1}}{\|w\|_{L_1}},\tag{4}$$

where $\Im:L_1\to L_1$ denotes the convolution operator, that is, $y(t)=(\Im*w)(t)$. We say that system (1) has L_1-induced performance at the level γ if, under zero initial conditions,

$$\|\Im\|_{(L_1,L_1)}<\gamma,\tag{5}$$

where $\gamma>0$ is a given scalar.

III. Performance Analysis

In this section, we compute the exact value of L_1-induced norm for positive system (1). Then, the performance characterization is provided for positive system (1) over the whole uncertain domain in (2).

First, we give the following theorem through which the value of L_1-induced norm of system (1) can be computed directly.

Theorem 1: For a stable positive linear system given in (1), the exact value of the L_1-induced norm is given by

$$\|\Im\|_{(L_1,L_1)}=\|D_w-CA^{-1}B_w\|_1.\tag{6}$$

Proof: For system (1), the impulse response $G(t)$ is given by

$$G(t)\triangleq\left\{\begin{array}{l}0,\ t<0,\\ Ce^{At}B_w+D_w\delta(t),\ t\geq 0.\end{array}\right.\tag{7}$$

Next, let $\Im:L_1\to L_1$ denote the convolution operator:

$$z(t)=(\Im*w)(t)\triangleq\int_0^\infty G(t-\tau)w(\tau)d\tau.\tag{8}$$

From [33], we have

$$\|\Im\|_{(L_1,L_1)}=\max_{j=1,\ldots,m}\|\mathrm{col}_j(\bar{G})\|_1,\tag{9}$$

where

$$\bar{G}_{ij}=\int_0^\infty G_{ij}(t)dt.\tag{10}$$

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Let c_i, b_{wj}, d_{wij} denote the ith row vector, the jth column vector and the (i,j)-element of matrices C, B_w and D_w, respectively. Equation (10) can be written as

$$\bar{G}_{ij} = \int_0^\infty \left(c_i e^{At} b_{wj} + d_{wij}\delta(t)\right) dt = -c_i A^{-1} b_{wj} + d_{wij}, \quad (11)$$

yielding (6). □

Next, we derive the following result which provides a fundamental characterization on the stability of system (1) with the performance in (5).

Theorem 2: The positive linear system in (1) is asymptotically stable and satisfies $\|y\|_{L_1} < \gamma\|w\|_{L_1}$ if and only if there exists a vector $p \geq\geq 0$ satisfying

$$\mathbf{1}_r^T C + p^T A \quad << \quad 0, \quad (12)$$
$$p^T B_w + \mathbf{1}_r^T D_w - \gamma \mathbf{1}_m^T \quad << \quad 0. \quad (13)$$

Proof: (*Sufficiency*) In the following, we consider two cases: $x(t) \equiv 0$ and there exists a t such that $x(t) \neq 0$. First, for $x(t) \equiv 0$, from (12), the asymptotic stability of system (1) is proved. It is easy to see that if $x(t) \equiv 0$, we have $y(t) = D_w w(t)$ and from (13), $\|y\|_{L_1} < \gamma\|w\|_{L_1}$ holds.

Next, we assume that there exists a t such that $x(t) \neq 0$. From (12), we can see that (3) holds, and thus the asymptotic stability of system (1) is proved.

Consider the linear Lyapunov function candidate $V(x(t)) = p^T x(t)$ and we have

$$\frac{dV(x(t))}{dt} = p^T(Ax(t) + B_w w(t)).$$

Let

$$\begin{aligned} J &= \int_0^T \|y(t)\|_1 dt - \int_0^T \gamma\|w(t)\|_1 dt \\ &= \int_0^T \left(\sum_{i=1}^r y_i(t) - \gamma\sum_{i=1}^m w_i(t)\right) dt \\ &= \int_0^T \left(\sum_{i=1}^r y_i(t) - \gamma\sum_{i=1}^m w_i(t) + \frac{dV(t)}{dt}\right) dt - V(T) \\ &= \int_0^T \left(\left[\mathbf{1}_r^T C + p^T A\right] x(t) + (\mathbf{1}_r^T D_w + p^T B_w \right. \\ &\quad \left. -\gamma \mathbf{1}_m^T) w(t)\right) dt - V(T) \\ &= \int_0^T \left(\left[\mathbf{1}_r^T C + p^T A + \varepsilon \mathbf{1}_n^T\right] x(t) + (\mathbf{1}_r^T D_w \right. \\ &\quad \left. + p^T B_w - \gamma \mathbf{1}_m^T) w(t)\right) dt - \int_0^T \varepsilon \mathbf{1}_n^T x(t) dt - V(T), \end{aligned}$$

where $\varepsilon > 0$ is sufficiently small such that $\mathbf{1}_r^T C + p^T A + \varepsilon \mathbf{1}_n^T << 0$ holds.

From (12) and (13), we have

$$J + \int_0^T \varepsilon \mathbf{1}_n^T x(t) dt + V(T) < 0,$$

which equals to

$$\int_0^T \left(\sum_{i=1}^r y_i(t)\right) dt + \varepsilon \int_0^T \left(\sum_{i=1}^n x_i(t)\right) dt < \gamma \int_0^T \left(\sum_{i=1}^m w_i(t)\right) dt - p^T x(T).$$

Since the system is asymptotically stable, when $T \to \infty$ we have

$$\int_0^\infty \left(\sum_{i=1}^r y_i(t)\right) dt + \varepsilon \int_0^\infty \left(\sum_{i=1}^n x_i(t)\right) dt \leq \gamma \int_0^\infty \left(\sum_{i=1}^m w_i(t)\right) dt,$$

which implies

$$\|y\|_{L_1} < \gamma\|w\|_{L_1}. \quad (14)$$

This proves sufficiency.

(*Necessity*) Assume that system (1) is asymptotically stable and satisfies $\|y\|_{L_1} < \gamma\|w\|_{L_1}$. Then, according to Theorem 1, the following inequality holds

$$\|D_w - CA^{-1}B_w\|_1 < \gamma, \quad (15)$$

which implies

$$\mathbf{1}_r^T D_w - \mathbf{1}_r^T C A^{-1} B_w - \gamma \mathbf{1}_m^T << 0. \quad (16)$$

Define $\tilde{p} \triangleq (-\mathbf{1}_r^T C A^{-1})^T \geq\geq 0$ and $p \triangleq \tilde{p} + \epsilon\xi >> 0$, where $\xi >> 0$ satisfies $\xi^T(-A) >> 0$, and $\epsilon > 0$ is a sufficiently small number. We have

$$\begin{aligned} \mathbf{1}_r^T C + p^T A &= \mathbf{1}_r^T C + (\tilde{p}^T + \epsilon\xi^T)A \\ &= \mathbf{1}_r^T C - \mathbf{1}_r^T C + \epsilon\xi^T A \\ &= \epsilon\xi^T A \\ &<< 0. \end{aligned} \quad (17)$$

On the other hand,

$$\begin{aligned} &\mathbf{1}_r^T D_w + p^T B_w - \gamma \mathbf{1}_m^T \\ &= \mathbf{1}_r^T D_w - \mathbf{1}_r^T C A^{-1} B_w + \epsilon\xi^T B_w - \gamma \mathbf{1}_m^T. \end{aligned}$$

From (16) and $\epsilon > 0$ is sufficiently small, we have that (13) holds. This completes the whole proof. □

Remark 1: Theorem 2 presents a necessary and sufficient condition on the L_1-induced performance of a stable positive system in terms of linear programming. A similar result, proven by virtue of dissipativity theory, can be found in [24]. In the following, we provide a theorem as the performance characterization for positive system (1) over the whole uncertain domain in (2).

Theorem 3: The positive linear system in (1) is robustly stable and satisfies $\|y\|_{L_1} < \gamma\|w\|_{L_1}$ for any $A \in [\underline{A}, \overline{A}]$, $C \in [\underline{C}, \overline{C}]$, $B_w \in [\underline{B}_w, \overline{B}_w]$ and $D_w \in [\underline{D}_w, \overline{D}_w]$ if and only if there exists a vector $p \geq\geq 0$ satisfying

$$\mathbf{1}_r^T \overline{C} + p^T \overline{A} \quad << \quad 0, \quad (18)$$
$$p^T \overline{B}_w + \mathbf{1}_r^T \overline{D}_w - \gamma \mathbf{1}_m^T \quad << \quad 0. \quad (19)$$

Proof: (*Sufficiency*) For any $A \in [\underline{A}, \overline{A}]$, $C \in [\underline{C}, \overline{C}]$, $B_w \in [\underline{B}_w, \overline{B}_w]$ and $D_w \in [\underline{D}_w, \overline{D}_w]$,

$$\begin{aligned} \mathbf{1}_r^T C + p^T A &\leq\leq \mathbf{1}_r^T \overline{C} + p^T \overline{A}, \\ p^T B_w + \mathbf{1}_r^T D_w - \gamma \mathbf{1}_m^T &\leq\leq p^T \overline{B}_w + \mathbf{1}_r^T \overline{D}_w - \gamma \mathbf{1}_m^T, \end{aligned}$$

which, by Theorem 2, implies that system (1) is robust stable and satisfies $\|y\|_{L_1} < \gamma\|w\|_{L_1}$ over all admissible uncertainty domain. This proves sufficiency.

(*Necessity*) Assume that system (1) is robustly stable and satisfies $\|y\|_{L_1} < \gamma\|w\|_{L_1}$. From Theorem 2, we have (12) and (13) hold, which implies that (18) and (19) hold. This completes the whole proof. □

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IV. Sparse State-feedback Controller Design

This section deals with the robust sparse state-feedback stabilization problem for positive systems with interval uncertainties.

The problem of L_1-induced sparse controller design (L1SCD) is formulated as follows.

Problem L1SCD: *Given a positive system*

$$\mathcal{S}: \begin{cases} \dot{x}(t) &= Ax(t)+Bu(t)+B_w w(t), \\ y(t) &= Cx(t)+Du(t)+D_w w(t), \end{cases} \quad (20)$$

where $A \in [\underline{A}, \overline{A}]$, $B \in [\underline{B}, \overline{B}]$, $B_w \in [\underline{B}_w, \overline{B}_w]$, $C \in [\underline{C}, \overline{C}]$, $D \in [\underline{D}, \overline{D}]$ *and* $D_w \in [\underline{D}_w, \overline{D}_w]$, *find a state-feedback controller* $u(t) = Kx(t)$ *such that*

1) *the closed-loop system*

$$\begin{cases} \dot{x}(t) &= (A+BK)x(t)+B_w w(t), \\ y(t) &= (C+DK)x(t)+D_w w(t), \end{cases} \quad (21)$$

is positive and robustly stable.

2) *the* ℓ_0*-norm of the controller gain* K *is minimized subject to the* L_1*-induced performance, that is,*

$$\begin{aligned} &\min \|K\|_0 \\ &\text{subject to } \|y\|_{L_1} < \gamma \|w\|_{L_1}. \end{aligned} \quad (22)$$

This is a common sense approach which simply seeks the sparsest controller gain K satisfying the constraint. However, the optimization problem is non-convex and NP-hard. From Lemma 2, we know that the ℓ_0-optimization problem (22) can be relaxed by the following ℓ_1-minimization problem:

$$\begin{aligned} &\min \|\text{vec}(K)\|_1 \\ &\text{subject to } \|y\|_{L_1} < \gamma \|w\|_{L_1}. \end{aligned} \quad (23)$$

Before solving the ℓ_1-minimization problem (23), a necessary and sufficient condition is first presented for the existence of a solution to the L_1-induced state-feedback stabilization problem. We suppose $K \leq\leq 0$ and the detailed proof is omitted here.

Theorem 4: Suppose $K \leq\leq 0$. The closed-loop system (21) is positive, robustly stable and satisfies $\|z\|_{L_1} < \gamma \|w\|_{L_1}$ for any $A \in [\underline{A}, \overline{A}]$, $B \in [\underline{B}, \overline{B}]$, $B_w \in [\underline{B}_w, \overline{B}_w]$, $C \in [\underline{C}, \overline{C}]$, $D \in [\underline{D}, \overline{D}]$ and $D_w \in [\underline{D}_w, \overline{D}_w]$ if and only if there exist a matrix K and a vector $p \geq\geq 0$ satisfying

$$\underline{A} + \overline{B}K \text{ is Metzler}, \quad (24)$$

$$\underline{C} + \overline{D}K \geq\geq 0, \quad (25)$$

$$\mathbf{1}_r^T(\overline{C} + \underline{D}K) + p^T(\overline{A} + \underline{B}K) << 0, \quad (26)$$

$$p^T \overline{B}_w + \mathbf{1}_r^T \overline{D}_w - \gamma \mathbf{1}_m^T << 0. \quad (27)$$

Note that the Lyapunov vector p is coupled with the controller matrix K in (26), which cannot be easily solved. However, when matrix K is fixed, (26) turns out to be linear with respect to the remaining variables. Therefore, a natural way is to fix K, and solve (24)–(27) by linear programming. Thus, the following iterative algorithm can be proposed to solve Problem L1SCD.

Algorithm:

- Step 1. Set $i = 1$. Select an initial matrix K_1 such that the system

$$\begin{cases} \dot{x}(t) &= Ax(t)+Bu(t)+B_w w(t), \\ y(t) &= Cx(t)+Du(t)+D_w w(t), \end{cases} \quad (28)$$

with

$$u(t) = K_1 x(t) \quad (29)$$

is positive and robustly stable.

- Step 2. For fixed K_i, solve the following feasibility problem for p_i.
FP: Find p_i subject to the following constraints:

$$\begin{aligned} \mathbf{1}_r^T(\overline{C} + \underline{D}K_i) + p_i^T(\overline{A} + \underline{B}K_i) &<< 0, \\ p_i^T \overline{B}_w + \mathbf{1}_r^T \overline{D}_w - \gamma \mathbf{1}_m^T &<< 0. \\ p_i &\geq\geq 0. \end{aligned}$$

Denote K_i^*, p_i as the solution to the feasibility problem. If $\left|\left(K_i^* - K_{i-1}^*\right)/K_i^*\right| < \varepsilon_1$, where ε_1 is a prescribed bound, then $K^* = K_i$, $p = p_i$. STOP.

- Step 3. For fixed p_i, solve the following optimization problem for K_i.
OP: Minimize $\|\text{vec}(K)\|_1$ subject to the following constraints:

$$\begin{aligned} \underline{A} + \overline{B}K_i &\text{ is Metzler}, \\ \underline{C} + \overline{D}K_i &\geq\geq 0, \\ \mathbf{1}_r^T(\overline{C} + \underline{D}K_i) + p_i^T(\overline{A} + \underline{B}K_i) &<< 0. \end{aligned}$$

Denote K_i^* as the solution to the optimization problem.

- Step 4. If $\left|\left(K_i^* - K_{i-1}^*\right)/K_i^*\right| < \varepsilon_2$, where ε_2 is a prescribed tolerance, STOP; else set $i = i+1$ and $K_i = K_{i-1}$, then go to Step 2.

Remark 2: The initial matrix K_1 can be viewed as a state-feedback controller matrix, and can be constructed by existing convex optimization approaches. From [13], we know that system (28) with (29) is positive and robustly stable if and only if there exist matrices $P \triangleq \text{diag}(p_1, p_2, \ldots, p_n)$ and $Q \triangleq [q_{ij}] \in \mathbb{R}^{l \times n}$ such that

$$\begin{aligned} P\overline{A}^T + Q^T \overline{B}^T + \overline{A}P + \overline{B}Q &< 0, \\ \underline{a}_{ij} p_j + \sum_{z=1}^{l} \overline{b}_{iz} q_{zj} &\geq 0, \ (1 \leq i \neq j \leq n) \\ \underline{c}_{ij} p_j + \sum_{z=1}^{l} \overline{d}_{iz} q_{zj} &\geq 0. \end{aligned}$$

Under this condition, an initial choice of K can be given by $K_1 = QP^{-1}$.

V. Illustrative Example

In this section, we present an illustrative example to demonstrate the applicability of the proposed results.

Consider the following positive system:

$$\begin{cases} \dot{x}(t) &= Ax(t)+Bu(t)+B_w w(t), \\ y(t) &= Cx(t)+Du(t)+D_w w(t), \end{cases} \quad (30)$$

where $A \in [\underline{A}, \overline{A}]$, $B \in [\underline{B}, \overline{B}]$, $B_w \in [\underline{B}_w, \overline{B}_w]$, $C \in$

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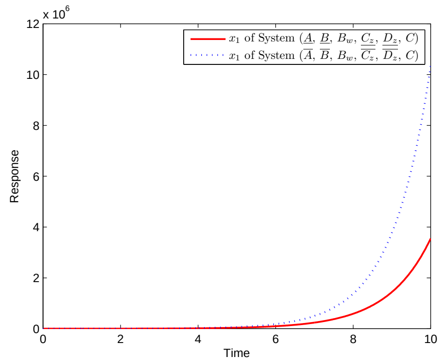

Fig. 1: Time Response of Open-loop System

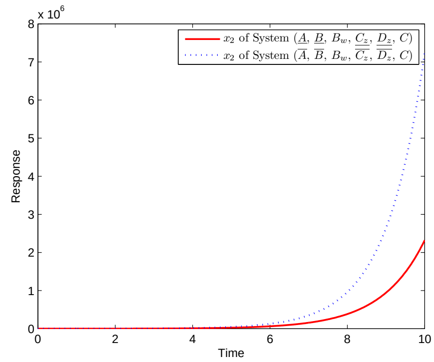

Fig. 2: Time Response of Open-loop System

$[\underline{C}, \overline{C}]$, $D \in [\underline{D}, \overline{D}]$, $D_w \in [\underline{D}_w, \overline{D}_w]$ with

$$\underline{A} = \begin{bmatrix} -2.00 & 1.30 & 2.00 \\ 0.50 & -3.00 & 2.00 \\ 2.00 & 1.50 & -2.00 \end{bmatrix},$$

$$\overline{A} = \begin{bmatrix} -1.98 & 1.31 & 2.00 \\ 0.60 & -2.96 & 2.10 \\ 2.00 & 1.50 & -1.92 \end{bmatrix},$$

$$\underline{B} = \begin{bmatrix} 1.00 & 0.00 \\ 0.00 & 1.00 \\ 1.00 & 0.5 \end{bmatrix}, \ \overline{B} = \begin{bmatrix} 1.04 & 0.01 \\ 0.04 & 1.00 \\ 1.02 & 0.50 \end{bmatrix},$$

$$\underline{B}_w = \begin{bmatrix} 0.48 & 0.20 \\ 0.08 & 0.00 \\ 0.00 & 0.50 \end{bmatrix}, \ \overline{B}_w = \begin{bmatrix} 0.50 & 0.21 \\ 0.10 & 0.00 \\ 0.02 & 0.52 \end{bmatrix},$$

$$\underline{C} = \begin{bmatrix} 1.00 & 0.60 & 1.00 \\ 0.80 & 0.80 & 1.00 \end{bmatrix}, \ \underline{D} = \begin{bmatrix} 0.48 & 0.00 \\ 0.50 & 1.00 \end{bmatrix},$$

$$\overline{C} = \begin{bmatrix} 1.02 & 0.61 & 1.00 \\ 0.80 & 0.82 & 1.00 \end{bmatrix}, \ \overline{D} = \begin{bmatrix} 0.50 & 0 \\ 0.52 & 1.01 \end{bmatrix},$$

$$\underline{D}_w = \begin{bmatrix} 0.10 & 0.00 \\ 0.10 & 0.20 \end{bmatrix}, \ \overline{D}_w = \begin{bmatrix} 0.12 & 0.01 \\ 0.11 & 0.22 \end{bmatrix}.$$

For $\gamma = 0.7$, by solving the conditions in Theorem 4 using Algorithm, we obtain the sparse controller gain

$$K^* = \begin{bmatrix} -0.6413 & -0.5003 & -1.7548 \\ -0.0013 & -0.0008 & -0.0000 \end{bmatrix}$$

after 10 iterations and a feasible solution is achieved with

$$p = \begin{bmatrix} 0.9761 & 0.7829 & 0.5888 \end{bmatrix}.$$

To illustrate the disturbance attenuation performance, the external disturbance $w(t)$ is assumed to be

$$w(t) = \begin{cases} \begin{bmatrix} 300 + 200\cos 2t & e^{-3t} \end{bmatrix}^T, & t \leq 5, \\ \begin{bmatrix} 0 & 0 \end{bmatrix}^T, & \text{otherwise}, \end{cases}$$

and the initial condition used in the simulation is

$$x(0) = \begin{bmatrix} 500 & 200 & 300 \end{bmatrix}^T.$$

Figures 1–3 show the response of open-loop system and Figure 4–6 show the state response of the closed-loop system, from which we can see that the system can be stabilized by the designed controller.

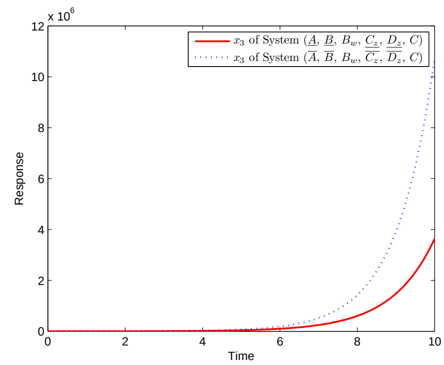

Fig. 3: Time Response of Open-loop System

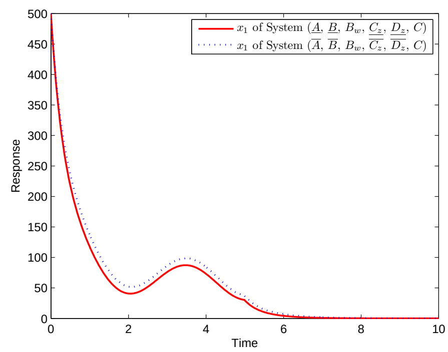

Fig. 4: Time Response of Closed-loop System

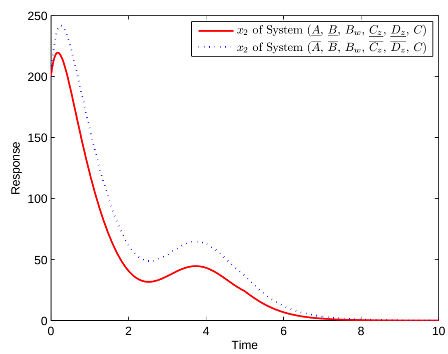

Fig. 5: Time Response of Closed-loop System

VI. Conclusion

In this paper, the L_1-induced performance analysis and the sparse state-feedback stabilization problem for continuous-

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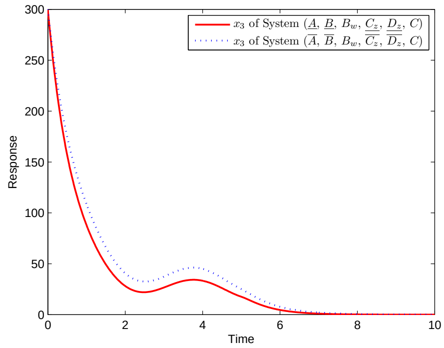

Fig. 6: Time Response of Closed-loop System

time interval positive systems have been studied. A method has been derived to compute the exact value of the L_1-induced norm for positive systems and a characterization has been proposed to ensure the asymptotic stability of the positive system with a prescribed L_1-induced performance level. In addition, the sparse controller design problem has been formulated by minimizing the ℓ_0-norm of the controller gain K, which has been relaxed by an ℓ_1-minimization problem. Moreover, the resulting problem has been solved by an iterative convex optimization algorithm. Finally, an illustrative example is provided to show the effectiveness and applicability of the theoretical results.

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