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Block Method for Solving Pantograph-type Functional Differential Equations

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***Abstract*— In this paper, we describe the development of a two-point block method for solving pantograph-type functional differential equations. The block method, implemented in variable stepsize technique produces two approximations simultaneously using the same back values. The grid-point formulae for the variable steps are derived, calculated and stored at the start of the program for greater efficiency. The delay solutions for the unknown function and its derivative at earlier times are interpolated using the previous computed values. Stability regions for the block method are illustrated. Numerical results are given to demonstrate the accuracy and efficiency of the block method.**

***Index Terms*—Block method, functional differential equation, pantograph equation, polynomial interpolation, stability region**

I. Introduction

FUNCTIONAL differential equation of the form

$$y'(x) = f(x, y(x), y(\alpha(x)), y'(\alpha(x))) \qquad (1)$$

appears in many real life applications and has been investigated by many authors in recent years. The classical case is when $\alpha(x) = x - \tau$. When the right hand side of (1) does not depend on the derivative of the unknown function y, the equation is known as delay differential equation. Otherwise, it is known as neutral delay differential equation.

In this paper, we consider numerical solution for functional differential equation of the form:

$$\begin{cases} y'(x) = f(x, y(x), y(qx), y'(qx)), & 0 < x \le T, \\ y(0) = y_0, \end{cases} \qquad (2)$$

where $0 < q < 1$. Equation (2), known as the pantograph equation arises in many physical applications such as number theory, electrodynamics, astrophysics, etc.. Detailed explanations can be found in [1] – [3]. Numerical solutions for (2) have been studied extensively, see for example [4] – [7] and the references cited therein. These methods produce one approximation in a single integration step. Block methods, however produce more than one approximation in a step. Block methods have been used to solve wide range of ordinary differential equations as well as delay differential equations (see [8] – [11] and the references cited therein).

In this paper, we solve (2) using a two-point block method in variable step. In a single integration step, two new approximates for the function y in (2) are obtained while keeping a constant stepsize, doubling or halving. The coefficients of the method need to be recalculate whenever the stepsize changes. In order to avoid the tedious calculation, the coefficients based on the stepsize ratio are calculated beforehand and stored at the start of the program.

The organization of this paper is as follows. In section II, we briefly describe the development of the variable step block method. Stability region for the block method is discussed in section III. Numerical results for some pantograph equations are presented in section IV and finally section V is the conclusion.

II. Method Development

Referring to (2), we seek a set of discrete solutions for the unknown function y in the interval $[0, T]$. The interval is divided into a sequence of mesh points $\{x_i\}_{i=0}^{t}$ of different lengths, such that $0 = x_0 < x_1 < \cdots < x_t = T$. Let the approximated solution for $y(x_n)$ be denoted as y_n. Suppose that the solutions have been obtained up to x_n. At the current step, two new solutions y_{n+1} and y_{n+2} at x_{n+1} and x_{n+2} respectively are simultaneously approximated using the same back values by taking the same stepsize. The points x_{n+1} and x_{n+2} are contained in the current block. The length of the current block is $2h$. We refer to this particular block method as two-point one-block method. The block method is shown in Fig 1.

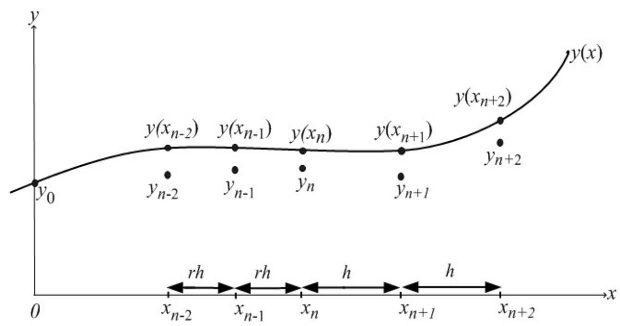

Fig 1. Two-point one-block method

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In Fig 1, the stepsize of the previous step is viewed in the multiple of the current stepsize. Thus, $x_{n+1}-x_n=h$, $x_{n+2}-x_{n+1}=h$ and $x_{n-1}-x_{n-2}=x_n-x_{n-1}=rh$. The value of r is either 1, 2, or $\frac{1}{2}$, depending upon the decision to change the stepsize. In this algorithm, we employ the strategy of having the stepsize to be constant, halved or doubled.

The formulae for the block method can be written as the pair,

$$\begin{aligned} y_{n+1} &= y_n + h\sum_{i=0}^{4}\beta_i(r) f(x_{n-2+i}, y_{n-2+i}, \bar{y}_{n-2+i}, \hat{y}_{n-2+i}), \\ y_{n+2} &= y_n + h\sum_{i=0}^{4}\beta_i^*(r) f(x_{n-2+i}, y_{n-2+i}, \bar{y}_{n-2+i}, \hat{y}_{n-2+i}), \end{aligned} \tag{3}$$

where $\bar{y}_n$ and $\hat{y}_n$ are the approximations to $y(qx_n)$ and $y'(qx_n)$ respectively. For simplicity, from now on we refer to $f(x_n, y_n, \bar{y}_n, \hat{y}_n)$ as f_n. The coefficient functions $\beta_i(r)$ and $\beta_i^*(r)$ will give the coefficients of the method when r is either 1, 2, or $\frac{1}{2}$.

The first formula in (3) is obtained by integrating (2) from x_n to x_{n+1} while replacing the function f with the polynomial P where $P(x)$ is given by

$$P(x) = \sum_{j=0}^{4} L_{4,j}(x) f_{n+2-j},$$

and

$$L_{4,j}(x) = \prod_{\substack{i=0 \\ i\neq j}}^{4} \frac{(x - x_{n+2-i})}{(x_{n+2-j} - x_{n+2-i})}, \quad \text{for } j = 0,1,\ldots,4.$$

Similarly, the second formula in (3) is obtained by integrating (2) from x_n to x_{n+2} while replacing the function f with the polynomial P. The value of $\bar{y}_n$ is obtained by the interpolation function such as,

$$\begin{aligned} \bar{y}_n &= y[x_j] + (qx_n - x_j) y[x_j, x_{j-1}] + \cdots \\ &\quad + (qx_n - x_j)\cdots(qx_n - x_{j-3}) y[x_j, \ldots, x_{j-4}], \end{aligned}$$

where

$$y[x_j, x_{j-1}, \ldots, x_{j-4}] = \frac{y[x_j, \ldots, x_{j-3}] - y[x_{j-1}, \ldots, x_{j-4}]}{x_j - x_{j-4}},$$

provided that $x_{j-1} \le qx_n \le x_j$, $n \ge j$, $j \ge 1$. We approximate the value of $\hat{y}_n$ by interpolating the values of f, that is,

$$\begin{aligned} \hat{y}_n &= f[x_j] + (qx_n - x_j) f[x_j, x_{j-1}] + \cdots \\ &\quad + (qx_n - x_j)\cdots(qx_n - x_{j-3}) f[x_j, \ldots, x_{j-4}], \end{aligned}$$

where

$$f[x_j, x_{j-1}, \ldots, x_{j-4}] = \frac{f[x_j, \ldots, x_{j-3}] - f[x_{j-1}, \ldots, x_{j-4}]}{x_j - x_{j-4}}.$$

The formulae in (3) are implicit, thus a set of predictors are derived similarly using the same number of back values. The corrector formulae in (3) are iterated until convergence.

For greater efficiency while achieving the required accuracy, the algorithm is implemented in variable stepsize scheme. The stepsize is changed based on the local error that is controlled at the second point. A step is considered successful if the local error is less than a specified tolerance. If the current step is successful, we consider either doubling or keeping the same stepsize. If the same stepsize had been used for at least two blocks, we double the next stepsize. Otherwise, the next stepsize is kept the same. If the current step fails, the next stepsize is reduced by half. For repeated failures, a restart with the most optimal stepsize with one back value is required. For variable step algorithms, the coefficients of the methods need to be recalculated whenever a stepsize changes. The recalculation cost of these coefficients is avoided by calculating the coefficients beforehand and storing them at the start of the program. With our stepsize changing strategy, we store the coefficients $\beta_i(r)$ and $\beta_i^*(r)$ for r is 1, 2 and $\frac{1}{2}$.

III. Region of Absolute Stability

In the development of a numerical method, it is of practical importance to study the behavior of the global error. The numerical solution y_n is expected to behave as the exact solution $y(x_n)$ does as x_n approaches infinity. In this section, we present the result of stability analysis of the two-point one-block method when they are applied to the neutral delay differential equations with real coefficients.

For the sake of simplicity and without the lost of generality, we consider the equation

$$\begin{aligned} y'(x) &= ay(x) + by(x-\tau) + cy'(x-\tau), \quad x \ge 0, \\ y(x) &= \varphi(x), \quad -\tau \le x < 0, \end{aligned} \tag{4}$$

where $a, b, c \in R$, τ is the delay term such as $\tau = mh$, h is a constant stepsize such that $x_n = x_0 + nh$ and $m \in Z^+$. If $i \in Z^+$, we define vectors $\mathbf{Y}_{N+i} = \begin{bmatrix} y_{n-3+2i} \\ y_{n-2+2i} \end{bmatrix}$ and $\mathbf{F}_{N+i} = \begin{bmatrix} f_{n-3+2i} \\ f_{n-2+2i} \end{bmatrix}$. Then, the block method (3) can be written in matrix form such as,

$$A_1\mathbf{Y}_{N+1} + A_2\mathbf{Y}_{N+2} = h\sum_{i=0}^{2} B_i(r)\mathbf{F}_{N+i}, \tag{5}$$

where $A_1 = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $B_i(r)$ is a matrix that contains the coefficients $\beta_i(r)$ and $\beta_i^*(r)$. Applying method (5) to (4), we get

$$\begin{aligned} A_1\mathbf{Y}_{N+1} + A_2\mathbf{Y}_{N+2} &= H_1\sum_{i=0}^{2} B_i(r)\mathbf{Y}_{N+i} + H_2\sum_{i=0}^{2} B_i(r)\mathbf{Y}_{N+i-m} \\ &\quad + cA_2\mathbf{Y}_{N+2-m} + cA_1\mathbf{Y}_{N+1-m}, \end{aligned}$$

where $H_1 = ha$ and $H_2 = hb$. Rearranging, we have

$$\sum_{i=0}^{2}(A_i - H_1 B_i(r))\mathbf{Y}_{N+i} = \sum_{i=0}^{2}(H_2 B_i(r) + cA_i)\mathbf{Y}_{N+i-m}, \tag{6}$$

where A_0 is the null matrix. Characteristic polynomial for

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(6) is given by $C_m(H_1, H_2, c; \zeta)$ where C_m is the determinant of

$$\sum_{i=0}^{2}(A_i - H_1 B_i(r))\zeta^{m+i} - \sum_{i=0}^{2}(H_2 B_i(r) + cA_i)\zeta^i = 0. \quad (7)$$

The numerical solution (6) is asymptotically stable if and only if for all m, all zeros of the characteristic polynomial (7) lie within the open unit disk in the plane. The stability region is defined as follows:

Definition 1: For a fixed stepsize h, $a, b \in R$, and for any, but fixed c, the region S in the H_1 - H_2 plane is called the stability region of the method if for any $(H_1, H_2) \in S$, the numerical solution of (4) vanishes as x_n approaches infinity.

In Fig 2, the stability regions for $m = 1$ and $c = 0.5$ are illustrated. We use the boundary locus technique as described in [12] and [13]. The regions are sketched for $r = 1$, $r = 2$, and $r = \frac{1}{2}$. The coefficient matrices are given as follows:

For $r = 1$:

$$B_0 = \begin{bmatrix} 0 & \frac{11}{720} \\ 0 & \frac{-1}{90} \end{bmatrix}, \quad B_1 = \begin{bmatrix} \frac{-74}{720} & \frac{456}{720} \\ \frac{4}{90} & \frac{24}{90} \end{bmatrix}, \text{ and } B_2 = \begin{bmatrix} \frac{346}{720} & \frac{-10}{720} \\ \frac{124}{90} & \frac{29}{90} \end{bmatrix}.$$

For $r = 2$:

$$B_0 = \begin{bmatrix} 0 & \frac{137}{14400} \\ 0 & \frac{-1}{900} \end{bmatrix}, \quad B_1 = \begin{bmatrix} \frac{-335}{14400} & \frac{7455}{14400} \\ \frac{5}{900} & \frac{285}{900} \end{bmatrix}, \text{ and}$$

$$B_2 = \begin{bmatrix} \frac{7808}{14400} & \frac{-565}{14400} \\ \frac{1216}{900} & \frac{295}{900} \end{bmatrix}.$$

For $r = \frac{1}{2}$:

$$B_0 = \begin{bmatrix} 0 & \frac{145}{1800} \\ 0 & \frac{-20}{225} \end{bmatrix}, \quad B_1 = \begin{bmatrix} \frac{-704}{1800} & \frac{1635}{1800} \\ \frac{64}{225} & \frac{15}{225} \end{bmatrix}, \text{ and } B_2 = \begin{bmatrix} \frac{755}{1800} & \frac{-31}{1800} \\ \frac{320}{225} & \frac{71}{225} \end{bmatrix}.$$

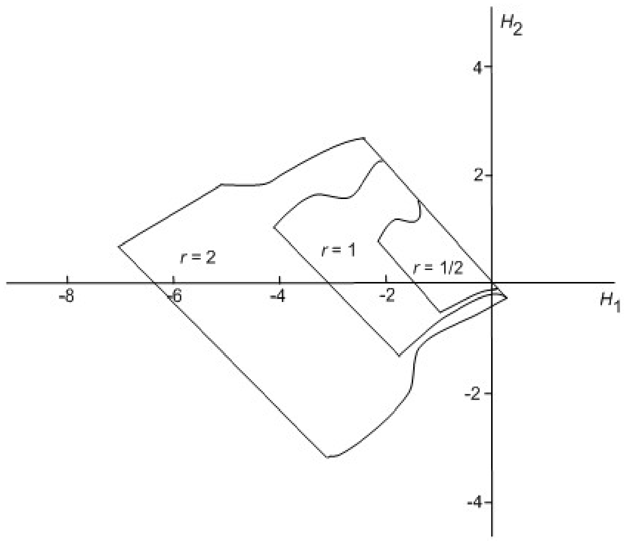

Fig 2. Stability region for the block method

Referring to Fig 2, the stability regions are closed region bounded by the corresponding boundary curves. It is observed that the stability region shrinks as the stepsize increases.

IV. Numerical Results

In this section, we present some numerical examples in order to illustrate the accuracy and efficiency of the block method. The examples taken and cited from [7] are as follows:

Example 1:

$$y'(x) = \frac{1}{2}y(x) + \frac{1}{2}e^{x/2}y\left(\frac{x}{2}\right), \quad 0 \le x \le 1,$$

$$y(0) = 1.$$

The exact solution is $y(x) = e^x$.

Example 2:

$$y'(x) = -\frac{5}{4}e^{-x/4}y\left(\frac{4}{5}x\right), \quad 0 \le x \le 1,$$

$$y(0) = 1.$$

The exact solution is $y(x) = e^{-1.25x}$.

Example 3:

$$y'(x) = -y(x) + \frac{q}{2}y(qx) - \frac{q}{2}e^{-qx}, \quad 0 \le x \le 1,$$

$$y(0) = 1.$$

The exact solution is $y(x) = e^{-x}$.

Example 4:

$$y'(x) = ay(x) + by(qx) + \cos x - a\sin x - b\sin(qx), \quad 0 \le x \le 1,$$

$$y(0) = 0.$$

The exact solution is $y(x) = \sin x$.

Example 5:

$$y'(x) = -y(x) + \frac{1}{2}y\left(\frac{x}{2}\right) + \frac{1}{2}y'\left(\frac{x}{2}\right), \quad 0 \le x \le 1,$$

$$y(0) = 1.$$

The exact solution is $y(x) = e^{-x}$.

Example 6:

$$y'(x) = -y(x) + 0.1y(0.8x) + 0.5y'(0.8x)$$
$$+ (0.32x - 0.5)e^{-0.8x} + e^{-x}, \quad 0 \le x \le 10,$$

$$y(0) = 0.$$

The exact solution is $y(x) = xe^{-x}$.

Numerical results for Example 1 – Example 6 are given in Table I – Table VIII. The following abbreviations are used in the tables, TOL – the chosen tolerance, STEP – the total number of steps taken, FS – the number of failed steps, AVERR – the average error, and MAXE – the maximum error. The notation 7.02683E-01 means 7.02683×10^{-1}.

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Table I: Numerical Results for Example 1

TOL	STEP	FS	AVERR	MAXE
10^{-2}	20	0	7.02683E-01	9.68012E-05
10^{-4}	27	0	4.15905E-07	9.05425E-07
10^{-6}	35	0	3.09498E-07	4.21310E-07
10^{-8}	48	0	9.55941E-09	1.28084E-08
10^{-10}	75	0	7.68855E-11	9.88663E-11

Table II: Numerical Results for Example 2

TOL	STEP	FS	AVERR	MAXE
10^{-2}	21	0	9.76063E-08	1.17093E-07
10^{-4}	27	0	4.95202E-09	1.27731E-07
10^{-6}	35	0	1.06955E-09	1.21167E-08
10^{-8}	50	0	3.12606E-11	1.43148E-10
10^{-10}	79	0	5.60297E-13	1.51457E-12

Table III: Numerical Results for Example 3, $q = 0.2$

TOL	STEP	FS	AVERR	MAXE
10^{-2}	20	0	1.54360E-05	1.91824E-05
10^{-4}	27	0	1.15315E-07	5.14934E-07
10^{-6}	35	0	6.83967E-08	8.52745E-08
10^{-8}	47	0	2.03160E-09	2.57931E-09
10^{-10}	74	0	1.50180E-11	2.00260E-11

Table IV: Numerical Results for Example 3, $q = 0.8$

TOL	STEP	FS	AVERR	MAXE
10^{-2}	20	0	1.57507E-07	2.30455E-06
10^{-4}	27	0	2.90818E-08	4.90683E-07
10^{-6}	35	0	4.03006E-10	3.90288E-09
10^{-8}	48	0	2.46267E-11	1.51318E-10
10^{-10}	74	0	3.09422E-13	1.74260E-12

Table V: Numerical Results for Example 4, $a = -1$, $b = 0.5$, $q = 0.1$

TOL	STEP	FS	AVERR	MAXE
10^{-2}	20	0	1.61996E-06	1.15114E-05
10^{-4}	27	0	1.08039E-07	1.15418E-06
10^{-6}	35	0	8.23406E-09	1.15448E-07
10^{-8}	48	0	5.74175E-10	1.15451E-08
10^{-10}	79	0	4.12730E-11	1.15451E-09

Table VI: Numerical Results for Example 4, $a = -1$, $b = 0.5$, $q = 0.5$

TOL	STEP	FS	AVERR	MAXE
10^{-2}	20	0	6.25587E-06	4.77895E-05
10^{-4}	27	0	4.53165E-07	4.78257E-06
10^{-6}	35	0	3.48942E-08	4.78294E-07
10^{-8}	50	0	2.91205E-09	4.78298E-08
10^{-10}	79	0	1.83823E-10	4.78299E-09

Table VII: Numerical Results for Example 5

TOL	STEP	FS	AVERR	MAXE
10^{-2}	20	0	7.31358E-04	1.83509E-03
10^{-4}	27	0	1.88577E-05	2.92294E-05
10^{-6}	71	0	9.02824E-06	2.12659E-05
10^{-8}	166	2	1.14939E-06	2.13569E-06
10^{-10}	236	4	4.56047E-08	5.25142E-08

Table VIII: Numerical Results for Example 6

TOL	STEP	FS	AVERR	MAXE
10^{-2}	70	0	1.22749E-02	4.54860E-02
10^{-4}	97	0	3.92866E-04	1.14032E-03
10^{-6}	118	0	1.61615E-06	4.86641E-06
10^{-8}	173	0	2.72657E-07	4.87304E-07
10^{-10}	300	3	1.72160E-08	3.97650E-08

From Table 1 – Table VIII, it is observed that for the given tolerances, the two-point block method achieves the desired accuracy. When the tolerance becomes smaller, the total number of steps increases. In order to achieve the desired accuracy, smaller stepsizes are taken, thus resulting in the increase number of total steps taken.

V. Conclusion

In this paper, we have discussed the development of a two-point block method for solving pantograph-type functional differential equations. The block method produces two approximate solutions in a single integration step by using the same back values. The algorithm is implemented in variable stepsize technique where the coefficients for the various stepsizes are stored at the beginning of the program for greater efficiency. Stability regions for a general linear test equation are obtained for a fixed, but variable stepsizes. The numerical results indicate that the two-point block method achieves the desired accuracy as efficiently as possible.

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