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Discrete and Continuous Growth of Hollow Cylinder. Finite Deformations

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Abstract—**The finite deformations of the growing cylinder fabricated of an incompressible elastic material of Mooney–Rivlin type are under consideration. We assume that the deformations are axisymmetric and constant along the cylinder axis. The discrete and continuous types of growing are studied. The analytical solutions of the corresponding boundary-value problems are derived.**

Index Terms—**additive technologies, growing solids, finite deformations, hyperelasticity, continuous growth, discrete growth.**

I. Introduction

A VAST majority of objects around us arise from some growth processes. Many natural phenomena such as growth of biological tissues, glaciers, blocks of sedimentary and volcanic rocks, and space objects may serve as an example. Similar additive processes determine the specific features of many industrial technologies including well-known crystal growth, welding, laser deposition, melt solidification, electrolytic formation, pyrolytic deposition, polymerization, and concreting technologies [1]–[3]. Recent research indicates that growing solids exhibit properties dramatically different from those of conventional solids, so that classical solid mechanics cannot be used to model their behavior.

It is essential that in growing bodies residual stresses can occur through a variety of mechanisms. For example, in layer-by-layer welding technology a heat from parts being welded may cause localized expansion. When the finished melt cools the incompatible distortion appears that cause the residual stresses. Another example occurs during an additive manufacturing when thin film materials with different thermal and crystalline properties are deposed sequentially under different process conditions. In general it is impossible to avoid residual stresses. It leads to undesirable consequences, such as a shape distortion, local discontinuity, loss of stability. In particular, the estimation (and minimization) of possible distortions in stereolithography, the analysis of stability of epitaxial thin-walled structures applicable in micro-electromechanical systems (MEMS) are significant.

In the design of mentioned above additive technologies it is often desirable to minimize distortion and residual stresses, or to produce structures with a predefined distributions of initial stresses. This may be achieved through mathematical modeling of evolution stress-strain state of a growing body. The present paper deals with the development of this theory.

In the paper the concept of the *growth of a solid* is used. This concept refers to a new branch of continuum mechanics [4]–[7], therefore it seems appropriate here to clarify the definition of a growing solid. In a broad sense growing process defines the alteration of the body composition occurring in the course of deformation. The growing process may be accompanied by a change of topological properties of the body. We say that the altering of the body composition is the accession of new material points and (or) formation of new constrains between particles already included into the composition. It also should be noted that the change of topological properties can occur without the influx of material and can be caused by the transition of the boundary points into the interior.

In modern continuum mechanics there are a number of different approaches to the studying of the growth phenomenon. For today a large number of papers devoted to mechanics of growing solids have been published. References may be found in the review [8]. Some works of direct relevance to the issues discussed in the paper are mentioned below. In the paper [9] the volumetric grows, in particular the growth of biological tissues, is studied. Article [10] is devoted to the development of geometric methods adopted for the mechanics of incompatible strains arising as the result of the growing process. In the works [4]–[7], [11] growth is investigated as the continuous process of deposition of strained material surfaces to a deformable 3D body.

It is known that under certain additional assumptions on the continuity of functions defining the stress-strain state of adhered material surfaces the continuous growing process can be considered as the limit of a sequence of discrete processes. However, one can find only few examples concerning the finite deformations of growing solids and comparison of discrete and continuous growth. The aim of the paper is to give such example.

II. Common Definitions

In what follows the geometrical concept of a body that is represented in terms of smooth manifolds is used [12]–[16]. Let the *body* $\mathcal{B}$ be the connected abstract subset of a topological space such that the image of $\mathcal{B}$ may be imbedded into physical space as a region with regular boundary. Furthermore, assume that the body $\mathcal{B}$ exhibit properties of differentiable manifold [17]. We say that $\mathfrak{p} \in \mathcal{B}$ are material points. Assume that they are *simple*, i.e. the local response of the body depends only on first deformation gradient [18].

In the classical continuum mechanics bodies are treated as permanent sets of material points. In mechanics of growing

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solids one consider the *evolution* of the set $\mathcal{B}$. Note that the evolution of growing body in the abstract topological (material) space can be very complex and may be described in general in terms of discontinuous mappings [19]. However, under some restrictions on the smoothness of functions describing the growing process the evolution of the body can be presented by a continuous family of bodies ordered with respect to inclusion. In general this family can be associated with a smooth bundle. The dimension of a base of this bundle defines the class of a growing body [5]. In present paper we will considered the simplest class that corresponds to the three-dimensional bundle over one-dimensional base. It has the following interpretation in the terms of continuum mechanics. For a sufficiently large class of additive technologies the growing process may be modelled as a continuous influx of prestressed material surfaces [20] to a growing three-dimensional body. Due to this assumption the growing body can be represented by a one-parameter family of smooth bodies

$$\mathfrak{C} = \{\mathcal{B}_\alpha\}_{\alpha\in\mathcal{I}}\,.$$

Here $\mathcal{I}$ is a set of indices that can be finite, countable or continual.

We introduce the notion of *total body* $\mathcal{B}^*$ and *initial body* $\mathcal{B}_*$ as follows

$$\mathcal{B}^* = \bigcup_{\alpha\in\mathcal{I}} \mathcal{B}_\alpha, \quad \mathcal{B}_* = \bigcap_{\alpha\in\mathcal{I}} \mathcal{B}_\alpha.$$

We shall say that the elements of $\mathfrak{C}$ corresponding to interior points of the interval $\mathcal{I}$ are the *intermediate* bodies.

We will distinguish discrete and continuous growth. In the case of a discrete growth the family $\mathfrak{C}$ is a finite sequence of nested sets:

$$\mathfrak{C}\ :\ \mathcal{B}_1 \subset \mathcal{B}_2 \subset \ldots \subset \mathcal{B}_N. \tag{1}$$

If the growth is continuous then the family $\mathfrak{C}$ is represented by continuous family of bodies over the interval $\mathcal{I} = (\alpha,\beta) \subset \mathbb{R}$, that satisfy the following condition. There are two-dimensional smooth manifolds Ω_k and no more than a countable set of homeomorphisms such that

$$\begin{gathered}\Psi_k\ :\ (\Omega_k,\alpha) \to \mathcal{B}^*, \quad \alpha\in\mathbb{R},\\ \forall\alpha<\beta \quad \mathcal{B}_\alpha \subset \mathcal{B}_\beta, \quad \forall\alpha\ \exists k\ \partial\mathcal{B}_\alpha = \Psi_k(\Omega_k,\alpha),\\ \bigcup_k \Psi_k(\Omega_k\times\mathcal{I}_k) = \mathcal{B}^*, \quad \bigcup_k \mathcal{I}_k = \mathcal{I}.\end{gathered} \tag{2}$$

Obviously, the set $\mathfrak{C}$ has countable cardinality $|\mathfrak{C}| = \aleph_1$. Note that manifolds Ω_k represent the preimage of growth boundary. Relations (2) introduce on the manifold $\mathcal{B}^*$ the structure of a smooth bundle [5], [17]. The interval $\mathcal{I}$ represents the base of the bundle while the manifolds Ω_k represent the fibers. If there is a single (universal) homeomorphism $\Psi = \Psi_1$ and a single manifold $\Omega = \Omega_1$, then the bundle becomes trivial. Otherwise topological structure of the growing boundary can vary. Changes in the topology of the preimage of growth boundaries correspond to the phenomenon of selfcontact of the image Ω_k, e.g. the transformation of a cylinder to a torus under the joining of the bases of the cylinder.

Arguing as above we see that growing body can be represented as a bundle over the total body $\mathcal{B}^*$. Considering the fact that the base of the bundle is one-dimensional we denote the fiber by $\mathfrak{M}_\alpha$. where α is the base coordinate of the fiber. Structural properties of the bundle implies that the fibers are disjoint, and their union coincides with total body $\mathcal{B}^*$, i.e. $\mathcal{B}^* = \bigcup\limits_{\gamma\in\mathcal{I}} \mathfrak{M}_\gamma$.

In the process of growth the body $\mathcal{B}_\alpha$ is presented by open subsets of total body $\mathcal{B}^*$, whose boundary $\partial\mathcal{B}_\alpha$ is a union of two separate fibers $\mathfrak{M}_{\alpha'}$ and $\mathfrak{M}_{\beta'}$, i.e. $\partial\mathcal{B}_\alpha = \mathfrak{M}_{\alpha'} \cup \mathfrak{M}_{\beta'}$. In this case the body $\mathcal{B}_\alpha$ can be presented as the union of fibers over an open interval $(\alpha',\beta') \subset \mathcal{I}$:

$$\mathcal{B}_\alpha = \mathcal{B}(\alpha',\beta') = \bigcup_{\gamma\in(\alpha',\beta')} \mathfrak{M}_\gamma. \tag{3}$$

Under the above mentioned assumptions we can define the growing body as the one-parameter family

$$\mathfrak{C} = \left\{\mathcal{B}_\alpha = \mathcal{B}(\alpha_0,\alpha)\middle|\alpha\in\mathcal{I}\right\}.$$

Here α is a continuous parameter that characterizes the evolution of the growing body. As $\alpha \to \alpha_0$ the body degenerates into an infinitely thin film or a point. The obvious generalization of this definition is follows

$$\mathfrak{C} = \left\{\mathcal{B}_\gamma = \mathcal{B}(\alpha_\gamma,\beta_\gamma)\middle|(\alpha_\gamma,\beta_\gamma)\subset\mathcal{I}\right\},$$

where $(\alpha_\gamma,\beta_\gamma)$ is a family of nested intervals.

According to the above definition the boundary of the growing body should be topologically equivalent to a typical fiber, which should be a smooth manifold. Hence the growing boundary must be topologically equivalent to a geometrically closed surface. If the growing boundary is topologically equivalent to a manifold with edge, then the growing body can be defined as follows

$$\mathfrak{C} = \left\{\mathcal{B}_\gamma = \mathcal{B}_0 \cap \mathcal{B}(\alpha_\gamma,\beta_\gamma)\middle|(\alpha_\gamma,\beta_\gamma)\subset\mathcal{I}\right\}.$$

Here $\mathcal{B}_0$ is a fixed subset of the material manifold with smooth edge.

In present paper we consider the growth of a hollow circular cylinder of fixed height h. Suppose that the additional material is attached to the lateral surface of the growing cylinder. If the coordinate charts correspond to the placement coordinates in an actual configuration then the the image of the set $\mathcal{B}_0$ corresponds to a sufficiently large parallelepiped which height is equal to the height of the growing cylinder.

III. Stress-strain state of the body-fiber

The stress-strain state of a growing body fundamentally differs from the stress-strain state of solids considered in classical solid mechanics. The most important is the fact that growing body has no natural (stress-free) configuration. Stress-strain state for growing bodies may be modelled in the framework of the theory of inhomogeneity developed in [12]–[14].

The representation of a body as a bundle of a smooth manifold allows one to use additional hypothesis concerning the properties of the fibers. In particular one can assume that each individual fiber has a natural configuration. Such hypothesis is adopted in present paper.

In order to describe stress-strain of a growing body it is necessary to determine the stress-strain state for a fiber as its structural element. In the case of discrete growth this structural element is a three-dimensional body $\mathcal{B}_{n+1}\setminus\mathcal{B}_n$, corresponding to the increment of the sequence (1). In the case of continuous growth the material surface $\mathfrak{M}_\gamma$, which corresponds to a fiber of a bundle (3), plays the role of structural element. Within the present work we assume that each separate body-fiber has a natural configuration immersed in Euclidean space. It is clear that the assembly of body-fibers have no such configuration.

In the case of discrete growth each assembly consists of a finite number of nested hollow cylindrical bodies. The second case is more abstract. It corresponds to the assembly of the continuum family of two-dimensional *material surfaces*.

We assume that the material of a body-fibers is hyperelastic and incompressible. Then the stress-strain state can be determined analytically by universal solutions of Rivlin–Ericksen type [18].

Let the image of stress-free (natural) configuration of the body-fiber $\mathcal{B}_{n+1} \setminus \mathcal{B}_n$ is embedded into physical (Euclidean) space $\mathcal{E}$. This embedding can be defined by the vector field of placements presented in Cartesian basis $\{\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$ by decomposition $\boldsymbol{X} = X^m \boldsymbol{i}_m$. Here $\{X^1, X^2, X^3\}$ are Cartesian coordinates. Suppose that the deformation of the body-fiber is defined by the map $\boldsymbol{X} \mapsto \boldsymbol{x}$. We assume that this map has a symmetry relative to the axial axis of the cylindrical fibers and does not depend on coordinate X^3.

For a more compact formulation of the kinematic relations we use cylindrical coordinates $\{R, \Theta, Z\}$:

$$X^1 = R\cos\Theta, \quad X^2 = R\sin\Theta, \quad X^3 = Z.$$

The cylindrical coordinates define the local basis $\{\boldsymbol{e}_R, \boldsymbol{e}_\Theta, \boldsymbol{e}_Z\}$ and reciprocal basis $\{\boldsymbol{e}^R, \boldsymbol{e}^\Theta, \boldsymbol{e}^Z\}$. Elements of them can be presented by the decompositions

$$\boldsymbol{e}_R = \boldsymbol{e}^R = \boldsymbol{i}_1 \cos\Theta + \boldsymbol{i}_2 \sin\Theta,$$
$$\boldsymbol{e}_\Theta = -\boldsymbol{i}_1 R\sin\Theta + \boldsymbol{i}_2 R\cos\Theta,\ \boldsymbol{e}^\Theta = \frac{\boldsymbol{e}_\Theta}{R^2},\ \boldsymbol{e}^Z = \boldsymbol{e}_Z = \boldsymbol{i}_3.$$

The reference positions of material points in simplest form can be written as $\boldsymbol{X} = R\boldsymbol{e}_R + Z\boldsymbol{e}_Z$. Taking into account the central symmetry, the independence with respect to vertical coordinate Z, and the condition of incompressibility $|d\boldsymbol{x}/d\boldsymbol{X}| = 1$ we arrive at the following family of mapping (universal deformations belonging to the family 3 according to the classification given in [18])

$$\boldsymbol{x}(\boldsymbol{X}) = \boldsymbol{e}_R \sqrt{(\boldsymbol{e}^R \cdot \boldsymbol{X})^2 + a} + \boldsymbol{e}_Z \otimes \boldsymbol{e}^Z \cdot \boldsymbol{X}. \qquad (4)$$

Here a is a deformation parameter that represents the change of the outer cylindrical surface radius. The deformation gradient $\boldsymbol{F}$ and left Cauchy–Green tensor $\boldsymbol{B} = \boldsymbol{F} \cdot \boldsymbol{F}^*$ (hereinafter the symbol $*$ denotes the transpose) are determined in terms of local basis corresponded to the reference position as follows

$$\boldsymbol{F} = \frac{R}{\sqrt{R^2+a}}\, \boldsymbol{e}_R \otimes \boldsymbol{e}_R + \frac{\sqrt{R^2+a}}{R^3}\, \boldsymbol{e}_\Theta \otimes \boldsymbol{e}_\Theta + \boldsymbol{e}_Z \otimes \boldsymbol{e}_Z.$$

The decomposition of tensor $\boldsymbol{B}$ and its inverse in the terms of the elements of the local basis corresponded to the actual position, i.e.

$$\boldsymbol{e}_R = \boldsymbol{e}_r, \quad \boldsymbol{e}_\Theta = \frac{\sqrt{r^2-a}}{r}, \quad \boldsymbol{e}_\theta, \boldsymbol{e}_Z = \boldsymbol{e}_z,$$

have the forms

$$\boldsymbol{B} = \frac{r^2-a}{r^2}\, \boldsymbol{e}_r \otimes \boldsymbol{e}_r + \frac{1}{r^2-a}\, \boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta + \boldsymbol{e}_z \otimes \boldsymbol{e}_z, \qquad (5)$$

$$\boldsymbol{B}^{-1} = \frac{r^2}{r^2-a}\, \boldsymbol{e}_r \otimes \boldsymbol{e}_r + \frac{r^2-a}{r^4}\, \boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta + \boldsymbol{e}_z \otimes \boldsymbol{e}_z.$$

If the cylindrical body-fiber is produced from an incompressible material of Mooney–Rivlin type then the strain energy can be presented as a linear function of the first $I_1 = I_1(\boldsymbol{B})$ and second $I_2 = I_2(\boldsymbol{B})$ invariants of tensor $\boldsymbol{B}$, i.e.:

$$W(I_1, I_2) = C_1(I_1 - 3) + C_2(I_2 - 3),$$
$$I_1 = \mathrm{Tr}\boldsymbol{B} = 3 + \frac{a^2}{r^2(r^2-a)}, \quad I_2 = I_1.$$

Here C_1, C_2 are material constants. Under the conditions of incompressibility we have the following decomposition of Cauchy stress tensor [21]

$$\boldsymbol{T} = -p\boldsymbol{I} + J_1\boldsymbol{B} + J_{-1}\boldsymbol{B}^{-1},$$

where p is hydrostatic pressure, $J_1 = 2\partial W/\partial I_1 = 2C_1$ and $J_{-1} = -2\partial W/\partial I_2 = -2C_2$ are coefficients of reaction, and $\boldsymbol{I}$ is a unit tensor. Note that constants C_1, C_2 can be defined by pair of engineering constants μ, β, i.e.:

$$C_1 = \mu(1+\beta)/4, \quad C_2 = \mu(1-\beta)/4.$$

Here μ corresponds to the shear modulus and β defines the additional parameter for nonlinear response. From thermodynamical restriction it follows that $-1 < \beta < 1$ [18].

After simple calculations we obtain the following:

$$\boldsymbol{T} = T^{rr}\boldsymbol{e}_r \otimes \boldsymbol{e}_r + T^{\theta\theta}\boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta + T^{zz}\boldsymbol{e}_z \otimes \boldsymbol{e}_z,$$
$$T^{rr} = -p + J_1\frac{r^2-a}{r^2} + J_{-1}\frac{r^2}{r^2-a},$$
$$T^{\theta\theta} = -\frac{p}{r^2} + J_1\frac{1}{r^2-a} + J_{-1}\frac{r^2-a}{r^4},$$
$$T^{zz} = -p + J_1 + J_{-1}.$$

Hydrostatic stress component p can be determined by the equilibrium equation $\nabla \cdot \boldsymbol{T} = \boldsymbol{0}$. Integrating this equation with respect to r we get

$$T^{rr} = \frac{\mu}{2}\left(\ln\frac{r^2-a}{r^2} - \frac{a}{r^2}\right) + p_0, \qquad (6)$$
$$T^{\theta\theta} = \frac{T^{rr}}{r^2} + \frac{\mu}{r^2}\left(\frac{r^2}{r^2-a} - \frac{r^2-a}{r^2}\right),$$
$$T^{zz} = T^{rr} + \mu\, a\,\frac{r^2 - (1+\beta)\,a/2}{r^2(r^2-a)}, \qquad (7)$$

where p_0 is the constant of integration. Note that in the terms of physical basis $\mathbf{e}_{\langle r\rangle} = \mathbf{e}_r$, $\mathbf{e}_{\langle\theta\rangle} = \mathbf{e}_\theta/r$, $\mathbf{e}_{\langle z\rangle} = \mathbf{e}_z$ the stresses have the form

$$T_{\langle rr\rangle} = T^{rr}, \quad T_{\langle\theta\theta\rangle} = T^{\theta\theta}r^2, \quad T_{\langle zz\rangle} = T^{zz}.$$

Thus, the deformations and stresses can be defined up to the parameters a and p_0. This implies that the boundary conditions may be satisfied exactly only on the cylindrical surfaces if the constant hydrostatic load intensity p_i and p_e are given

$$\boldsymbol{T} \cdot \boldsymbol{e}^r\big|_{r=r_i} = p_i\boldsymbol{e}_r, \quad \boldsymbol{T} \cdot \boldsymbol{e}^r\big|_{r=r_e} = p_e\boldsymbol{e}_r, \qquad (8)$$

Here r_i, r_e are the radii of the inner and outer cylindrical boundary surfaces.

Substituting expressions for the radial component of the stress (7) to the boundary conditions (8) and taking into

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account the kinematic relations (4) we obtain the system of equations

$$\begin{cases} \frac{\mu}{2}\left(\ln\frac{R_i^2}{R_i^2+a} - \frac{a}{R_i^2+a}\right) + p_0 = p_i, \\ \frac{\mu}{2}\left(\ln\frac{R_e^2}{R_e^2+a} - \frac{a}{R_e^2+a}\right) + p_0 = p_e, \end{cases} \tag{9}$$

where $R_i = \sqrt{r_i^2 - a}$, $R_e = \sqrt{r_e^2 - a}$ are reference values of the radii of the boundary surfaces. After eliminating of the parameter p_0 from the resulting system we obtain the equation with respect to the parameter a:

$$\ln\left(\frac{R_i^2}{R_e^2}\frac{R_e^2+a}{R_i^2+a}\right) = 2\,\frac{p_i - p_e}{\mu} + a\frac{R_e^2 - R_i^2}{(R_i^2+a)(R_e^2+a)}.$$

Let $x = a/R_e^2$ be a new variable that can be interpreted as a relative deformation parameter. Potentiating the left and right hand-sides of the resulting expression we obtain the equation with respect to x

$$F = 0, \quad F = \frac{1+x}{\gamma+x} - \frac{A}{\gamma}\, e^{x\frac{1-\gamma}{(1+x)(\gamma+x)}}. \tag{10}$$

Here $\gamma = R_i^2/R_e^2$, $A = \exp[2(p_i - p_e)/\mu]$. Because the internal radius in the reference configuration has always positive value then $0 < \gamma < 1$. Furthermore $x > -\gamma$. Limit relations

$$\lim_{x\to-\gamma} F = \infty, \quad \lim_{x\to\infty} F = 1 - \frac{A}{\gamma}$$

show that equation (10) has a solution only if $A > \gamma$, i.e. there is a limit for the difference of hydrostatic load intensities: $p_i - p_e > \mu/2\ln\gamma$.

If the value of x is determined then the absolute deformation parameter $a = R_e^1 x$ can be calculated and the corresponding value p_0 may be also determined. Thus, for given values of hydrostatic loads p_i, p_e and radii of the boundary surfaces R_i, R_e one can define the parameters a, p_0 and all components of strain tensors (5) and stresses (7) as well.

IV. Discrete growth

Consider a finite set of bodies. Let the elements of this set be the circular hollow cylinders of equal height h (in natural configuration). The motion (4) transform them to the hollow cylinders of the same height, but of another radii. Such deformation can be realized, e.g. by expanding the hollow cylinder which base lie on the smooth rigid slabs. We assume that the images of the actual configuration of the cylinders are pairwise disjoint and their union is a connected set. The final composite body can be treated as a result of discrete growth because cylindrical parts cannot deform independently after joining.

Let N be the number of cylindrical parts. Assume that the following scenario of growth is realized. On the first step the joining of the 1-st and 2-d body-fibers is performed. A composite body appears which we call the first assembly. Then the third body is joint to the composite body, etc. On the internal $r = r_{i,n}^1$ and the outer boundary $r = r_{e,n}^n$ of this composite bodies the uniformly distributed pressure $p_{i,n}$ and $p_{e,n}$ are defined

$$\boldsymbol{T}\cdot\boldsymbol{e}^r\big|_{r=r_{i,n}^1} = p_{i,n}\boldsymbol{e}_r, \quad \boldsymbol{T}\cdot\boldsymbol{e}^r\big|_{r=r_{e,n}^n} = p_{e,n}\boldsymbol{e}_r. \tag{11}$$

Index n indicates the number of assembly. The indexing in the notation of intensity of hydrostatic loads $p_{i,n}$, $p_{e,n}$ shows that they may vary during the growing process. Suppose that the contact between body-fibers is ideal, i.e. inner surface of k-th fiber and the outer surface of $k+1$-th fiber in the actual configuration are the same and stresses on them are in equilibrium:

$$\boldsymbol{T}\boldsymbol{e}^r\big|_{r=r_{e,n}^k} = \boldsymbol{T}\boldsymbol{e}^r\big|_{r=r_{i,n}^{k+1}}, \quad r_{e,n}^k = r_{i,n}^{k+1}, \; k = 1,2,...n{-}1. \tag{12}$$

The deformation parameters a_n^k and parameters $p_{0,n}^k$, $k = 1,2,\ldots,n$ may be found from the system of $2n$ nonlinear equations (11) and (12). Taking into account (4) and (7) we get

$$\begin{aligned} &\frac{\mu}{2}\left[\ln\frac{(R_i^1)^2}{(R_i^1)^2+a_n^1} - \frac{a_n^1}{(R_i^1)^2+a_n^1}\right] + p_{0,n}^1 = p_{i,n}, \\ &\frac{\mu}{2}\left[\ln\frac{(R_e^n)^2}{(R_e^n)^2+a_n^n} - \frac{a_n^n}{(R_e^n)^2+a_n^n}\right] + p_{0,n}^n = p_{e,n}, \\ &\frac{\mu}{2}\left[\ln\frac{(R_e^k)^2}{(R_e^k)^2+a_n^k} - \frac{a_n^k}{(R_e^k)^2+a_n^k}\right] + p_{0,n}^k = \\ &\quad = \frac{\mu}{2}\left[\ln\frac{(R_i^{k+1})^2}{(R_i^{k+1})^2+a_n^{k+1}} - \frac{a_n^{k+1}}{(R_i^{k+1})^2 + a_n^{k+1}}\right] + p_{0,n}^{k+1}, \\ &(R_e^k)^2 + a_n^k = (R_i^{k+1})^2 + a_n^{k+1}, \; k = 1,2,...,n-1. \end{aligned} \tag{13}$$

Let $\alpha_k = 1 + A_k/(R_e^k)^2$, $\beta_k = \gamma_k + A_k/(R_e^k)^2$, $\nu_k = (R_e^1)^2/(R_e^k)^2$, $x_n = a_n^1/(R_e^1)^2$, $W_n = e^{2\frac{p_{i,n}-p_{e,n}}{\mu}}$

$$A_1 = 0, \; A_k = \sum_{p=2}^{k}\left((R_e^{p-1})^2 - (R_i^p)^2\right), \; k = 2,3,\ldots,n,$$

$$\gamma_k = \left(\frac{R_i^k}{R_e^k}\right)^2.$$

Potentiating of the left and right hand sides of above equation, we obtain

$$\prod_{k=1}^{n}\gamma_k\frac{\alpha_k + \nu_k x_n}{\beta_k + \nu_k x_n} = W_n \exp\left[\sum_{k=1}^{n}(1-\gamma_k)\frac{\nu_k x_n + \alpha_k - 1}{(\beta_k + \nu_k x_n)(\alpha_k + \nu_k x_n)}\right]. \tag{14}$$

We distinguish the following types of growth

1) Growth with a prescribed reference geometry. Here we suppose that the geometrical characteristics of body-fibers in the image of natural configuration are given, i.e. the reference radii of the unstrained body-fibers R_i^k and R_e^k are known.
2) Growth with a given actual geometry. The position of growing boundaries in the image of the actual configuration $\mathcal{R}_n$ and the thickness of body-fibers in the reference configuration are known, i.e. the values $\Delta^k = R_e^k - R_i^k$, $k = 1,\ldots,n$ are prescribed.

Let us consider these types of growth in detail.

Type 1. Growth with a given reference geometry. Using given values of the reference radius R_i^k, R_e^k, $k = 1,\ldots,n$ one can calculate values of α_k, γ_k, β_k, ν_k, and taking into account given values of hydrostatic load $p_{i,n}$, $p_{e,n}$ calculate the values of W_n. As a result one obtain a series of uncoupled non-linear equations (14). The solutions of this equations

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determines deformation parameters x_n independently. Thereafter one may calculate a_n^k, $k = 1, \dots, n$ and define stresses by the relations (7).

Type 2. Growth with a given actual geometry. In this case the reference radii of the body-fibers fibers are not known a priori, and the equations (14) have to be supplemented by additional equations that define the radius of growing boundary $\mathcal{R}_n$ in actual configurations

$$r_{e,n}^n = \sqrt{(R_e^n)^2 + a_n^n} = \mathcal{R}_n.$$

To analyze the system of equations firstly allocate in the left and right hand sides of the equation (14) the terms corresponding to the n-th body-fiber, i.e.:

$$\gamma_n \frac{\alpha_n + \nu_n x_n}{\beta_n + \nu_n x_n} \prod_{k=1}^{n-1} \gamma_k \frac{\alpha_k + \nu_k x_n}{\beta_k + \nu_k x_n} = \\ = W_n \exp\left[\frac{(1-\gamma_n)(\nu_n x_n + \alpha_n - 1)}{(\beta_n + \nu_n x_n)(\alpha_n + \nu_n x_n)} + \right. \\ \left. + \sum_{k=1}^{n-1} \frac{(1-\gamma_k)(\nu_k x_n + \alpha_k - 1)}{(\beta_k + \nu_k x_n)(\alpha_k + \nu_k x_n)}\right]. \quad (15)$$

Unlike *type 1* the values of γ_n, α_n, β_n, ν_n, can't be defined a priori, because they depend on the dimensionless deformation parameter x_n. In fact, since

$$(R_e^n)^n = \mathcal{R}_n^2 - a_n^n = \mathcal{R}_n^2 - (R_e^1)^2 x_n - A_n, \quad R_i^n = R_e^n - \Delta^n,$$

the expression for A_n

$$A_n = A_{n-1} + (R_e^{n-1})^2 - (R_i^n)^2$$

is the algebraic equation whose solution determines A_n through the parameters with indices $m < n$, actual radius of the growing border $\mathcal{R}_n$ and the deformation parameter x_n. Substituting these expressions into equation (15) leads to the explicit form of non-linear equations:

$$\left(1 + \frac{2\,\xi_n}{\sigma_n - x_n}\right)^2 \frac{\zeta_n}{\phi_{n-1} + x_n} \prod_{k=1}^{n-1} \gamma_k \frac{\alpha_k + \nu_k x_n}{\beta_k + \nu_k x_n} = \\ = W_n \exp\left[\frac{\sigma_n - x_n - \xi_n}{\phi_{n-1} + x_n}\left(1 - \frac{(\sigma_n - x_n)^2}{4\zeta_n \xi_n}\right) + \right. \\ \left. + \sum_{k=1}^{n-1} \frac{(1-\gamma_k)(\nu_k x_n + \alpha_k - 1)}{(\beta_k + \nu_k x_n)(\alpha_k + \nu_k x_n)}\right]. \quad (16)$$

$$\xi_n = \left(\frac{\Delta^n}{R_e^1}\right)^2, \quad \sigma_n = \frac{\mathcal{H}_n}{(R_e^1)^2}, \quad \zeta_n = \left(\frac{\mathcal{R}_n}{R_e^1}\right)^2, \\ \phi_{n-1} = \frac{A_{n-1} + (R_e^{n-1})^2}{(R_e^1)^2}, \\ \mathcal{H}_n = \mathcal{R}_n^2 - A_{n-1} - (R_e^{n-1})^2 + (\Delta_n)^2.$$

V. Continuous growth

In the case of continuous growth it is convenient to introduce an intermediate configuration which image is not free from stresses but it may be immersed in Euclidean space. If such intermediate configurations perform additional conditions, i.e. the local configuration in the neighborhood of any interior point does not change during the growing process, then the total local deformation, which transforms the neighborhood of material point to the actual state, can be presented as a multiplicative decomposition

$$\boldsymbol{H} = \boldsymbol{F} \cdot \boldsymbol{K}, \quad \mathrm{rot}\boldsymbol{F} = \boldsymbol{0}, \quad \mathrm{rot}\boldsymbol{K} \neq \boldsymbol{0}, \quad \dot{\boldsymbol{K}} = \boldsymbol{0},$$

where $\dot{\boldsymbol{K}}$ denotes the derivative with respect to time, or to a time like parameter.

In general, the intermediate configuration is not compatible with actual external fields acting on the growing body. Thus one must attach a system of fictitious mass and surface forces which have the character of Eshelby forces [15].

Bearing in mind the idea of a bundle as a continual family of material surfaces, which separately has natural (stress-free) configuration in Euclidean space, we can present the system of fictitious forces by continuous family of surface loads that hold the material surfaces in assembly.

Thus, with each material surface one can associate the deformation that transforms the surface from the intermediate configuration to the unstressed state

$$\mathbf{K} = \frac{R}{\sqrt{R^2 + \alpha(R)}} \mathbf{e}_R \otimes \mathbf{e}^R + \frac{\sqrt{R^2 + \alpha(R)}}{R} \mathbf{e}_\Theta \otimes \mathbf{e}^\Theta + \\ + \mathbf{e}_Z \otimes \mathbf{e}^Z.$$

Note that the union of these fields determines a single field of linear transformations (a three-dimensional field of second-rank tensors) which are not gradients of any vector field in Euclidean space.

The body deforms from the intermediate configuration to the actual in conventional sense. So it is subjected to the deformation $\mathbf{F}$ which has the form

$$\mathbf{F} = \frac{\tilde{r}}{\sqrt{\tilde{r}^2 + A(t)}} \mathbf{e}_{\tilde{r}} \otimes \mathbf{e}^{\tilde{r}} + \frac{\sqrt{\tilde{r}^2 + A(t)}}{\tilde{r}} \mathbf{e}_{\tilde{\theta}} \otimes \mathbf{e}^{\tilde{\theta}} + \mathbf{e}_{\tilde{z}} \otimes \mathbf{e}^{\tilde{z}}.$$

In this case the total distortion and corresponding strain are the following

$$\mathbf{H} = \mathbf{F} \cdot \mathbf{K}, \ \mathbf{B} = \mathbf{H} \cdot \mathbf{H}^*, \\ \boldsymbol{H}(\tilde{r}, t) = \frac{\sqrt{\tilde{r}^2 - \alpha(\tilde{r})}}{\sqrt{\tilde{r}^2 + A(t)}} \mathbf{e}_{\tilde{r}} \otimes \mathbf{e}^{\tilde{r}} + \frac{\sqrt{\tilde{r}^2 + A(t)}}{\sqrt{\tilde{r}^2 - \alpha(\tilde{r})}} \mathbf{e}_{\tilde{\theta}} \otimes \mathbf{e}^{\tilde{\theta}} + \\ + \mathbf{e}_{\tilde{z}} \otimes \mathbf{e}^{\tilde{z}}.$$

Considering the general case we assume that growth starts on a non-empty initial body which is a hollow cylinder that is free of stresses at initial instant. Its inner and outer radii are $\tilde{r}_i$, $\tilde{r}_e$ respectively. Cylindrical material surfaces are attached to the outer surface of the body continuously increasing its external radius in the intermediate configuration. Let actual value of this radius is r_g. Suppose that on cylindrical surfaces of the growing body we have hydrostatic load p_e and p_i, i.e.

$$\boldsymbol{T} \cdot \boldsymbol{e}^{\tilde{r}}\Big|_{\tilde{r}=\tilde{r}_i} = p_i \boldsymbol{e}_{\tilde{r}}, \quad \boldsymbol{T} \cdot \boldsymbol{e}^{\tilde{r}}\Big|_{\tilde{r}=\tilde{r}_e} = p_e \boldsymbol{e}_{\tilde{r}}, \quad (17)$$

Then physical components of stresses can be presented by the formulas

$$T_{<\tilde{r}\tilde{r}>} = \begin{cases} I(\tilde{r}, A) + p_i, \ \tilde{r}_0 \le \tilde{r} \le \tilde{r}_1 \\ I(\tilde{r}_1, A) + p_i + \int\limits_{\tilde{r}_1}^{\tilde{r}} \rho\left(\frac{1}{\rho^2 - \alpha} - \frac{\rho^2 - \alpha}{(\rho^2 + A)^2}\right) d\rho, \ \tilde{r}_1 \le \tilde{r} \le \tilde{r}_g \end{cases}$$

$$T_{<\tilde{\theta}\tilde{\theta}>} = T_{<\tilde{r}\tilde{r}>} + \begin{cases} \frac{\tilde{r}^2 + A}{\tilde{r}^2} - \frac{\tilde{r}^2}{\tilde{r}^2 + A}, \ \tilde{r}_0 \le \tilde{r} \le \tilde{r}_1 \\ \frac{\tilde{r}^2 + A}{\tilde{r}^2 - \alpha} - \frac{\tilde{r}^2 - \alpha}{\tilde{r}^2 + A}, \ \tilde{r}_1 \le \tilde{r} \le \tilde{r}_g \end{cases} \quad (18)$$

$$T_{<\tilde{z}\tilde{z}>} = T_{<\tilde{r}\tilde{r}>} + \begin{cases} \frac{\tilde{r}^2 + A - (1+\beta)A/2}{(\tilde{r}^2 + A)\tilde{r}^2} A, \ \tilde{r}_0 \le \tilde{r} \le \tilde{r}_1 \\ \frac{\tilde{r}^2 + A - (1+\beta)(A+\alpha)/2}{(\tilde{r}^2 + A)(\tilde{r}^2 + \alpha)} (A + \alpha), \ \tilde{r}_1 \le \tilde{r} \le \tilde{r}_g \end{cases}$$

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$$I(\tilde{r},A)=\int\limits_{\tilde{r}_0}^{\tilde{r}}\rho\Big(\frac{1}{\rho^2}-\frac{\rho^2}{(\rho^2+A)^2}\Big)\,d\rho=$$
$$=\ln\frac{\tilde{r}\sqrt{A+\tilde{r}_0^2}}{r_0\sqrt{A+\tilde{r}^2}}-\frac{A}{2}\Big(\frac{1}{A+\tilde{r}_0^2}+\frac{1}{A+\tilde{r}^2}\Big). \quad (19)$$

Radial stresses $T_{<\tilde{r}\tilde{r}>}$ in the neighborhood of growing boundary are defined by the formula

$$T_{<\tilde{r}\tilde{r}>}=I(\tilde{r}_1,A)+p_i+\int\limits_{\tilde{r}_1}^{\tilde{r}_g}\rho\Big(\frac{1}{\rho^2-\alpha}-\frac{\rho^2-\alpha}{(\rho^2+A)^2}\Big)d\rho. \quad (20)$$

Circumferential stresses $T_{<\tilde{\theta}\tilde{\theta}>}$ in the neighborhood of growing boundary can be defined by the relation

$$T_{<\tilde{\theta}\tilde{\theta}>|_{\tilde{r}=\tilde{r}_g}}=p_e+\frac{\tilde{r}^2+A}{\tilde{r}^2-\alpha(\tilde{r}_g)}-\frac{\tilde{r}^2-\alpha(\tilde{r}_g)}{\tilde{r}^2+A}. \quad (21)$$

Radius of the outer cylindrical boundary in the actual configuration is defined by

$$r_g=\sqrt{\tilde{r}_g^2+A}. \quad (22)$$

The rate of change of the material composition of the body can be given a function $V(t)$, which determines the increasing of the volume of the growing body during the growing process. For incompressible material it is an invariant with respect to the change of configuration. Under the assumption that the inner radius of $\tilde{r}_i$ does not change, the outer one can be defined as follows

$$\tilde{r}_g=\sqrt{V(t)/(\pi h)+\tilde{r}_0^2}. \quad (23)$$

Consider the types of growth like in the case considered for discrete growth. We assume that in all cases the dependence of the volume $V(t)$ is known.

*Type 1. Growth with given distortion.*The distortion function $\alpha=\alpha(\tilde{r})$ is prescribed. To determine stress field one must find parameter $A(t)$ which is defined implicitly by the equation

$$F(A)=\Delta p,$$
$$F(A)=I(\tilde{r}_1,A)+\int\limits_{\tilde{r}_1}^{\tilde{r}_g}\rho\Big(\frac{1}{\rho^2-\alpha}-\frac{\rho^2-\alpha}{(\rho^2+A)^2}\Big)d\rho.$$

Here $\Delta p=(p_e-p_i)/\mu$. Stresses that arise in the body in the process of growth can be determined by relation (18).

Type 2. Growth with a predefined displacements of growing boundary. The position of the image of growing boundary in the actual configuration is known, i.e. the function $z=r_g(t)$ is given. Since the radial coordinate of the growing boundary in intermediate configuration is given by (23), the parameter A can be determined from equation (22), i.e.

$$A=z^2-\tilde{r}_g^2. \quad (24)$$

Substituting expressions (24) and (20) into the boundary conditions (17) we get integral equation with respect to $\alpha(\tilde{r})$

$$\int\limits_{\tilde{r}_0}^{x}\rho\Big(\frac{1}{\rho^2-\alpha}-\frac{\rho^2-\alpha(\rho)}{(\rho^2+z^2(x)-x^2)^2}\Big)d\rho=$$
$$=\Delta p-I(\tilde{r},z^2(x)-x^2),\quad \Delta p=(p_e-p_i)/\mu.$$

As a result of change of variables $\xi=\rho^2$, $y=\rho^2-\alpha(\rho)$, $\zeta=x^2$ we obtain the integral equation

$$\int\limits_{a}^{\zeta}\Big(\frac{1}{y(\xi)}-\frac{y(\xi)}{(\xi-A(\zeta))^2}\Big)d\xi=Q(\xi) \quad (25)$$

with respect to function $y(\xi)$. Here the functions $A(\zeta)=z^2(\sqrt{\xi})-\zeta$ $Q(\zeta)=2(\Delta p(\sqrt{\xi})-I(\tilde{r}_1,A(\zeta)))$ are prescribed.

The solution of this equation defines function $\alpha(\rho)$ and therefore all stresses (18).

The computational examples show the convergence of solutions obtained for the discrete growth to corresponding solutions for continuous growth under the following conditions: the number of discrete body-fibers increases while their thickness decreases such that the final volume of growing solid is fixed.

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