Proceedings of the World Congress on Engineering 2014 Vol II,
WCE 2014, July 2 - 4, 2014, London, U.K.

A Mixed Variational Formulation for a Slip-dependent Frictional Contact Model

Andaluzia Cristina Matei *†

Abstract—**A 3D slip-dependent frictional contact problem in elastostatics is discussed. We deliver a variational formulation as a mixed variational problem whose Lagrange multipliers set is solution-dependent. Then, the existence and the boundedness of the solutions is investigated. The proof is based on a recent result for an abstract mixed variational problem with solution-dependent set of Lagrange multipliers.**

Keywords: slip-dependent frictional contact problem, mixed variational problems, solution-dependent set of Lagrange multipliers, weak solutions.

1 Introduction

The weak formulations of contact problems are related to the theory of variational inequalities, see e.g. [3, 9], or to the theory of saddle point problems, see e.g. [2, 4].

The first mathematical results on contact problem with slip displacements dependent friction in elastostatics were obtained in [6].

In the present work we focus on a 3D contact model with slip dependent coefficient of friction, for linearly elastic materials. This model was already analyzed into the framework of quasi-variational inequalities, see [1]. The novelty in the present paper consists in the variational approach we use; herein, a mixed variational formulation is proposed, in a form of a generalized saddle point problem, the set of the Lagrange multipliers being solution-dependent.

The mixed variational formulations are related to modern numerical techniques in order to approximate the weak solutions of contact models and this motivates the present study. Referring to numerical techniques for approximating weak solutions of contact problems via saddle point technique, we send the reader to, e.g., [5, 10, 11].

*Manuscript received February 5, 2014; revised March 19, 2014. This work was supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PN-II-RU-TE-2011-3-0223.

†Department of Mathematics, University of Craiova, A. I. Cuza 13, 200585, Craiova, Romania, e-mail: andaluziamatei@inf.ucv.ro (see also http://www.math.ucv.ro/∼amatei).

2 The model

The classical model for a 3D slip-dependent contact process is the following one.

Problem 1 *Find* $\boldsymbol{u} : \overline{\Omega} \to R^3$ *and* $\boldsymbol{\sigma} : \overline{\Omega} \to S^3$ *such that*

$$
\begin{aligned}
&Div\, \boldsymbol{\sigma}(\boldsymbol{x}) + \boldsymbol{f}_0(\boldsymbol{x}) = \boldsymbol{0} && in\ \Omega, && (1)\\
&\boldsymbol{\sigma}(\boldsymbol{x}) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})) && in\ \Omega, && (2)\\
&\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{0} && on\ \Gamma_1, && (3)\\
&\boldsymbol{\sigma}(\boldsymbol{x})\,\boldsymbol{\nu}(\boldsymbol{x}) = \boldsymbol{f}_2 && on\ \Gamma_2, && (4)\\
&u_\nu(\boldsymbol{x}) = 0 && on\ \Gamma_3, && (5)\\
&\|\boldsymbol{\sigma}_\tau(\boldsymbol{x})\| \le g(\boldsymbol{x}, \|\boldsymbol{u}_\tau(\boldsymbol{x})\|), && && \\
&\boldsymbol{\sigma}_\tau(\boldsymbol{x}) = -g(\boldsymbol{x}, \|\boldsymbol{u}_\tau(\boldsymbol{x})\|)\,\frac{\boldsymbol{u}_\tau(\boldsymbol{x})}{\|\boldsymbol{u}_\tau(\boldsymbol{x})\|} && && \\
&if\ \boldsymbol{u}_\tau(\boldsymbol{x}) \ne \boldsymbol{0} && on\ \Gamma_3. && (6)
\end{aligned}
$$

Problem 1 has the following structure: (1) represents the equilibrium equation, (2) represents the constitutive law for linearly elastic materials, (3) represents the displacements boundary condition, (4) represents the traction boundary condition and (5)-(6) model the bilateral contact with slip-dependent coefficient of friction g. Notice that $u_\nu = \boldsymbol{u}\cdot\boldsymbol{\nu}$, $\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu\boldsymbol{\nu}$, $\sigma_\nu = (\boldsymbol{\sigma\nu})\cdot\boldsymbol{\nu}$, $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu\boldsymbol{\nu}$, where " $\cdot$ " denotes the inner product of two vectors and $\boldsymbol{\nu}$ is the unit outward normal vector. The domain Ω is a bounded domain in R^3 and Γ_1, Γ_2, Γ_3 is a partition of the boundary $\partial\Omega := \Gamma$. For details on this model we refer to [1].

3 Assumptions

In order to weakly solve Problem 1 we make the following assumptions.

Assumption 1 $\mathcal{E} = (\mathcal{E}_{ijls}) : \Omega \times S^3 \to S^3$,

- $\mathcal{E}_{ijls} = \mathcal{E}_{ijsl} = \mathcal{E}_{lsij} \in L^\infty(\Omega)$,
- *There exists* $m_\mathcal{E} > 0$ *such that* $\mathcal{E}_{ijls}\varepsilon_{ij}\varepsilon_{ls} \ge m_\mathcal{E}\,|\boldsymbol{\varepsilon}|^2$, $\boldsymbol{\varepsilon} \in S^3$, *a.e. in* Ω.

Assumption 2 $\boldsymbol{f}_0 \in L^2(\Omega)^3$, $\quad \boldsymbol{f}_2 \in L^2(\Gamma_2)^3$.

ISBN: 978-988-19253-5-0
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Assumption 3 $g : \Gamma_3 \times R_+ \to R_+$

- *there exists* $L_g > 0$:
  $|g(\boldsymbol{x}, r_1) - g(\boldsymbol{x}, r_2)| \leq L_g |r_1 - r_2| \quad r_1, r_2 \in R_+$, *a.e.* $\boldsymbol{x} \in \Gamma_3$;
- *the mapping* $\boldsymbol{x} \mapsto g(\boldsymbol{x}, r)$ *is Lebesgue measurable on* Γ_3, *for all* $r \in R_+$;
- *the mapping* $\boldsymbol{x} \mapsto g(\boldsymbol{x}, 0)$ *belongs to* $L^2(\Gamma_3)$.

4 Weak formulation

Let us introduce the following functional space.

$$V = \{\boldsymbol{v} \in H^1(\Omega)^3 \,|\, \boldsymbol{\gamma v} = 0 \text{ on } \Gamma_1,\, v_\nu = 0 \text{ on } \Gamma_3\}. \quad (7)$$

Notice that, everywhere in this paper, for each $\boldsymbol{w} \in V$, $w_\nu = \boldsymbol{\gamma w} \cdot \boldsymbol{\nu}$ and $\boldsymbol{w}_\tau = \boldsymbol{\gamma w} - w_\nu \boldsymbol{\nu}$ a.e. on Γ, where $\boldsymbol{\gamma}$ denotes the Sobolev trace operator for vectors.

Define $\boldsymbol{f} \in V$ using Riesz's representation theorem,

$$(\boldsymbol{f}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, dx + \int_{\Gamma_2} \boldsymbol{f}_2(\boldsymbol{x}) \cdot \boldsymbol{\gamma v}(\boldsymbol{x})\, d\Gamma \quad (8)$$

for all $\boldsymbol{v} \in V$.

Let $\boldsymbol{u}$ be a sufficiently regular solution of Problem 1. By a Green formula we get, for all $\boldsymbol{v} \in V$,

$$a(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})_V + \int_{\Gamma_3} \boldsymbol{\sigma}_\tau(\boldsymbol{x}) \cdot \boldsymbol{v}_\tau(\boldsymbol{x})\, d\Gamma \quad (9)$$

where $a(\cdot,\cdot) : V \times V \to R$,

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})) : \boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x}))\, dx. \quad (10)$$

Notice that " : " denotes the inner product of two tensors.

Let us introduce the spaces

$$S = \{\boldsymbol{\gamma w} \,|\, \boldsymbol{w} \in V\}; \quad (11)$$
$$D = S'. \quad (12)$$

For each $\boldsymbol{\varphi} \in V$ we define

$$\boldsymbol{\Lambda}(\boldsymbol{\varphi}) = \{\boldsymbol{\mu} \in D \,|\, <\boldsymbol{\mu}, \boldsymbol{\gamma v}> \leq \int_{\Gamma_3} g(\boldsymbol{x}, \|\boldsymbol{\varphi}_\tau(\boldsymbol{x})\|)\|\boldsymbol{v}_\tau(\boldsymbol{x})\|\, d\Gamma \quad \boldsymbol{v} \in V\}; \quad (13)$$

here and below $< \cdot, \cdot >$ denotes the duality pairing between D and S.

Let us define a Lagrange multiplier $\boldsymbol{\lambda} \in D$,

$$<\boldsymbol{\lambda}, \boldsymbol{\zeta}> = -\int_{\Gamma_3} \boldsymbol{\sigma}_\tau(\boldsymbol{x}) \cdot [\boldsymbol{\zeta} - (\boldsymbol{\zeta} \cdot \boldsymbol{\nu})\boldsymbol{\nu}](\boldsymbol{x})\, d\Gamma \quad (14)$$

for all $\boldsymbol{\zeta} \in D$.

By (14) and (6) we deduce that $\boldsymbol{\lambda} \in \Lambda(\boldsymbol{u})$.

We also define

$$b : V \times D \to R \quad b(\boldsymbol{v}, \boldsymbol{\mu}) = <\boldsymbol{\mu}, \boldsymbol{\gamma v}>. \quad (15)$$

Let us rewrite (9) as

$$a(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})_V - \langle \boldsymbol{\lambda}, \boldsymbol{\gamma v} \rangle \quad \text{for all } \boldsymbol{v} \in V.$$

By the definition of the form $b(\cdot,\cdot)$, we obtain

$$a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, \boldsymbol{\lambda}) = (\boldsymbol{f}, \boldsymbol{v})_V \quad \text{for all } \boldsymbol{v} \in V. \quad (16)$$

The friction law (6) leads us to the identity

$$\int_{\Gamma_3} \boldsymbol{\sigma}_\tau(\boldsymbol{x}) \cdot \boldsymbol{u}_\tau(\boldsymbol{x})\, d\Gamma = -\int_{\Gamma_3} g(\boldsymbol{x}, \|\boldsymbol{u}_\tau(\boldsymbol{x})\|)\|\boldsymbol{u}_\tau(\boldsymbol{x})\|\, d\Gamma.$$

Thus,

$$b(\boldsymbol{u}, \boldsymbol{\lambda}) = \int_{\Gamma_3} g(\boldsymbol{x}, \|\boldsymbol{u}_\tau(\boldsymbol{x})\|)\|\boldsymbol{u}_\tau(\boldsymbol{x})\|\, d\Gamma. \quad (17)$$

By (13) with $\boldsymbol{\varphi} = \boldsymbol{u}$ we are led to

$$b(\boldsymbol{u}, \boldsymbol{\zeta}) \leq \int_{\Gamma_3} g(\boldsymbol{x}, \|\boldsymbol{u}_\tau(\boldsymbol{x})\|)\|\boldsymbol{u}_\tau(\boldsymbol{x})\|\, d\Gamma \quad (18)$$

for all $\boldsymbol{\zeta} \in \Lambda(\boldsymbol{u})$. Subtract now (17) from (18) to obtain the inequality

$$b(\boldsymbol{u}, \boldsymbol{\zeta} - \boldsymbol{\lambda}) \leq 0 \quad \text{for all } \boldsymbol{\zeta} \in \Lambda(\boldsymbol{u}). \quad (19)$$

Therefore, Problem 1 has the following weak formulation.

Problem 2 *Find* $\boldsymbol{u} \in V$ *and* $\boldsymbol{\lambda} \in \Lambda(\boldsymbol{u}) \subset D$ *such that* (16) *and* (19) *hold true.*

Each solution of Problem 2 is called *weak solution* of Problem 1.

5 Abstract auxiliary result

Let us consider the following abstract mixed variational problem.

Problem 3 *Given* $f \in X$, $f \neq 0_X$, *find* $(u, \lambda) \in X \times Y$ *such that* $\lambda \in \Lambda(u) \subset Y$ *and*

$$a(u, v) + b(v, \lambda) = (f, v)_X \text{ for all } v \in X, \quad (20)$$
$$b(u, \mu - \lambda) \leq 0 \text{ for all } \mu \in \Lambda(u). \quad (21)$$

We made the following assumptions.

Assumption 4 *$(X,(\cdot,\cdot)_X,\|\cdot\|_X)$ and $(Y,(\cdot,\cdot)_Y,\|\cdot\|_Y)$ are two Hilbert spaces.*

Assumption 5 *$a(\cdot,\cdot): X\times X\to R$ is a symmetric bilinear form such that*

(i_1) there exists $M_a>0$:

$|a(u,v)|\le M_a\|u\|_X\|v\|_X$ *for all $u,v\in X$,*

(i_2) there exists $m_a>0$:

$a(v,v)\ge m_a\,\|v\|_X^2$ *for all $v\in X$.*

Assumption 6 *$b(\cdot,\cdot): X\times Y\to R$ is a bilinear form such that*

(j_1) there exists $M_b>0$:

$|b(v,\mu)|\le M_b\|v\|_X\|\mu\|_Y$ *for all $v\in X$, $\mu\in Y$,*

(j_2) there exists $\alpha>0$: $\inf_{\mu\in Y,\mu\neq 0_Y}\sup_{v\in X,v\neq 0_X}\dfrac{b(v,\mu)}{\|v\|_X\|\mu\|_Y}\ge\alpha$.

Assumption 7 *For each $\varphi\in X$, $\Lambda(\varphi)$ is a closed convex subset of Y such that $0_Y\in\Lambda(\varphi)$.*

Assumption 8 *Let $(\eta_n)_n\subset X$ and $(u_n)_n\subset X$ be two weakly convergent sequences, $\eta_n\rightharpoonup\eta$ in X and $u_n\rightharpoonup u$ in X, as $n\to\infty$.*

(k_1) For each $\mu\in\Lambda(\eta)$, there exists a sequence $(\mu_n)_n\subset Y$ such that $\mu_n\in\Lambda(\eta_n)$ and

$\liminf_{n\to\infty} b(u_n,\mu_n-\mu)\ge 0.$

(k_2) For each subsequence $(\Lambda(\eta_{n'}))_{n'}$ of the sequence

$(\Lambda(\eta_n))_n$, if $(\mu_{n'})_{n'}\subset Y$ such that

$\mu_{n'}\in\Lambda(\eta_{n'})$ and $\mu_{n'}\rightharpoonup\mu$ in Y as $n'\to\infty$, then $\mu\in\Lambda(\eta)$.

Theorem 1 *If Assumptions 4-8 hold true, then Problem 3 has a solution. In addition, if $(u,\lambda)\in X\times\Lambda(u)$ is a solution of Problem 3, then*

$$(u,\lambda)\in K_1\times\big(\Lambda(u)\cap K_2\big),$$

where

$$\begin{aligned}K_1 &= \{v\in X\,|\,\|v\|_X\le\frac{1}{m_a}\|f\|_X\};\\ K_2 &= \{\mu\in Y\,|\,\|\mu\|_Y\le\frac{m_a+M_a}{\alpha\, m_a}\|f\|_X\},\end{aligned}$$

m_a, α and M_a being the constants in Assumptions 5-6.

For the proof of this theorem we refer to [7].

6 Existence and boundedness results

Theorem 2 (An existence result) *If Assumptions 1 -3 hold true, then Problem 1 has a solution.*

Proof. As the spaces V and D are real Hilbert spaces then Assumption 4 is fulfilled with $X=V$ and $Y=D$.

The form $a(\cdot,\cdot)$ defined in (10) verifies Assumption 5 with

$$M_a=\|\mathcal{E}\|_\infty \text{ and } m_a=m_{\mathcal{E}},\tag{22}$$

where

$$\|\mathcal{E}\|_\infty=\max_{0\le i,j,k,l\le d}\|E_{ijkl}\|_{L^\infty(\Omega)}.$$

Let us prove (j_1) in Assumption 6. Since S is a closed subspace of H_Γ, see [8], we can write

$$|b(\boldsymbol{v},\boldsymbol{\mu})|\le\|\boldsymbol{\mu}\|_D\|\boldsymbol{\gamma v}\|_{H_\Gamma}.$$

We recall that $H_\Gamma=\boldsymbol{\gamma}(H^1(\Omega)^3)$ and the Sobolev trace operator $\boldsymbol{\gamma}: H^1(\Omega)^3\to H_\Gamma$ is a linear and continuous operator. Due to the fact that $\|\cdot\|_V$ and $\|\cdot\|_{H^1(\Omega)^3}$ are equivalent norms, we deduce that there exists $M_b>0$ such that (j_1) holds true.

We also recall that there exists a linear and continuous operator $\mathcal{Z}$ such that

$$\mathcal{Z}: H_\Gamma\to H^1(\Omega)^3\quad \boldsymbol{\gamma}(\mathcal{Z}(\boldsymbol{\zeta}))=\boldsymbol{\zeta}\quad\text{ for all }\boldsymbol{\zeta}\in H_\Gamma.$$

The operator $\mathcal{Z}$ is called *the right inverse* of the operator $\boldsymbol{\gamma}$. Notice that,

$$\boldsymbol{\gamma}(\mathcal{Z}(\boldsymbol{\gamma w}))=\boldsymbol{\gamma w}\qquad\text{for all }\boldsymbol{w}\in V.$$

Since, for each $\boldsymbol{w}\in V$, $\mathcal{Z}(\boldsymbol{\gamma w})$ has the same trace as $\boldsymbol{w}$, we deduce that for each $\boldsymbol{w}\in V$, $\mathcal{Z}(\boldsymbol{\gamma w})\in V$.

Let us prove now (j_2) in Assumption 6.

$$\begin{aligned}\|\boldsymbol{\mu}\|_D &= \sup_{\boldsymbol{\gamma w}\in S,\,\boldsymbol{\gamma w}\neq 0_S}\frac{<\boldsymbol{\mu},\boldsymbol{\gamma w}>}{\|\boldsymbol{\gamma w}\|_{H_\Gamma}}\\ &\le c\sup_{\boldsymbol{\gamma w}\in S,\,\boldsymbol{\gamma w}\neq 0_S}\frac{b(\mathcal{Z}(\boldsymbol{\gamma w}),\boldsymbol{\mu})}{\|\mathcal{Z}(\boldsymbol{\gamma w})\|_V}\\ &\le c\sup_{\boldsymbol{v}\in V,\,\boldsymbol{v}\neq\boldsymbol{0}_V}\frac{b(\boldsymbol{v},\boldsymbol{\mu})}{\|\boldsymbol{v}\|_V},\end{aligned}$$

where $c>0$. We can take

$$\alpha=\frac{1}{c}.\tag{23}$$

Obviously, $0_D\in\Lambda(\boldsymbol{\varphi})$. Also, $\Lambda(\boldsymbol{\varphi})$ is a closed convex subset of the space D. Hence, Assumption 7 is fulfilled.

Let us verify Assumption 8. To start, let $(\boldsymbol{\eta}_n)_n\subset V$ and $(\boldsymbol{u}_n)_n\subset V$ be two weakly convergent sequences, $\boldsymbol{\eta}_n\rightharpoonup\boldsymbol{\eta}$ in V and $\boldsymbol{u}_n\rightharpoonup\boldsymbol{u}$ in V, as $n\to\infty$. Let us take $\boldsymbol{\mu}\in\Lambda(\boldsymbol{\eta})$.

In order to check Assumption 8 a crucial point is the construction of an appropriate sequence in (k_1). Let us define $(\boldsymbol{\mu}_n)_n$ as follows: for each $n \geq 1$,

$$\begin{aligned} &< \boldsymbol{\mu}_n, \boldsymbol{\zeta} > \\ &= \int_{\Gamma_3} g(\boldsymbol{x}, \|\boldsymbol{\eta}_{\tau n}(\boldsymbol{x})\|) \boldsymbol{\psi}(\boldsymbol{u}_{\tau n}(\boldsymbol{x})) \\ &[\boldsymbol{\zeta} - (\boldsymbol{\zeta} \cdot \boldsymbol{\nu})\boldsymbol{\nu}](\boldsymbol{x})\, d\Gamma \\ &- \int_{\Gamma_3} g(\boldsymbol{x}, \|\boldsymbol{\eta}_\tau(\boldsymbol{x})\|) \|\boldsymbol{u}_{\tau n}(\boldsymbol{x})\|\, d\Gamma \\ &+ < \boldsymbol{\mu}, \boldsymbol{\gamma} \boldsymbol{u}_n >, \qquad \boldsymbol{\zeta} \in D, \end{aligned} \tag{24}$$

where

$$\boldsymbol{\psi}(\boldsymbol{r}) = \begin{cases} \frac{\boldsymbol{r}}{\|\boldsymbol{r}\|} & \text{if } \boldsymbol{r} \neq \boldsymbol{0}; \\ \boldsymbol{0} & \text{if } \boldsymbol{r} = \boldsymbol{0}. \end{cases}$$

Taking into account the definition in (13), we deduce that, for each positive integer n, we have $\boldsymbol{\mu}_n \in \Lambda(\boldsymbol{\eta}_n)$.

We recall here that $\boldsymbol{\gamma} : H^1(\Omega)^3 \to L^2(\Gamma)^3$ is a compact operator. Thus, since $\boldsymbol{\eta}_n \rightharpoonup \boldsymbol{\eta}$ in V and $\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}$ in V as $n \to \infty$, using the compactness of the trace operator we can write

$$\boldsymbol{\gamma}\boldsymbol{\eta}_n \to \boldsymbol{\gamma}\boldsymbol{\eta} \text{ in } L^2(\Gamma)^3 \text{ as } n \to \infty;$$
$$\boldsymbol{\gamma}\boldsymbol{u}_n \to \boldsymbol{\gamma}\boldsymbol{u} \text{ in } L^2(\Gamma)^3 \text{ as } n \to \infty.$$

Therefore,

$$\boldsymbol{u}_{\tau n}(\boldsymbol{x}) \to \boldsymbol{u}_\tau(\boldsymbol{x}) \text{ a.e. on } \Gamma_3 \text{ as } n \to \infty$$

and

$$g(\boldsymbol{x}, \|\boldsymbol{\gamma}\boldsymbol{\eta}_n(\boldsymbol{x})\|) \to g(\boldsymbol{x}, \|\boldsymbol{\gamma}\boldsymbol{\eta}(\boldsymbol{x})\|) \text{ a.e. on } \Gamma_3 \text{ as } n \to \infty.$$

Setting $\boldsymbol{\zeta} = \boldsymbol{\gamma}\boldsymbol{u}_n$ in (24) we can write

$\langle \boldsymbol{\mu}_n - \boldsymbol{\mu}, \boldsymbol{\gamma}\boldsymbol{u}_n \rangle = \int_{\Gamma_3} \Big(g(\boldsymbol{x}, \|\boldsymbol{\gamma}\boldsymbol{\eta}_n(\boldsymbol{x})\|) - g(\boldsymbol{x}, \|\boldsymbol{\gamma}\boldsymbol{\eta}(\boldsymbol{x})\|) \Big)$ $\|\boldsymbol{u}_{\tau n}(\boldsymbol{x})\|\, d\Gamma$.

Hence, passing to the inferior limit as $n \to \infty$, we get

$\liminf_{n\to\infty} b(\boldsymbol{u}_n, \boldsymbol{\mu}_n - \boldsymbol{\mu})$

$= \liminf_{n\to\infty} \int_{\Gamma_3} \Big(g(\boldsymbol{x}, \|\boldsymbol{\eta}_{\tau n}(\boldsymbol{x})\|) - g(\boldsymbol{x}, \|\boldsymbol{\eta}_\tau(\boldsymbol{x})\|) \Big) \|\boldsymbol{u}_{\tau n}(\boldsymbol{x})\| d\Gamma$

$= 0.$

Using again the properties of the trace operator and the assumptions on the friction bound we deduce that (k_2) in Assumption 8 is also verified.

We apply now Theorem 1.

Let us introduce

$$\boldsymbol{K}_1 = \{\boldsymbol{v} \in V \,|\, \|\boldsymbol{v}\|_V \leq \frac{1}{m_a} \|\boldsymbol{f}\|_V\}; \tag{25}$$

$$\boldsymbol{K}_2 = \{\boldsymbol{\mu} \in D \,|\, \|\boldsymbol{\mu}\|_D \leq \frac{m_a + M_a}{\alpha\, m_a} \|\boldsymbol{f}\|_V\}, \tag{26}$$

Theorem 3 (A boundedness result) *If $(\boldsymbol{u}, \boldsymbol{\lambda})$ is a weak solution of Problem 1, then*

$$(\boldsymbol{u}, \boldsymbol{\lambda}) \in \boldsymbol{K}_1 \times \big(\Lambda(\boldsymbol{u}) \cap \boldsymbol{K}_2\big)$$

where $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are given by (25)-(26), V given by (7), D given by (12), $\boldsymbol{f}$ given by (8), m_a and M_a being the constants in (22) and α being the constant in (23).

Proof. The proof is a straightforward consequence of Theorem 1.

References

[1] C. Ciulcu, D. Motreanu, M. Sofonea, Analysis of an elastic contact problem with slip dependent coefficient of friction , *Mathematical Inequalities and Applications*, pp. 465-479, 4/2001.

[2] I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*, Classics in Applied Mathematics **28** SIAM, Philadelphia, PA, 1999.

[3] W. Han and M. Sofonea, *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity*, in "Studies in Advanced Mathematics," American Mathematical Society, International Press, USA, 2002.

[4] J. Haslinger, I. Hlaváček and J. Nečas, *Numerical Methods for Unilateral Problems in Solid Mechanics*, in "Handbook of Numerical Analysis", J.-L. L. P. Ciarlet, ed., **IV**, North-Holland, Amsterdam, pp. 313–485, 1996.

[5] P. Hild, Y. Renard, A stabilized Lagrange multiplier method for the finite element approximation of contact problems in elastostatics. *Numer. Math.*, pp. 101–129, 115/2010.

[6] I.R. Ionescu, J.-C. Paumier, On the contact problem with slip displacements dependent friction in elastostatics, *Int. J. Eng. Sci.*, pp. 471-491, 34/96.

[7] A. Matei, On the solvability of mixed variational problems with solution-dependent sets of Lagrange multipliers, Proceedings of The Royal Society of Edinburgh,Section: A Mathematics, pp. 1047-1059, 143(05)/2013.

[8] A. Matei and R. Ciurcea, Contact problems for nonlinearly elastic materials: weak solvability involving dual Lagrange multipliers, *ANZIAMJ*, pp. 160-178, 52/2010.

[9] M. Sofonea and A. Matei, Mathematical Models in Contact Mechanics, London Mathematical Society, Lecture Note Series 398, 280 pages, Cambridge University Press, 2012.

ISBN: 978-988-19253-5-0
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

[10] B. Wohlmuth, A Mortar Finite Element Method Using Dual Spaces for the Lagrange Multiplier, *SIAM Journal on Numerical Analysis*, pp. 989-1012, 38/2000.

[11] B. Wohlmuth, *Discretization Methods and Iterative Solvers Based on Domain Decomposition*, in "Lecture Notes in Computational Science and Engineering", 17, Springer, 2001.

ISBN: 978-988-19253-5-0
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)