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General Parameterization of Stabilizing Controllers with Coprime Factorizations

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***Abstract*—This paper is concerned with the factorization approach to control systems. The factorization approach we use here assumes that the plant admits coprime factorizations. On the ohter hand, the set of stable causal transfer functions is a general commutative ring. The objective of this paper is to present that even in the case where the set of stable causal transfer functions is a general commutative ring, we can employ the Youla-parameterization for the parametrization of stabilizing controllers.**

***Index Terms*—Linear systems, Feedback stabilization, Coprime factorization over commutative rings Parametrization of stabilizinng controllers**

I. Introduction

IN the factorization approach[2], [7], [9], [10], a transfer function is given as the ratio of *two stable causal transfer functions* and the set of stable causal transfer functions forms a *commutative ring*.

Since stabilizing controllers are not unique in general, the choice of stabilizing controllers is important for the resulting closed loop. In the classical case such as continuous-time LTI systems and discrete-time LTI systems, the stabilizing controllers can be parameterized by the method called "Youla-parameterization"[2], [7], [10], [11] (also called Youla-Kučera-parameterization). However, there exist models in which some stabilizable transfer matrices do not have their right-/left-coprime factorizations in general[1], [3]. In such models, we cannot employ the Youla-parameterization in general.

The objective of this paper is to present that in the factorization approach, if a plant has both right-/left-coprime factorizations (even if some other stabilizable plants in the same model do not have right-/left-coprime factorizations), we can still employ the Youla-parameterization for the parameterization of stabilizing controllers of the plant.

II. Preliminaries

In the following we begin by introducing notations used in this paper. Then we give the formulation of the feedback stabilization problem.

A. Notations

a) Commutative Rings: We will consider that *the set of all stable causal transfer functions* is a commutative ring, denoted by $\mathcal{A}$. The total ring of fractions of $\mathcal{A}$ is denoted by $\mathcal{F}$; that is, $\mathcal{F} = \{n/d \,|\, n, d \in \mathcal{A},\ d \text{ is a nonzerodivisor}\}$. This will be considered to be *the set of all possible transfer functions*. If the commutative ring $\mathcal{A}$ is an integral domain, $\mathcal{F}$ becomes a field of fractions of $\mathcal{A}$. However, if $\mathcal{A}$ is not an integral domain, then $\mathcal{F}$ is not a field, because any nonzero zerodivisor of $\mathcal{F}$ is not a unit.

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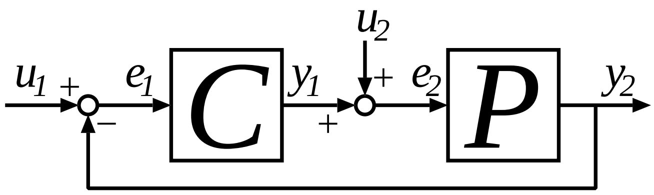

Fig. 1. Feedback system Σ.

b) Matrices: Suppose that x and y denote sizes of matrices.

The set of matrices over $\mathcal{A}$ of size $x \times y$ is denoted by $\mathcal{A}^{x\times y}$. In particular, the set of square matrices over $\mathcal{A}$ of size x is denoted by $(\mathcal{A})_x$. A square matrix is called *singular* over $\mathcal{A}$ if its determinant is a zerodivisor of $\mathcal{A}$, and *nonsingular* otherwise. The identity and the zero matrices are denoted by I_x and $O_{x\times y}$, respectively, if the sizes are required, otherwise they are denoted simply by I and O.

Matrices A and B over $\mathcal{A}$ are *right-coprime over* $\mathcal{A}$ if there exist matrices $\widetilde{X}$ and $\widetilde{Y}$ over $\mathcal{A}$ such that $\widetilde{X}A + \widetilde{Y}B = I$. Analogously, matrices $\widetilde{A}$ and $\widetilde{B}$ over $\mathcal{A}$ are *left-coprime over* $\mathcal{A}$ if there exist matrices X and Y over $\mathcal{A}$ such that $\widetilde{A}X + \widetilde{B}Y = I$. Further, pair (N, D) of matrices N and D is said to be a *right-coprime factorization of* P *over* $\mathcal{A}$ if (i) the matrix D is nonsingular over $\mathcal{A}$, (ii) $P = ND^{-1}$ over $\mathcal{F}$, and (iii) N and D are right-coprime over $\mathcal{A}$. Also, pair $(\widetilde{N}, \widetilde{D})$ of matrices $\widetilde{N}$ and $\widetilde{D}$ is said to be a *left-coprime factorization of* P *over* $\mathcal{A}$ if (i) $\widetilde{D}$ is nonsingular over $\mathcal{A}$, (ii) $P = \widetilde{D}^{-1}\widetilde{N}$ over $\mathcal{F}$, and (iii) $\widetilde{N}$ and $\widetilde{D}$ are left-coprime over $\mathcal{A}$. As we have seen, in the case where a matrix is potentially used to express *left* fractional form and/or *left* coprimeness, we usually attach a tilde '$\widetilde{\ }$' to a symbol; for example $\widetilde{N}$, $\widetilde{D}$ for $P = \widetilde{D}^{-1}\widetilde{N}$ and $\widetilde{Y}$, $\widetilde{X}$ for $\widetilde{Y}N + \widetilde{X}D = I$.

B. Feedback Stabilization Problem

The stabilization problem considered in this paper follows that of Sule in [8] and Mori and Abe in [6] who consider the feedback system Σ [9, Ch.5, Figure 5.1] as in Figure 1. For further details the reader is referred to [9], [6]. Throughout this paper, the plant we consider has m inputs and n outputs, and its transfer matrix, which itself is also called simply a *plant*, is denoted by P and belongs to $\mathcal{F}^{n\times m}$.

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Definition 1: Define $\widehat{F}_{\rm ad}$ by

$$\begin{aligned}\widehat{F}_{\rm ad} &= \{(X,Y) \in \mathcal{F}^{x\times y} \times \mathcal{F}^{y\times x} \mid \\ &\qquad \det(I_x + XY) \text{ is a unit of } \mathcal{F}, \\ &\qquad x \text{ and } y \text{ are positive integers}\}.\end{aligned}$$

For $P \in \mathcal{F}^{n\times m}$ and $C \in \mathcal{F}^{m\times n}$, the matrix $H(P,C) \in (\mathcal{F})_{m+n}$ is defined by

$$H(P,C) = \begin{bmatrix} (I_n+PC)^{-1} & -P(I_m+CP)^{-1} \\ C(I_n+PC)^{-1} & (I_m+CP)^{-1} \end{bmatrix} \quad (1)$$

provided $(P,C) \in \widehat{F}_{\rm ad}$. This $H(P,C)$ is the transfer matrix from $[\,u_1^t \quad u_2^t\,]^t$ to $[\,e_1^t \quad e_2^t\,]^t$ of the feedback system Σ. If (i) $(P,C) \in \widehat{F}_{\rm ad}$ and (ii) $H(P,C) \in (\mathcal{A})_{m+n}$, then we say that the plant P is *stabilizable*, P is *stabilized* by C, and C is a *stabilizing controller* of P. ▪

It is known that $W(P,C)$ defined below is over $\mathcal{A}$ if and only if $H(P,C)$ is over $\mathcal{A}$:

$$W(P,C) := \begin{bmatrix} C(I_n+PC)^{-1} & -CP(I_m+CP)^{-1} \\ PC(I_n+PC)^{-1} & P(I_m+CP)^{-1} \end{bmatrix}.$$

This $W(P,C)$ is the transfer matrix from u_1 and u_2 to y_1 and y_2. Then, we have

$$H(P,C) = I_{m+n} - FW(P,C),$$

where

$$F = \begin{bmatrix} O & I_n \\ -I_m & O \end{bmatrix}.$$

The matrix F is unimodular; in fact,

$$F^{-1} = \begin{bmatrix} O & -I_m \\ I_n & O \end{bmatrix},$$

which is over $\mathcal{A}$. Thus, $W(P,C)$ can be expressed in terms of F and $H(P,C)$:

$$W(P,C) = F^{-1}(I_{m+n} - H(P,C)).$$

Here we define the causality of transfer functions, which is an important physical constraint, used in this paper. We employ the definition of causality from Vidyasagar *et al.*[10, Definition 3.1] and Mori and Abe[6].

Definition 2: Let $\mathcal{Z}$ be a prime ideal of $\mathcal{A}$, with $\mathcal{Z} \neq \mathcal{A}$, including all zerodivisors. Define the subsets $\mathcal{P}$ and $\mathcal{P}_{\rm S}$ of $\mathcal{F}$ as follows:

$$\begin{aligned}\mathcal{P} &= \{n/d \in \mathcal{F} \mid n \in \mathcal{A},\ d \in \mathcal{A}\backslash\mathcal{Z}\}, \\ \mathcal{P}_{\rm S} &= \{n/d \in \mathcal{F} \mid n \in \mathcal{Z},\ d \in \mathcal{A}\backslash\mathcal{Z}\}.\end{aligned}$$

A transfer function in $\mathcal{P}$ $\left(\mathcal{P}_{\rm S}\right)$ is called *causal* $\left(\textit{strictly causal}\right)$. Similarly, if every entry of a transfer matrix over $\mathcal{F}$ is in $\mathcal{P}$ $\left(\mathcal{P}_{\rm S}\right)$, the transfer matrix is called *causal* $\left(\textit{strictly causal}\right)$. ▪

It should be noted that when using "a stabilizing controller," we do not guarantee the causality. However, in the classical case of the factorization approach, once we restrict ourselves to strictly proper plants, it is known that any stabilizing controller of strictly causal plant is causal (cf. Corollary 5.2.20 of [9], Theorem 4.1 of [10], and Proposition 6.2 of [6]). One can see, in fact, that many practical systems are strictly causal. On the other hand, including noncausal stabilizing controllers seems to make the theory easy and simple in the mathematical viewpoint. From these observations, we have accepted the possibility of the noncausality of stabilizing controllers in the parametrization.

III. Parametrization without Coprime Factorizability

Here we review the parametrization method without considering the coprime factorizability[4], [5]. Let $\mathcal{H}$ be the set of $H(P,C)$'s with all stabilizing controllers C of the plant P. This set $\mathcal{H}$ and all stabilizing controllers are obtained as in the following way.

Let H_0 be $H(P,C_0)$, where C_0 is a stabilizing controller of p. Let $\Omega(Q)$ be a matrix defined as follows:

$$\begin{aligned}\Omega(Q) := (H_0 - \begin{bmatrix} I_n & O \\ O & O \end{bmatrix})Q \quad (2) \\ \times(H_0 - \begin{bmatrix} O & O \\ O & I_m \end{bmatrix}) + H_0\end{aligned}$$

with a stable causal and square matrix Q of size $(m+n)\times(m+n)$. Using this matrix Q, we have the following theorem, the controller parametrization, as follows.

Theorem 1 ([4], [5]): The set of all $H(P,C)$'s with all stabilizing controllers is given as follows

$$\mathcal{H} = \{\Omega(Q) \mid Q \text{ is stable causal and } \Omega(Q) \text{ is nonsingular}\} \quad (3)$$

Furthermore, any stabilizing controller has the following form:

$$-[\,O \quad I_m\,]\,\Omega(Q)^{-1}\begin{bmatrix} I_n \\ O \end{bmatrix}, \quad (4)$$

provided that $\Omega(Q)$ is nonsingular.
The parameterization above is given by a parameter matrix Q without the coprime factorizability of the plant. The parameter matrix Q is of size $(m+n)\times(m+n)$. That is, in order to archive the parametrization, we need $(m+n)^2$ parameters.

IV. Main Result

Suppose that the plant P is stabilizable. Suppose further that P has right-/left-coprime factorizations over $\mathcal{A}$ of P. Let (N,D) and $(\widetilde{D},\widetilde{N})$ be right-/left-coprime factorizations over $\mathcal{A}$ of P and (Y_0,X_0) and $(\widetilde{X}_0,\widetilde{Y}_0)$ be right-/left-coprime factorizations over $\mathcal{A}$ of C_0, a stabilizing controller of P, such that

$$\widetilde{Y}_0 N + \widetilde{X}_0 D = I_m, \quad \widetilde{N} Y_0 + \widetilde{D} X_0 = I_n.$$

The following is the parameterization of stabilizing controllers presented as a Youla-parameterization.

Theorem 2: (cf. Theorems 5.2.1 and 8.3.12 of [9]) All matrices X, Y, $\widetilde{X}$, $\widetilde{Y}$ over $\mathcal{A}$ satisfying

$$\widetilde{Y}N + \widetilde{X}D = I_m, \quad \widetilde{N}Y + \widetilde{D}X = I_n$$

are expressed as $X = X_0 - NS$, $Y = Y_0 + DS$, $\widetilde{X} = \widetilde{X}_0 - R\widetilde{N}$ and $\widetilde{Y} = \widetilde{Y}_0 + R\widetilde{D}$ for R and S in $\mathcal{A}^{m\times n}$.

Further the set of all $\mathcal{A}$-stabilizing controllers, denoted by $\mathcal{S}(P)$, is given as

$$\begin{aligned}\mathcal{S}(P) &= \{(\widetilde{X}_0 - R\widetilde{N})^{-1}(\widetilde{Y}_0 + R\widetilde{D}) \mid \\ &\qquad R \in \mathcal{A}^{m\times n},\ \widetilde{X}_0 - R\widetilde{N} \text{ is nonsingular}\} \\ &= \{(Y_0 + DS)(X_0 - NS)^{-1} \mid \\ &\qquad S \in \mathcal{A}^{m\times n},\ X_0 - NS \text{ is nonsingular}\}.\end{aligned}$$

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The "integral domain version" of Theorem 2 was already shown in Section 8 of [9] without the proof. Nevertheless, we need to give the proof because the proofs have some differences.

Proof of Theorem 2: Observe first that Lemma 8.3.2 of [9] holds over commutative rings as well as integral domains. Hence, any stabilizing controller has both right-/right-coprime factorizations over $\mathcal{A}$.

In the proof of Theorem 5.2.1 of [9] three intermediate results (Lemma 4.1.32, Corollaries 4.1.67, 5.1.30 of [9]) was used. Modifying them in order to hold over $\mathcal{A}$, we can prove analogously to the proof of Theorem 5.2.1 of [9].

First, observe that Corollary 4.1.67 of [9] as well as Theorem 4.1.60 of [9] directly holds over $\mathcal{A}$.

Next, we consider Corollary 5.1.30 of [9]. Observe that Lemma 3.1 of [10] also holds over $\mathcal{A}$. By virtue of Lemma 3.1 of [10] instead of Theorem 5.1.25 of [9], we can see that Corollary 5.1.30 of [9] holds over $\mathcal{A}$ (We have avoided to use the notion of the characteristic determinant).

To finish the proof we generalize Lemma 4.1.32 of [9] over $\mathcal{A}$. It should be noted that in Lemma 4.1.32 of [9], only the right-coprimeness is required. On the other hand, in our case, both the right-/left-coprimenesses will be required since in the case of a general commutative ring the existence of the left-coprime factorization is not guaranteed even if there exists a right-coprime factorization. Fortunately, this new requirement does not affect the proof of Theorem 5.2.1 of [9] over $\mathcal{A}$. Presenting Lemma 2 below as a generalization of Lemma 4.1.32 of [9], we finish this proof. ∎

Before presenting the generalization of Lemma 4.1.32 of [9], we should give a lemma which is a generalization of Corollary 4.1.26 of [9].

Lemma 1: (cf. Corollary 4.1.26 of [9]) Suppose $P \in \mathcal{F}(\mathcal{A})^{n\times m}$ and that $P = ND^{-1} = \widetilde{D}^{-1}\widetilde{N}$ with the matrices $N, D, \widetilde{N}, \widetilde{D}$ over $\mathcal{A}$. Let $F_1 = [\,-\widetilde{N} \quad \widetilde{D}\,]$ and $F_2 = [\,D^t \quad N^t\,]^t$. Then the following are equivalent:

(i) N and D are right-coprime over $\mathcal{A}$, and $\widetilde{N}$ and $\widetilde{D}$ left-coprime over $\mathcal{A}$.

(ii) There exist unimodular matrices U_1 and U_2 of the forms $U_1 = [\,G_1^t \quad F_1^t\,]^t$ and $U_2 = [\,F_2 \quad G_2\,]$ for some matrices G_1 and G_2 over $\mathcal{A}$.

Proof: The "(ii)→(i)" part is proved analogously to the "if" part of Corollary 4.1.26 of [9]. On the other hand, the "(i)→(ii)" part is directly from Corollary 4.1.67 of [9] which holds over $\mathcal{A}$ as stated above. ∎

Lemma 2: (cf. Lemma 4.1.32 of [9]) Suppose $P \in \mathcal{F}(\mathcal{A})^{n\times m}$. Let (N, D) and $(\widetilde{N}, \widetilde{D})$ be right-/left-coprime factorizations over $\mathcal{A}$ of P, respectively. Let U_1 and U_2 be the unimodular matrices of the form in Lemma 1. Then the set of matrices $\widetilde{Y} \in \mathcal{A}^{m\times n}$ and $\widetilde{X} \in \mathcal{A}^{m\times m}$ with $\widetilde{Y}N + \widetilde{X}D = I_m$ is given by

$$[\,\widetilde{X} \quad \widetilde{Y}\,] = [\,I_m \quad R\,]\,U_2^{-1}, \tag{5}$$

where $R \in \mathcal{A}^{m\times n}$. Similarly the set of matrices $Y \in \mathcal{A}^{m\times n}$ and $X \in \mathcal{A}^{n\times n}$ with $\widetilde{N}Y + \widetilde{D}X = I_n$ is given by

$$\begin{bmatrix} -Y \\ X \end{bmatrix} = U_1^{-1} \begin{bmatrix} S \\ I_n \end{bmatrix}, \tag{6}$$

where $S \in \mathcal{A}^{m\times n}$.

The proof of Lemma 2 is analogous to that of Lemma 4.1.32 of [9], in which Lemma 1 above is used instead of Corollary 4.1.26 of [9].

Proof of Lemma 2: It is necessary to show that (i) every $\widetilde{Y}$ and $\widetilde{X}$ of the form (5) satisfies $\widetilde{Y}N + \widetilde{N}D = I_m$, and (ii) every $\widetilde{Y}$ and $\widetilde{X}$ satisfy $\widetilde{Y}N + \widetilde{X}D = I_m$ are of the form (5) for some R.

To prove (i), observe that $U_2^{-1}U_2 = I_{m+n}$. Hence

$$\begin{aligned} \widetilde{Y}N + \widetilde{X}D &= [\,\widetilde{X} \quad \widetilde{Y}\,] \begin{bmatrix} D \\ N \end{bmatrix} \\ &= [\,I_m \quad R\,]\,U_2^{-1} \begin{bmatrix} D \\ N \end{bmatrix} \\ &= [\,I_m \quad R\,] \begin{bmatrix} I_m \\ O \end{bmatrix} \\ &= I_m. \end{aligned}$$

To prove (ii), suppose that $\widetilde{Y}'$ and $\widetilde{X}'$ satisfies $\widetilde{Y}'N + \widetilde{X}'D = I_m$. Decompose U_2 as follows:

$$\begin{bmatrix} D & G_{21} \\ N & G_{22} \end{bmatrix} := U_2.$$

Define $R = \widetilde{Y}'G_{22} + \widetilde{X}'G_{21}$. Then

$$[\,\widetilde{X}' \quad \widetilde{Y}'\,]\,U_2 = [\,\widetilde{X}' \quad \widetilde{Y}'\,] \begin{bmatrix} D & G_{21} \\ N & G_{22} \end{bmatrix} = [\,I_m \quad R\,].$$

The proof concerning Y and X can be given analogously. It is necessary to show that (i) every Y and X of the form (6) satisfies $\widetilde{N}Y + \widetilde{D}X = I_n$, and (ii) every Y and X satisfy $\widetilde{N}Y + \widetilde{D}X = I_n$ are of the form (6) for some S.

To prove (i), observe that $U_1^{-1}U_1 = I_{m+n}$. Hence

$$\begin{aligned} \widetilde{N}Y + \widetilde{D}X &= [\,-\widetilde{N} \quad \widetilde{D}\,] \begin{bmatrix} -Y \\ X \end{bmatrix} \\ &= [\,-\widetilde{N} \quad \widetilde{D}\,]\,U_1^{-1} \begin{bmatrix} S \\ I_n \end{bmatrix} \\ &= [\,O \quad I_n\,] \begin{bmatrix} S \\ I_n \end{bmatrix} \\ &= I_n. \end{aligned}$$

To prove (ii), suppose that Y' and X' satisfies $\widetilde{N}Y' + \widetilde{D}X' = I_n$. Decompose U_1 as follows:

$$\begin{bmatrix} G_{11} & G_{12} \\ -\widetilde{N} & \widetilde{D} \end{bmatrix} := U_1.$$

Then, define $S = -G_{11}Y' + G_{12}X'$. Now we have

$$\begin{aligned} &U_1 \begin{bmatrix} -Y' \\ X' \end{bmatrix} \\ = &\begin{bmatrix} G_{11} & G_{12} \\ -\widetilde{N} & \widetilde{D} \end{bmatrix} \begin{bmatrix} -Y' \\ X' \end{bmatrix} \\ = &\begin{bmatrix} S \\ I_m \end{bmatrix}. \end{aligned}$$

∎

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