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An Embedding Theorem for Difference Weighted Spaces

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***Abstract*—We introduce a weighted space $w_p^2(\upsilon)$ which is a difference analogue of weighted Sobolev space $W_p^2(\upsilon)$ $(1 \le p < \infty)$. An embedding theorem on $w_p^2(\upsilon)$ embedded to the space of sequences $l_q(u)$ $(1 < p \le q < \infty)$ is obtained. In creation of proof of this theorem there are clear main ideas allowing to pass to studying of embedding questions for multiweight differential spaces of high orders.**

***Index Terms*—difference weighted spaces, weighted Sobolev spaces, embedding.**

I. Introduction

It is well known that in many cases the numerical solution of the ordinary differential equation is reduced to the solution of the corresponding differential equation. Here a knowledge of the corresponding embedding inequalities is often useful.

There are many unsolved problems in the theory of embedding of spaces of functions with discrete argument. Now the embedding theorems for weighted spaces of functions with discrete argument are of great interest. It is promoted by various useful applications of this theory as well. Note here that an extensive literature is devoted to the theory of weighed shift difference operators and to their spectra. And this theory recently of causes interest [1-14].

Our researches were inspired by those of M. Otelbayev [1-3], E. S. Smailov [4]. Also the subject of this work was considered by G. Mukhamediyev [5], R. Oynarov [6], B. Musilimov, A. T. Bulabayev [7-9] and A. T. Mukhambetzhanov. M. Otelbayev introduced a special weight function averaging thanks to which he received, in particular, two-sided estimates of some embedding operators norms, description of spectrum of some semi-bounded operators and estimates of distribution function of the spectrum ([1-3]). In E. S. Smailov's works the method given in [1] was developed. Also results of work [1] from the point of view of differential theorems of an embedding were generalized. E. S. Smailov [4] obtained criteria of a compact embedding $w_p^1(\upsilon) \to l_q$. Some spectral questions for differential operators considered in [4] were studied too. In [7-9] the questions of the embedding $w_p^1(\upsilon) \to l_q(u)$ are investigated in a two-dimensional case.

Upon transition to differential operators of order $m > 1$ the constructions made in the above mentioned works [1-9], aren't applicable.

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II. Embedding theorem for weighted sequence spaces

In this work we consider a difference analogue of weighted Sobolev space $W_p^m(\upsilon)$ with finite norm

$$\|u\|_{W_p^m(\upsilon)} = \left\{ \int_0^\infty \left(|u^{(m)}|^p + \upsilon(x) |u|^p \right) dx \right\}^{1/p}$$

$(1 \le p < \infty)$. Here $u^{(m)} = \frac{d^m u}{dx^m}$, $m > 1$, υ is non-negative function defined on $I = (0, \infty)$. For the simplicity we consider the case $m = 2$. Difference weighted space $w_p^2(\upsilon)$ $(1 < p < \infty)$ defined as the space of all sequences with finite norm

$$\|y\|_{w_p^2(\upsilon)} = \left\{ \sum_{j=1}^\infty \left(|\triangle^2 y_j|^p + \upsilon_j |y_j|^p \right) \right\}^{1/p}. \quad (1)$$

Denote by l the space of all real numerical sequences $y = \{y_j\}_{j=1}^\infty$, l_+ – the space of all non-negative numerical sequences of l. In (1) $y \in l$, $\upsilon \in l_+$. Further we denote by $\triangle$ the difference operator $\triangle y = \{\triangle y_j\}_{j=1}^\infty$,

$$\triangle y_j = y_{j+1} - y_j \quad (j = 1, 2...). \quad (2)$$

Second order difference operator Δ^2 on $y = \{y_j\}_{j=1}^\infty$ we give by the symmetric equality

$$\Delta^2 y_j = y_{j+1} - 2y_j + y_{j-1} = \Delta y_{j+1} - \Delta y_j \qquad (y_0 = 0, \quad j = 1, 2, ...). \quad (3)$$

Let $u = \{u_j\} \in l_+$. Let the space of all sequences $y = \{y_j\}_{j=1}^\infty$ with finite norm

$$\|y\|_{l_q(u)} = \left(\sum_{j=1}^\infty u_j |y_j|^q \right)^{1/q} \quad (4)$$

be denoted by $l_q(u)$ $(1 \le q < \infty)$. Hereinafter we suppose nondegeneracy assumptions

$$\sum_{j=1}^\infty u_j > 0, \quad \sum_{j=1}^\infty \upsilon_j > 0$$

are done.

Definition 1. Let X, Y be spaces with norms $\|\cdot; X\|$, $\|\cdot; Y\|$ respectively. We will say that X is embedded into Y and will write $X \to Y$, if next conditions satisfied:
e1) $X \subset Y$,
e2) there exists constant $c > 0$ such that for all $z \in X$ $\|z; Y\| \le c \|z; X\|$.
It is easily seen from e2) that the identity operator $Ex = x$ is continuous as one from X to Y.

Our aim is to make exact conditions on the embedding

$$w_p^2(\upsilon) \to l_q(u). \quad (5)$$

$\mathbb{Z}_+$ is standard notation of set of all non-negative integers. Let $\Re_{n,k}$ $(n,\, k \in \mathbb{Z}_+)$ be a set of all $\beta \in l$ normalized by the equality

$$\sum_{j=n}^{n+k} |\beta_j|^p = 1.$$

Now we set

$$S\left(n,k\right) = S\left(n,k|\upsilon\right) = \inf_{\beta\in\Re_{n,k}} \left(\sum_{j=n}^{n+k} \upsilon_j \beta_j\right)^{1/p}.$$

Note that

$$S\left(n,0\right) = \upsilon_n^{1/p}.$$

Let's introduce a discrete function

$$k_n^* = \begin{cases} 0, & \text{if } \upsilon_n \geq 1, \\ \sup\left\{k \in \mathbb{Z}_+ : k^{1+1/p'} S\left(n,k\right) \leq 1\right\}, & \text{if } \upsilon_n < 1. \end{cases} \tag{6}$$

Remark. One can show that $k_n^* \geq 0$ for all $n = 0, 1, ...$ If $k = k_n^* < \infty$ then

$$k^{1+1/p'} S\left(n,k\right) \leq 1 < \left(k+1\right)^{1+1/p'} S\left(n,k+1\right). \tag{7}$$

Definition 2. We say that weighted sequence υ is admissible, if

$$k_n^* < \infty \text{ for all } n \in \mathbb{Z}_+.$$

The set of all admissible weights will be denoted by Π ($\upsilon \in \Pi$).

Let $1 < p, q < \infty$. Suppose that

$$\mathcal{A}_{p,q}\left(n,k|u\right) = \left(k+1\right)^{1+1/p'} \left(\sum_{j=n}^{n+k} u_j\right)^{1/q}.$$

Theorem. Let $1 < p \leq q < \infty$, $\upsilon = \{\upsilon_j\}_{j=1}^{\infty} \in \Pi$. Let us assume that there exists a number γ, $0 < \gamma < 1$, such that for all $n \in \mathbb{Z}_+$ and $k = k_n^*$

$$\gamma \left(\frac{1}{k+1} \sum_{j=n}^{n+k} \upsilon_j\right)^{1/p} \leq \inf_{\beta\in\Re_{n,k}} \left(\sum_{j=n}^{n+k} \upsilon_j \beta_j\right)^{1/p}. \tag{8}$$

Then

$$\|y\|_{l_q(u)} \leq 6\gamma^{-1} \mathrm{A} \|y\|_{w_p^2(\upsilon)} \quad \left(y \in l\right),$$

where $A = \sup\limits_{n\in\mathbb{Z}_+} \mathcal{A}_{p,q}\left(n,k|u\right)$.

In the proof of the theorem the local inequality of the following lemma is significantly used.

Lemma. Let $1 < p, q < \infty$. Assume that $\upsilon = \{\upsilon_j\}_{j=1}^{\infty} \in \Pi$ satisfies the condition (8). Then for all $k = k_n^*$ and $y = \{y_j\}_{j=1}^{\infty} \in l$ the estimate

$$\left(\sum_{j=n}^{n+k} u_j |y_j|^q\right)^{1/q} \leq 6\gamma^{-1} \mathcal{A}_{p,q}\left(n,k|u\right)$$
$$\times \left\{ \left(\sum_{j=n}^{n+k+1} |\Delta^2 y_j|^p\right)^{1/p} + \left(\sum_{j=n}^{n+k+1} \upsilon_j |y_j|^p\right)^{1/p} \right\}$$

holds, where $c = 6\gamma^{-1}$ is constant independent on υ and u.

III. Conclusion

The research methods used in this work allow to consider other questions relating to the proved embedding. They include questions of the embedding operator compactness and approximation numbers estimates. Besides, it is possible to consider applications of similar difference embedding theorems to the spectral theory of difference analogues of differential operators.

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