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New Parabolic Inequalities and Applications to Thermal Explosion and Diffusion Models

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Abstract— **In this paper we obtain new comparison principles for parabolic boundary value problems in u without assuming the usual monotonic condition with respect to u. Starting with the basic theory in section 1, we developed new comparison principles in section 2. Applications of the theory developed in this paper is used to discuss the monotonicity of the solution of a thermal explosion problem. We generalized the results for parabolic systems in section 4 and discuss its applications to diffusion model.**

Index Terms— **Parabolic boundary value problems, Comparison Principles, Differential Inequalities, Parabolic, Elliptic.**

I. Introduction

Extensive literature on parabolic boundary value problems, Walter, [14], Pao, [11], Protter and Weinberger, [12], Taylor [13], has established their dominant role in the field of applied and industrial Mathematics.

Such boundary value problems play an important role in the following fields: combustion theory, Arioli, Gazzola, [3], Kasture, [7], Thermal boundary layer, Dhaigude, Kasture, [5], chemical and neuclear engineering, Ladde and Lakshmikantham, [8], Population dynamics, Al. Fonso castro, Maya and Shivaji, [2], Boundary layer theory, Walter, [14]. Multi component diffusion, Mc Nabb, [10], Alexander L Lee etal, [1], detection of proteins, C. Amatorel, etal [4], lubrication slip model, K. Ait Hadi, [6], bioelectroanalytical system, M. Puida, F. Ivanauskas, I. Ignatjev, G. Valin, V. Razumas, [9].

It is interesting to discuss the monotonicity property of a solution for a given boundary value problem with respect to any one of the parameter when all other parameters remains fixed. Such a study is important for real world problems governed by parabolic boundary value problems.

Maximum principles and comparison theorems for linear and non-linear parabolic equations with respect to u are known, Protter and Weinberger, [12], Walter, [14]. However, a variety of non-linear operators where monotonicity condition is not satisfied, occur naturally in many physical applications. Therefore we extend the results without monotonicity condition with respect to u and obtain more general results.

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Let D be an open connected subset of R^N with boundary ∂D, $\overline{D}$ be the closure of D. The points of D are denoted by $x = (x_1, x_2, \ldots, x_N)$. Let G be the topological product of an open domain D of the x-space R^N and a t interval $0 < t < T$. Suppose $G = (0,T) \times D$, $D \subset R^{n+1}$, $N \geq 1$. Let ∂G be the boundary of G. The points of G are denoted by

$$(t,x) = (t, x_1, x_2, \cdots, x_N).$$

For a real valued function $u(t,x)$ which is continuously differentiable, u_x denotes

$$grad\ u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \cdots, \frac{\partial u}{\partial x_N}\right) = p$$

and

$$u_{xx} \text{ is a } N \times N \text{ matrix } r, \text{where}$$

$$r = \begin{bmatrix} u_{x_1x_1}\ u_{x_1x_2} \cdots u_{x_1x_N} \\ u_{x_2x_1}\ u_{x_2x_2} \cdots u_{x_2x_N} \\ \vdots \quad \cdots \cdots \quad \vdots \\ u_{x_Nx_1}\ u_{x_Nx_2} \cdots u_{x_Nx_N} \end{bmatrix}.$$

The boundary ∂G can be divided into three pairwise disjoint subsets $\partial_0 G$, $\partial_1 G$, $\partial_2 G$, where

$\partial_0 G = \{(t,x) \in \partial G \,/\, U_-(t,x) \notin G\} = \{0\} \times \overline{D}$,

$\partial_2 G = \{(t,x) \in \partial G \,/\, U_-(t,x) \in G\} = \{T\} \times D$,

$\partial_1 G = \partial G - \partial_0 G - \partial_2 G = [0,T] \times \partial D$,

and $U_-(t,x)$ is a lower half neighborhood of (t,x). Thus we get

$$\partial G = \partial_0 G + \partial_1 G + \partial_2 G,$$
$$G_p = G + \partial_2 G,\ R_p = \partial G - \partial_2 G.$$

Further details are contained in Walter, [14].

II. Comparison Principle

We consider non linear differential operator of a particular type given by

$$\mathrm{P}u \equiv F(t,x,u,p,q,r) \tag{2.1}$$

where F is a real valued function $t \geq 0$, is the time variable, $q = \frac{\partial u}{\partial t}, x \in D$ is the space variable, $u : \overline{D} \to R^n$ with $u \in C^2(D) \cap C(\overline{D})$.

Definition 1: The operator P, given by (2.1) is said to be parabolic at a point (t, x) if F is monotone decreasing in r and weakly monotone increasing in q, we say that $F \in \Re$.

Definition 2: A comparison principle with strict inequality for an operator P is said to hold if, for two functions $u \text{ and } v$, $\mathrm{P}u < \mathrm{P}v \text{ in } G_P$ and $u < v \text{ on } R_p^+ \text{ and on } R_\infty$ implies $u < v \text{ in } G_p$.

Definition 3: A comparison principle allowing equality sign (weak comparison principle) for operator P is said to hold if, for two functions $u \text{ and } v, \mathrm{P}u \leq \mathrm{P}v \text{ in } G_P$ and $u \leq v \text{ on } R_p^+ \text{ and on } R_\infty$ implies $u \leq v \text{ in } G_p$.

Such comparison principles are available in Protter, Weinberger, [12], for linear parabolic operator without assuming a monotonic condition with respect u and for non-linear parabolic operators, Walter W, [14], assuming monotonic conditions. Here we obtain a weak comparison principle for non-linear operator without assuming a monotonic condition with respect u, which is contained in Theorem 2.2. We require the following

Theorem 2.1: If $F \in \Re$, $u, v \in Z_0(F)$ and if $u < v \text{ on } R_p^+ \text{ and on } R_\infty$

$F(t,x,v,v_t,v_x,v_{xx}) < F(t,x,w,w_t,w_x,w_{xx}) \text{ in } G_P,$

then $u < v \text{ in } G_p$, [14].

Theorem 2.2: If $F \in \Re$, $u, v \in Z_0(F)$, F is continuously differentiable with respect to $v \text{ and } q$, for all points

$$(t,x) \in G_p, \frac{\partial F}{\partial v} \text{ is bounded below}, \frac{\partial F}{\partial q} > 0,$$

$u \leq v \text{ on } R_p^+ \text{ and on } R_\infty$, and

$F(t,x,v,v_t,v_x,v_{xx}) \leq F(t,x,w,w_t,w_x,w_{xx}) \text{ in } G_P,$

then $u \leq v \text{ in } G_p$.

Proof :- Let $\overline{v} = v + \varepsilon e^{\lambda t}, \lambda > 0$,

So that $\overline{v}_t = v_t + \varepsilon\lambda e^{\lambda t}, \overline{v}_x = v_x, \overline{v}_{xx} = v_{xx}$.

Then

$$\begin{aligned}
&F(t,x,\overline{v},\overline{v}_t,\overline{v}_x,\overline{v}_{xx}) - F(t,x,v,v_t,v_x,v_{xx}) \\
&= F(t,x,\overline{v},\overline{v}_t,\overline{v}_x,\overline{v}_{xx}) - F(t,x,v,v_t,\overline{v}_x,\overline{v}_{xx}) \\
&\quad + F(t,x,v,v_t,\overline{v}_x,\overline{v}_{xx}) - F(t,x,v,v_t,v_x,v_{xx}) \\
&= F(t,x,v+\varepsilon e^{\lambda t},v_t+\varepsilon\lambda e^{\lambda t},v_x,v_{xx}) - F(t,x,v,v_t,v_x,v_{xx}).
\end{aligned}$$

Applying the mean value theorem, we get

$$\begin{aligned}
&F(t,x,\overline{v},\overline{v}_t,\overline{v}_x,\overline{v}_{xx}) - F(t,x,v,v_t,v_x,v_{xx}) \\
&= \varepsilon e^{\lambda t}\left[\frac{\partial F}{\partial v}(t,x,v^*,v_t^*,v_x,v_{xx}) + \lambda\frac{\partial F}{\partial q}(t,x,v^*,v_t^*,v_x,v_{xx})\right],
\end{aligned}$$

where $(t,x,v^*,v_t^*,v_x,v_{xx})$ is a point on the segment joining (t,x,v,v_t,v_x,v_{xx}) to $(t,x,v+\varepsilon e^{\lambda t},v_t+\varepsilon\lambda e^{\lambda t},v_x,v_{xx})$.

By assumption $\frac{\partial F}{\partial v}$ is bounded below, and λ can be chosen so large that $\lambda\left(\frac{\partial F}{\partial q}\right) + \frac{\partial F}{\partial v} > 0$, so that $F(t,x,\overline{v},\overline{v}_t,\overline{v}_x,\overline{v}_{xx}) - F(t,x,v,v_t,v_x,v_{xx}) > 0$, in G_p.

Further $u \leq v \leq \overline{v} \text{ on } R_p^+ \text{ and on } R_\infty$, using Theorem 2.1,we get $u < v \text{ in } G_p$.

Taking the limit as $\varepsilon \to 0$, we get the required result.

We get the following Theorem as a special case of Theorem 2.2 with

$\mathrm{P}u \equiv u_t - f(t,x,u,u_x,u_{xx})$.

Theorem 2.3: For two functions

$u, v \in Z_0(f), \mathrm{P}u \leq \mathrm{P}v \text{ in } G_P,$

$u \leq v \text{ on } R_p^+ \text{ and on } R_\infty,$

implies $u \leq v \text{ in } G_p$, provided f is continuously differentiable with respect to $v \text{ and } \frac{\partial f}{\partial v}$ is bounded above for all points $(t,x) \in G_p$.

Now the next section is an applications of the above results to a thermal explosion problem.

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III. MONOTONICITY WITH RESPECT TO PARAMETERS AND APPLICATIONS TO THERMAL EXPLOSION PROBLEM

Comparison principles and positivity-negativity Theorems are important tools for obtaining monotonicity of a solution with respective parameters. It is interesting to discuss monotonicity property of a solution for a given boundary value problem with respect to any one parameter when all other parameters remain fixed, without finding the solution explicitly. Such a study is very important for real world problems. Here we consider the following

A. *Thermal Explosion problem*

We consider the following parabolic boundary value problem:

$$u_t - D\nabla^2 u = \sigma \exp(\gamma - \gamma/u) \ \ in \ G_P, \qquad (3.1)$$

$$u = g(t,x) \ \ on \ \partial_1 G, u(0,x) = u_0 \ \ on \ \partial_o G \qquad (3.2)$$

The values of $u \in R^1$ so prescribed on $R_p = \partial G - \partial_2 G$ will be denoted by $u = G(t,x) \ \ on \ R_p$. The problem (3.1), (3.2) with N=3 represents a physical problem of thermal explosion, Pao, [11]. Here $D > 0$ represents the thermal diffusivity, the constant $\gamma > 0$ is called the Arrhenius number, $\sigma > 0$ is a constant, $u(t,x)$ is the temperature at time t, and $\nabla^2 = \frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2} + \frac{\partial^2}{\partial x_3{}^2}$.

For the above problem, increase of $u(t,x)$ in one of the parameters D, σ, γ is an indication of a quicker blowup. Therefore the study of monotonicity with respect to a parameter for such parabolic inequalities is important.

We study the N dimensional analogue of the thermal explosion problem; for $x \in R^N, u \in R^1$. We require the following condition:

$(3\alpha_1)$: Let $m = \underset{R_p}{Min}\, \{G(x,t)\} > 0.$

Such a condition is legitimate because for blow up problems, the initial and boundary conditions are expected to be high. Now we have the following

Theorem 3.1: The solution of problem (3.1), (3.2) is strictly positive and bounded below under condition $(3\alpha_1)$.

Theorem 3.2: The solution $u(t,x)$ of problem (3.1), (3.2) increases with σ under condition $(3\alpha_1)$.

Proof: In this problem the solution $u(t,x)$ of the problem (3.1), (3.2) satisfies $u \geq m > 0 \ \ in \ G_p$ due to condition $(3\alpha_1)$.

Differentiating (3.1) partially with respect to σ and assuming $U = \frac{\partial u}{\partial \sigma}$, we get

$$U_t - D\nabla^2 U = \frac{\sigma\gamma}{u^2} U \left[\exp\left(\gamma - \frac{\gamma}{u} \right) \right] + \exp\left(\gamma - \frac{\gamma}{u} \right).$$

Taking

$$f(t,x,U,U_x,U_{xx}) = D\nabla^2 U + \frac{\sigma\gamma}{u^2} U \left[\exp\left(\gamma - \frac{\gamma}{u} \right) \right],$$

then

$\frac{\partial f}{\partial U} = \frac{\sigma\gamma}{u^2} \left[\exp\left(\gamma - \frac{\gamma}{u} \right) \right]$ which is bounded above by $\frac{\sigma\gamma}{m^2} \exp(\gamma)$. Thus

$$LU = U_t - f(t,x,U,U_x,U_{xx}) = \exp\left(\gamma - \frac{\gamma}{u} \right) > 0 \ \ in \ G_p,$$

$$U = 0 \geq 0 \ \ on \ R_P.$$

Hence it follows from Theorem 2.3 that $U > 0 \ \ in \ G_p$.

Thus $u(t,x)$ is an increasing function of σ.

Theorem 3.3: The solution $u(t,x)$ of problem (3.1), (3.2) increases with γ under condition $(3\alpha_1)$ *where* $m > 1$.

Proof: Differentiating (3.1) partially with respect to γ and assuming $V = \frac{\partial u}{\partial \gamma}$, we get

$$V_t - D\nabla^2 V = \sigma\left(1 - \frac{1}{u}\right) \exp\left(\gamma - \frac{\gamma}{u} \right) + \frac{\sigma}{u^2} V \left[\exp\left(\gamma - \frac{\gamma}{u} \right) \right].$$

Taking $F(t,x,V,V_x,V_{xx}) = D\nabla^2 V + \frac{\sigma}{u^2} V \left[\exp\left(\gamma - \frac{\gamma}{u} \right) \right]$,

$\frac{\partial F}{\partial V} = \frac{\sigma}{u^2} \left[\exp\left(\gamma - \frac{\gamma}{u} \right) \right] < \frac{\sigma}{m^2} \exp(\gamma)$, which is bounded above.

Now

$$PV = V_t - F(t,x,V,V_x,V_{xx}) = \sigma\left(1 - \frac{1}{u}\right) \exp\left(\gamma - \frac{\gamma}{u} \right) > 0 \ \ in \ G_p,$$

$$V = 0 \geq 0 \ \ on \ R_p.$$

Hence it follows from Theorem 2.3 that $V > 0 \ \ in \ G_p$.

Thus $u(t,x)$ is an increasing function of γ.

Now we will prove the monotonicity with respect to the third parameter D, for which we require the following preliminary lemma.

Lemma 3.1: Let function $g(t,x)$ satisfy the following condition

$(3\alpha_2)$: $g_t(t,x) < 0 \ \ on \ \partial_1 G.$

Then

$u_t(t,x) < 0 \ \ in \ G_p \ \ and \ \ \nabla^2 u(t,x) < 0 \ \ in \ G_p.$

Theorem 3.4: Let all the conditions of Theorem 3.2 hold. In addition suppose condition $(3\alpha_2)$ is satisfied, then the solution $u(t,x)$ of problem (3.1), (3.2) decreases with D.

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IV. NON-LINEAR PARABOLIC SYSTEM AND APPLICATION

Let $u = (u_1, u_2, .., u_n) = u(x) \in R^n, (n \geq 1)$ with components $u^v, v = 1, 2, 3, ..., n,$ given by $u^v = u^v(t, x),\ x = (x_1, x_2, ..., x_N) \in R^N, (N \geq 1).$

$u^v_{,t} = q^v = \dfrac{\partial u^v}{\partial t},\ t \in E^1 \quad for\ v = 1, 2, 3, ..., n,$

and

$u^v_{,x} = p^v = \dfrac{\partial u^v}{\partial x} = \left(\dfrac{\partial u^v}{\partial x_1}, \dfrac{\partial u^v}{\partial x_2}, ..., \dfrac{\partial u^v}{\partial x_N}\right) = grad\, u^v, \quad p = (p^1, p^2, ..., p^n).$

Suppose

$$\left(u^v_{,xx}\right) = r^v = \left(\frac{\partial^2 u^v}{\partial x_i \partial x_j}\right) = \begin{bmatrix} \frac{\partial^2 u^v}{\partial x_1 \partial x_1} & \frac{\partial^2 u^v}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 u^v}{\partial x_1 \partial x_N} \\ \frac{\partial^2 u^v}{\partial x_2 \partial x_1} & \frac{\partial^2 u^v}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 u^v}{\partial x_2 \partial x_N} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 u^v}{\partial x_N \partial x_1} & \frac{\partial^2 u^v}{\partial x_N \partial x_2} & \cdots & \frac{\partial^2 u^v}{\partial x_N \partial x_N} \end{bmatrix} \text{ is a } N \times N$$

Matrix for $i = 1, 2, .., N;\ j = 1, 2, .., N \ \ and \ \ v = 1, 2, .., n.$

We consider the following system of parabolic operators given by $\mathrm{F} = (F^1, F^2, ..., F^n)$, defined by

$$\left.\begin{aligned} F^1 &= (t, x, u, u^1_{,t}, u^1_{,x}, u^1_{,xx}), \\ F^2 &= (t, x, u, u^2_{,t}, u^2_{,x}, u^2_{,xx}), \\ F^n &= (t, x, u, u^n_{,t}, u^n_{,x}, u^n_{,xx}) \end{aligned}\right\} \qquad (4.1)$$

We need the following definition.

Definition 4.1: The function $F^v = (t, x, u, u^v_{,t}, u^v_{,x}, u^v_{,xx}) \in \Re$ if it is monotonic increasing in r^v and strictly increasing in q^v.

Definition 4.2: If each $F^v \in \Re$ for $v = 1, 2, .., n,$ then we say that function $\mathrm{F} \in \Re$.

We refer to Walter,[14], for further details of notations. We need the following

Comparison Principle 4.1: If $\mathrm{F} \in \Re$ and $u, v \in Z_0(f)$ we have

$(\alpha):\ u < v\ \ on\ \ R^+_p\ \ and\ on\ R_\infty,$

$(\beta):\ \mathrm{F}(t, x, u, u_{,t}, u_{,x}, u_{,xx}) < \mathrm{F}(t, x, v, v_{,t}, v_{,x}, v_{,xx})\ \ in\ \ G_P,$

imply $u < v\ \ in\ \ G_p$, provided F^v is quasimonotonic increasing in $u(t, x)$.

We extend the above result by obtaining a comparison principle allowing equality sign which is contained in the following

Comparison Principle 4.2: Let $\mathrm{F} \in \Re$, $u, v \in Z_0(f)$ and for $v = 1, 2, .., n.$ Then

$(\alpha): F^v(t, x, u, u^v_{,t}, u^v_{,x}, u^v_{,xx}) \leq F^v(t, x, v, v^v_{,t}, v^v_{,x}, v^v_{,xx})\ \ in\ \ G_P,$

$(\beta):\ u \leq v\ \ on\ \ R^+_p\ \ and\ on\ R_\infty,$

Implies $u \leq v\ \ in\ \ G_p$, provided F^v satisfies a Lipschitz's condition of the following type:

For $v = 1, 2, .., n$, there exist negative functions $c^v_1, c^v_2, ..., c^v_{v-1}, c^v_{v+1}, ...c^v_n, c^v_l = c^v_l(t, x), l = 1, 2, 3....$ such that

$\sum_{l=1}^{n} c^v_l$ is bounded below and

$$F^v(t, x, \overline{u}, u^v_{,t}, u^v_{,x}, u^v_{,xx}) - F^v(t, x, u, u^v_{,t}, u^v_{,x}, u^v_{,xx}) \geq \sum_{l=1}^{n} c^v_l (\overline{u_l} - u_l) \quad for\ \ v = 1, 2, ...n$$

and F is differentiable with respect to q, $\dfrac{\partial F^v}{\partial q} > 0\ \ on\ D(f).$

Proof: Let $\overline{v}^v = v^v + \varepsilon e^{\lambda t}, \lambda > 0$ therefore

$\overline{v}^v_{,t} = v^v_{,t} + \varepsilon e^{\lambda t}, \quad \overline{v}^v_{,x} = v^v_{,x}, \quad \overline{v}^v_{,xx} = v^v_{,xx}.$

Consider

$$\begin{aligned} &F^v(t, x, \overline{v}^v, \overline{v}^v_{,t}, \overline{v}^v_{,x}, \overline{v}^v_{,xx}) - F^v(t, x, v^v, v^v_{,t}, v^v_{,x}, v^v_{,xx}) \\ &= F^v(t, x, \overline{v}^v + \varepsilon e^{\lambda t}, \overline{v}^v_{,t} + \varepsilon\lambda e^{\lambda t}, \overline{v}^v_{,x}, \overline{v}^v_{,xx}) - F^v(t, x, v^v, v^v_{,t}, v^v_{,x}, v^v_{,xx}). \end{aligned}$$

Using the mean value theorem and condition (β) we get

$$\begin{aligned} &F^{\overline{v}}(t, x, \overline{v}^v, \overline{v}^v_{,t}, \overline{v}^v_{,x}, \overline{v}^v_{,xx}) - F^v(t, x, v^v, v^v_{,t}, v^v_{,x}, v^v_{,xx}) \qquad (4.2) \\ &\geq \sum_{l=1}^{n} c^v_l \varepsilon\lambda e^{\lambda t} + \varepsilon\lambda e^{\lambda t} \frac{\partial F^v(t, x, \overline{v}^{*v}, \overline{v}^{*v}_{,t}, v^v_{,x}, v^v_{,xx})}{\partial q} \\ &= \varepsilon e^{\lambda t}\left[\sum_{l=1}^{n} c^v_l + \lambda \frac{\partial F^v(t, x, \overline{v}^{*v}, \overline{v}^{*v}_{,t}, v^v_{,x}, v^v_{,xx})}{\partial q}\right]. \end{aligned}$$

Since $\lambda > 0, \dfrac{\partial F}{\partial q} > 0$ and $\sum_{l=1}^{n} c^v_l$ is bounded below we can choose λ sufficiently large so that right hand side of (4.2) is greater than zero in G_p. By assumptions $(\alpha)\ and\ (\beta)$ we have

$F^v(t, x, u, u^v_{,t}, u^v_{,x}, u^v_{,xx}) < F^v(t, x, v, v^v_{,t}, v^v_{,x}, v^v_{,xx})\ \ in\ \ G_P,$

for $v = 1, 2, .., n,$

and $u < \overline{v}\ \ on\ \ R^+_p\ \ and\ on\ R_\infty.$

So by comparison principle 4.1, $u < \overline{v}\ in\ G_P$. Taking the limit as $\varepsilon \to 0,$ we get $u \leq v\ \ in\ \ G_p.$

Many physical phenomena are governed by parabolic systems, such as the diffusion of ions and point defects in metals, conduction of heat in heterogeneous media, transport of water through plant tissue, and random walk model, Alexander L Lee and James M. Hills [1]. In the next section we consider an application of the above results to the random walk model.

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A. Applications to Diffusion Model

Here we consider the following system in R^N, which describes the diffusion in presence of three diffusion paths in $R^3,\ for\ N=3$; [1],

$$\frac{\partial u_1}{\partial t} = D_1\nabla^2 u_1 - a_{11}u_1 + a_{12}u_2 + a_{13}u_3$$

$$\frac{\partial u_2}{\partial t} = D_2\nabla^2 u_2 + a_{21}u_1 - a_{22}u_2 + a_{23}u_3$$

$$\frac{\partial u_3}{\partial t} = D_3\nabla^2 u_3 + a_{31}u_1 + a_{32}u_2 - a_{33}u_3 \quad in\ G_P, \tag{4.3}$$

$$u_1 = g_1(t,x), u_2 = g_2(t,x), u_3 = g_3(t,x)\ on\ R_p, \tag{4.4}$$

where $u_1(t,x), u_2(t,x), u_3(t,x)$, are the concentrations, D_1, D_2, D_3 are diffusivities and are non-negative constants. The non-negative constants a_{ij} represents the transition probabilities in the diffusion model.

Alexander L Lee and James M. Hills [1], obtained the conditions under which atleast one of the three components $u_1(t,x), u_2(t,x), u_3(t,x)$, attains its maximum on R_p. In what follows we obtain the conditions under which all three components $u_1(t,x), u_2(t,x), u_3(t,x)$, attains their maximum on R_p. Since ours is a maximum principle for a boundary value problem, the result depends, apart from the coefficients on the boundary values $g_1(t,x), g_2(t,x), g_3(t,x)$ also.

Theorem 4.3: For the system (4.3), (4.4) we assume the following conditions:

$(6\alpha_1): \{u_1(t,x), u_2(t,x), u_3(t,x)\}$ are continuous with their derivatives up to second order in G_p and satisfy conditions (4.1), (4.2).

$(6\alpha_2): g_1(t,x), g_2(t,x), g_3(t,x) > 0\ \ on\ R_p$ and are continuous there.

$$(6\alpha_3): k = Min\left\{\frac{a_{11}}{a_{12}+a_{13}}, \frac{a_{22}}{a_{21}+a_{23}}, \frac{a_{33}}{a_{31}+a_{32}}\right\} > \frac{\overline{m}}{m},$$

where

$\overline{m} = \underset{R_p}{Max}\{m_1, m_2, m_3\}, m = \underset{R_p}{Min}\{m_1, m_2, m_3\}$ are positive on account of $(6\alpha_2)$ and $M_i = \underset{R_p}{Max}\, u_i,\quad m_i = \underset{R_p}{Min}\, u_i \quad i = 1,2,3.$ Then $u_1(t,x), u_2(t,x), u_3(t,x)$ attain their maximum on R_p.

Proof: Let

$$L_1u = \frac{\partial u_1}{\partial t} - D_1\nabla^2 u_1 + a_{11}u_1 - a_{12}u_2 - a_{13}u_3$$

$$L_2u = \frac{\partial u_2}{\partial t} - D_2\nabla^2 u_2 - a_{21}u_1 + a_{22}u_2 - a_{23}u_3$$

$$L_3u = \frac{\partial u_3}{\partial t} - D_3\nabla^2 u_3 - a_{31}u_1 - a_{32}u_2 + a_{33}u_3$$

$$L = (L_1, L_2, L_3), M = (M_1, M_2, M_3), u = (u_1, u_2, u_3).$$

Now $L_1u = 0, L_2u = 0, L_3u = 0,$ as u_1, u_2, u_3 satisfy (4.3), (4.4)
hence $Lu = 0$.

Further by condition $(6\alpha_2), u > 0\ \ on\ R_p$. Thus $L0 < Lu = 0\ \ in\ G_p,\ 0 < u\ \ on\ R_p$ implies $u > 0\ \ in\ \ G_p$. Thus $u_1(t,x), u_2(t,x), u_3(t,x)$ are positive. Now

$$L_1M = a_{11}M_1 - a_{12}M_2 - a_{13}M_3$$

$$L_2M = -a_{21}M_1 + a_{22}M_2 - a_{23}M_3$$

$$L_3M = -a_{31}M_1 - a_{32}M_2 + a_{33}M_3.$$

By assumed condition $(6\alpha_3)$,

$$L_1M > a_{11}m_1 - a_{12}M_2 - a_{13}M_3 > a_{11}m - (a_{12}+a_{13})\overline{m} > 0 = L_1u.$$

Similarly $L_2M > 0 = L_2u\ \ and\ \ L_3M > 0 = L_3u$.

Hence by comparison principle 4.2 applied to $Lu = 0 \le LM\ \ in\ G_p,\ u = M\ \ on\ R_p$, follows that $u \le M\ \ in\ G_p$. Hence u_1, u_2, u_3 attain their maximum on R_p.

REMARK 4.1: As maximum values of u_1, u_2, u_3 depend on boundary values, if the boundary values are changed, then the maximum principle may or may not hold as the condition $(6\alpha_3)$, may be satisfied for certain boundary values and may not be satisfied for other boundary values.

V. CONCLUDING REMARKS

Thus known theory of parabolic inequalities is developed further by obtaining new comparison principles for single equations as well as for parabolic systems and showing their applications for thermal explosion and diffusion models.

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REFERENCES

[1] Alexander L.Lee and James M.Hills, Maximum principles for linear systems pervenuto in Redazione it 3 giugno, 1991, 36-66.
[2] Al Fonso Castro,Maya C. and Shivaji R., Non-linear eigen value problems with semipositone structure, Nonlinear differential equations, Elect.,J. Diffe. Equations conf.05,33-49.
[3] Arioli, Gazzola F., Grunau H. C., Mitidieri E., A semilinear fourth order elliptic problems with exponential nonlinearity murst project"Metodi variazionali ed Equazioni Differenziali nonlinear", Oct 2003, 1-37.
[4] C. Amatore, A. Oleinick, I. Svir, N. da Mota1, L. Thouin, Theoretical Modeling and Optimization of the Detection Performance: a New Concept for Electrochemical Detection of Proteins in Microfluidic Channels ,Nonlinear Analysis: Modelling and Control, 2006, Vol. 11, No. 4, 345–365.
[5] Dhaigude and Kasture, Non-linear time degenerate Parabolic inequalities and applications, J. of Math.Anal.Appln,Vol.175, No.2,1993,476-481.

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[6] Hadi K. Ait, Boulder, Parabolic Nonlinear Second Order Slip Reynolds Equation: Approximation and Existence Nonlinear Analysis: Modelling and Control, 2007, Vol. 12, No. 1, 3–20.
[7] Kasture D. Y. , Applications of inequalities to Mechanics, Lecture notes at I. I. T. Madras 1991.
[8] Ladde G. S. Lakshmikantham V. and Vatsala A. S., Monotone Iterative Techniques for nonlinear differential equations, Pitman Advanced Pub. 1985.
[9] M. Puida, F. Ivanauskas, I. Ignatjev, G. Valinˇcius, V. Razumas, Computational Modeling of the Amperometric Bioanalytical System for Lipase Activity Assaya Time-Dependent Response Nonlinear Analysis: Modelling and Control, 2007, Vol. 12, No. 2, 245–251.
[10] McNABB, Comparison and existence theorems for multi component diffusion systems, J.M.A.A.Vol.3, No.1,1961, 133-144.
[11] Pao C. V– Nonlinear parabolic and elliptic equations, Plemum, New York, 1992.
[12] Protter M. H. and Weinberger H. F., Maximum principles in differential equations, Prentice Hall, Inc., (1967). Taylor A. B.
[13] Taylor A. B.- Mathematical models in Applied Mechanics, Clarendon Press, Oxford 1986.
[14] Walter W., Differential inequalities, Fifty years on from Hardy, Littlewood and Polya (Edited by W. N. Everitt) 1991, 249-283.