On Algebraization of Classical First Order Logic

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Abstract - Algebraization of first order logic and its deduction are introduced according to Halmos approach. Application to functional polyadic algebra is done.

Index Term: Polyadic algebra, Polyadic ideal, Polyadic filter, Functional polyadic algebra.

I. INTRODUCTION

There are mainly three approaches to the algebraization of first order logic. One approach is to develop cylindric algebras[1]. The second approach is by polyadic algebra[2]. the third one is by category theory [3].

In 1956 P. R. Halmos introduced polyadic algebra to express first order logic algebraically.

Polyadic algebra is an extension of Boolean algebra with operators corresponding to the usual existential and universal quantifiers over several variables together with endomorphisms to represent first order logic algebraically.

Certain polyadic filters and ultra filters have been used to express deduction in polyadic algebra. These ideas are applied to the specific case functional polyadic algebra.

II. POLYADIC ALGEBRA

General definition [3]

Suppose that B is a complete Boolean algebra. An existential quantifier on B is a mapping $\exists : B \to B$ such that

i) $\exists(0) = 0$

ii) $a \le \exists(a)$ for any $a \in B$.

iii) $\exists(a \wedge \exists(b)) = \exists(a) \wedge \exists(b)$ for any $a, b \in B$.

$(B, \exists)$ is called monadic algebra.

Let $I^I = \{\tau | \tau : I \to I \ \ is \ a \ transformation\}$. Denote the set of all endomorphisms on B by $End(B)$ and the set of all quantifiers on B by $Qant(B)$.

A polyadic algebra is $(B, I, S, \exists)$ where

$S : I^I \to End(B)$ and $\exists : 2^I \to Qant(B)$ such that for any $J, K \in 2^I$ and $\sigma, \tau \in I^I$ we have

1) $\exists(\phi) = id$

2) $\exists(J \cup K) = \exists(J)\exists(K)$

3) $S(\sigma\tau) = S(\sigma)S(\tau)$

4) $S(id) = id$

5) $S(\sigma)\exists(J) = S(\tau)\exists(J)$ if $\sigma|_{I-J} = \tau|_{I-J}$

6) $\exists(J)S(\tau) = S(\tau)\exists(\tau^{-1}(J))$ if τ is one to one on $\tau^{-1}(J)$.The cardinal number $|I|$ is called the degree of the algebra. If $|I| = n$ we get the so called n-adic algebra. If $I = \phi$ then $2^I = \{\phi\}$ and $I^I = \{id\}$. Therefore we get only the identity quantifier $\exists(\phi) = id$. Thus 0-adic algebra is just the Boolean algebra B. If $|I| = 1$ then $I^I = \{id\}$. Therefore there are only two quantifiers $\exists(\phi)$ and $\exists(I)$. Thus 1-adic algebra is the monadic algebra.

Polyadic Ideals And Filters

A subset U of a Boolean algebra B is called a Boolean ideal if

i) $0 \in U$

ii) $a \vee b \in U$ for any $a, b \in U$

iii) If $a \in U \ and \ b \le a \ then \ b \in U$

A subset F of a Boolean algebra B is called a Boolean filter if

i) $1 \in F$

ii) $a \wedge b \in F$ for any $a, b \in F$

iii) If $a \in F \ and \ b \ge a \ then \ b \in F$.

A subset U of a polyadic algebra B is called a polyadic ideal of B if

i) U is a Boolean ideal

ii) If $J \subseteq I \ and \ a \in U \ then \ \exists(J)(a) \in U$

iii) If $a \in U \ and \ \sigma \in I^I \ then \ S(\sigma)(a) \in U$

A subset F of a polyadic algebra B is called a polyadic filter of B if

i) F is a Boolean filter

ii) If $J \subseteq I \ and \ a \in F \ then \ \forall(J)(a) \in F$

iii) If $a \in F \ and \ \sigma \in I^I \ then \ S(\sigma)(a) \in F$.

Proposition 1) [3]

A subset U of a polyadic algebra B is an ideal of it if and only if U is an ideal of the Boolean algebra B and

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$\exists(I)(a)\in U$ for every $a\in U$. F is a filter of B if and only if F is a filter of the Boolean algebra B and $\forall(I)(a)\in F$ whenever $a\in F$.

Proof

Let U be an ideal of a polyadic algebra , by the definition U is an ideal of a Boolean algebra and $\exists(I)(a)\in U$.

Let U be an ideal of a Boolean algebra B and $\exists(I)(a)\in U$ for every $a\in U$.

If $J\subseteq I$, then $\exists(J)(a)<\exists(I)(a)$, so that $\exists(J)(a)\in U$. Now we verify that $S(\sigma)(a)\in U$, if $a\in U$ and $\sigma\in I^{I}$. We have $a<\exists(I)(a)$ and $S(\sigma)(a)<S(\sigma)\exists(I)(a)$, so it means to show that every $S(\sigma)$ acts as identity transformation from I^{I} since there is nothing out of I .Then

$S(\sigma)\exists(I)(a)=(id)\exists(I)(a)=\exists(I)(a)$, therefore $S(\sigma)(a)\in U$.

A similar argument proves the second part.

Proposition 2) [3]

There is a one to one correspondence between ideals and filters : if U is an ideal , then the set $U'=F$ of all a' with $a\in U$ is a filter . Analogously, if F is a filter , then $U=F'=\{a':a\in F\}$ is an ideal.

Proposition 3)

The set of all polyadic ideals and the set of all polyadic filters are closed under the arbitrary intersections.

Let B be a polyadic algebra and $\Gamma\subseteq B$. Let $U(\Gamma)$ denote the least polyadic ideal containing Γ and $F(\Gamma)$ denote the least polyadic filter containing Γ .We say that $U(\Gamma)$ and $F(\Gamma)$ are generated by Γ.

Proposition 4)

Let B be a polyadic algebra and $\Gamma\subseteq B$. Then

i) $U(\Gamma)=\{b\in B: b\le x_1\vee x_2\vee\ldots\vee x_n \text{ for some } x_1,x_2,\ldots,x_n\in\Gamma\}\cup\{0\}$

ii) $F(\Gamma)=\{b\in B: b\ge x_1\wedge x_2\wedge\ldots\wedge x_n \text{ for some } x_1,x_2,\ldots,x_n\in\Gamma\}\cup\{1\}$

Proof

i) Let

$J=\{b\in B: b\le x_1\vee x_2\vee\ldots\vee x_n \text{ for some } x_1,x_2,\ldots,x_n\in\Gamma\}\cup\{0\}$

$0\in J$. Let $b_1,b_2\in J$. Then $b_1\le x_1\vee x_2\vee\ldots\vee x_n$ and $b_2\le y_1\wedge y_2\wedge\ldots\wedge y_m$

for some $x_i,y_i\in\Gamma$.

$b_1\vee b_2\le x_1\vee x_2\vee\ldots\vee x_n\vee y_1\vee y_2\vee\ldots\vee y_m$.

Therefore $b_1\vee b_2\in J$. If $a\le b\le x_1\vee x_2\vee\ldots\vee x_n$, then $a\in J$. Then J is a Boolean ideal containing Γ. Therefore $U(\Gamma)\subseteq J$.

If $b\in J$, then $b_1\le x_1\vee x_2\vee\ldots\vee x_n$ where $x_i\in\Gamma$ i.e. $x_i\in U(\Gamma)$. Then $b\in U(\Gamma)$. Thus $U(\Gamma)=J$ as a Boolean ideal. Let $a\in J$ then $a\le x_1\vee x_2\vee\ldots\vee x_n$

$\exists(I)(a)\le\exists(I)(x_1\vee x_2\vee\ldots\vee x_n)=\exists(I)(x_1)\vee\exists(I)(x_2)\vee\ldots\vee\exists(I)(x_n)$

Therefore

$\exists(I)(a)\in U(\Gamma)$.

ii) A similar argument leads to (ii).

A filter F of a polyadic algebra is called ultrafilter if F is maximal with respect to the property that $0\notin F$.

Ultrafilters satisfy the following important properties [4].

Proposition 5)

Let F be a filter of a polyadic algebra B. Then

i) F is an ultrafilter of B iff for any $a\in F$ exactly one of a,a' belongs to F.

ii) F is ultrafilter of B iff $0\in F$ and $a\vee b\in F$ iff $a\in F$ or $b\in F$ for any $a,b\in F$.

iii) If $a\in B-F$, then there is an ultrafilter L such that $F\subset L$ and $a\in L$.

Let $\Gamma\subseteq B$. The ultrafilter containing $F(\Gamma)$ is denoted by $UF(\Gamma)$.

A mapping $\mu: B_1\to B_2$ between two Boolean algebras is called a Boolean homomorphism if

i) $\mu(a\wedge b)=\mu(a)\wedge\mu(b)$

ii) $\mu(a')=(\mu(a))'$

Obviously, $\mu(a\vee b)=\mu(a)\vee\mu(b)$, $\mu(0)=0$, $\mu(1)=1$.

A mapping $\mu: B_1\to B_2$ between two polyadic algebras is called polyadic homomorphism if

i) μ is a Boolean homomorphism

ii) $\mu\exists=\exists\mu$

iii) $\mu\sigma=\sigma\mu$ for any $\sigma\in I^{I}$

Obviously, $\mu\forall=\forall\mu$.

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III. DEDUCTION

Notation: for $\Gamma \subseteq B$ and $b \in B$, $\Gamma \vdash b$ reads b is deduced from Γ in B. Now , we define $\Gamma \vdash b$ in a polyadic algebra B iff $b \in F(\Gamma)$.

By definitions of filter and deduction $\vdash$ we have :

Proposition 6)

i) $\Gamma \vdash 1$ and $\Gamma \nvdash 0$

ii) If $\Gamma \vdash x$, $\Gamma \vdash y$ then $\Gamma \vdash x \wedge y$.

General properties of deduction are given by the following:

Theorem 7)

i) If $b \in \Gamma$ then $\Gamma \vdash b$

ii) If $\Gamma \vdash b$ and $\Gamma \subseteq \Sigma$ then $\Sigma \vdash b$

iii) If $\Sigma \vdash a$ for any $a \in \Gamma$ and $\Gamma \vdash b$, then $\Sigma \vdash b$

iv) If $\Gamma \vdash b$, then $\sigma(\Gamma) \vdash \sigma(b)$ for any substitution σ

v) If $\Gamma \vdash b$, then $\Gamma_0 \vdash b$ for some finite $\Gamma_0 \subseteq \Gamma$.

Proof.

i) $b \in \Gamma \quad \therefore b \in F(\Gamma) \quad \therefore \Gamma \vdash b$

ii) $\Gamma \vdash b \quad \therefore b \in F(\Gamma) \quad \therefore b \geq x_1 \wedge x_2 \wedge \ldots \wedge x_n$ for some $x_i \in \Gamma$

$\Gamma \subseteq \Sigma \quad \therefore b \in F(\Sigma) \quad \therefore \Sigma \vdash b$

iii) Let $\Gamma \vdash b \quad \therefore b \in F(\Gamma)$

$\therefore b \geq x_1 \wedge x_2 \wedge \ldots \wedge x_n$ for some $x_i \in \Gamma$

$\Sigma \vdash x_1, \Sigma \vdash x_2, \ldots, \Sigma \vdash x_n$

$\Sigma \vdash x_1 \wedge x_2 \wedge \ldots \wedge x_n$ by proposition (6)

$x_1 \wedge x_2 \wedge \ldots \wedge x_n \in F(\Sigma) \qquad \therefore b \in F(\Sigma)$

$\therefore \Sigma \vdash b$

iv) $\Gamma \vdash b \quad \therefore b \in F(\Gamma) \quad \therefore b \geq x_1 \wedge x_2 \wedge \ldots \wedge x_n$ for some $x_i \in \Gamma$

$\sigma(b) \geq \sigma(x_1 \wedge x_2 \wedge \ldots \wedge x_n) = \sigma(x_1) \wedge \sigma(x_2) \wedge \ldots \wedge \sigma(x_n)$

for some $\sigma(x_i) \in F(\sigma(\Gamma))$

$\therefore \sigma(b) \in F(\sigma(\Gamma)) \quad \therefore \sigma(\Gamma) \vdash \sigma(b)$

v) By proposition 4 (ii) and proposition (6)

IV. FUNCTIONAL POLYADIC ALGEBRA

Let $B = (B, \vee, \wedge, ', 0, 1)$ be a complete Boolean algebra, $B^A = \{p \mid p : A \to B \quad is \quad a \quad function\}$ where A is an algebra of type F. For $p, q \in B^A$ define $p \vee q$, $p \wedge q$, p' and $0,1$ pointwise as follows:

$(p \vee q)(a) = p(a) \vee q(a)$,

$(p \wedge q)(a) = p(a) \wedge q(a)$, $p'(a) = (p(a))'$,

$0(a) = 0$, $1(a) = 1$ for any $a \in A$. We have

Proposition 8) [2]

$(B^A, \vee, \wedge, ', 0, 1)$ is a functional Boolean algebra.

Proposition 9) [5]

$(B^A, \exists)$ is a monadic algebra, where $\exists : B^A \to B^A$ is given by $\exists(p)(a) = \sup\{p(a) : a \in A\}$

Proposition 10) [2]

$(B^A, I, S, \exists)$ is a polyadic algebra, where $S : I^I \to End(B^A)$ and $\exists : 2^I \to Qant(B^A)$.

Now, let $B^{A^k} = \{p \mid p : A^k \to B \quad is \quad a \quad function\}$; $k = 0,1,2,\ldots$. Define $S(\sigma)p(a_1,\ldots,a_k) = p(a_{\sigma(1)},\ldots,a_{\sigma(k)})$ for any $\sigma \in I^I$ and $(a_1,\ldots,a_k) \in A^k$. we have

Proposition 11)

$(B^{A^k}, I, S, \exists)$ is a polyadic algebra

Proof

i) $\exists(\phi)p(a_1,\ldots,a_k) = \sup_{\phi}\{p(a_1,\ldots,a_k)\} = p(a_1,\ldots,a_k)$

$\therefore \exists(\phi) = id$

ii) $\exists(J \cup M) = \sup_{J \cup M}\{p(a_1,\ldots a_k)\}$

$= \sup_{J}\{p(a_1,\ldots a_k)\}\sup_{K}\{p(a_1,\ldots,a_k)\}$

$= \exists(J)p(a_1,\ldots,a_k)\exists(M)p(a_1,\ldots,a_k)$

$\therefore \exists(J \cup M) = \exists(J)\exists(M)$

iii) $S(id)p(a_1,\ldots,a_k) = p(a_{id(1)},\ldots a_{id(k)}) = p(a_1,\ldots,a_k)$

$\therefore S(id) = id$

iv) $S(\sigma\tau)p(a_1,\ldots a_k) = p(a_{\sigma\tau(1)},\ldots,a_{\sigma\tau(k)}) = p(a_{\sigma(\tau(1))},\ldots,a_{\sigma(\tau(k))})$

$= S(\sigma)p(a_{\tau(1)},\ldots,a_{\tau(k)}) = S(\sigma)S(\tau)p(a_1,\ldots,a_k)$

$\therefore S(\sigma\tau) = S(\sigma)S(\tau)$

v) $S(\sigma)\exists(J)p(a_1,\ldots,a_k) = S(\sigma)\sup_{J}\{p(a_1,\ldots,a_k)\}$

$= \sup_{J}\{p(a_{\sigma(1)},\ldots a_{\sigma(k)})\}$

$= \sup_{J}\{p(a_{\tau(1)},\ldots,a_{\tau(k)})\}$

if $\sigma|_{I-J} = \tau|_{I-J}$

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$$= S(\tau)\sup_J\{p(a_1,...,a_k)\}$$

$$= S(\tau)\exists(J)\, p(a_1,...,a_k)$$

$$S(\sigma)\exists(J) = S(\tau)\exists(J) \quad \text{if } \sigma\big|_{I-J} = \tau\big|_{I-J}$$

vi) $\exists(J)S(\tau)\, p(a_1,...,a_k) = \exists(J)\, p(a_{\tau(1)},...,a_{\tau(k)})$

$$= \sup_J\{p(a_{\tau(1)},...,a_{\tau(k)})\}$$

if τ is $1-1$ on $\tau^{-1}(J)$

$$= \sup_{\tau^{-1}(J)}\{p(a_{\tau(1)},...,a_{\tau(k)})\}$$

$$= \sup_{\tau^{-1}(J)}\{S(\tau)\, p(a_1,...,a_k)\}$$

$$= S(\tau)\exists(\tau^{-1}(J))\, p(a_1,...,a_k)$$

$$\therefore \exists(J)S(\tau) = S(\tau)\exists(\tau^{-1}(J))$$

Finally, let $I = \{1,2,...\}$ be a countable set,

$B^{A^I} = \{p \mid p : A^I \to B \text{ is a function}\}$,

$a \in A^I$; $a : I \to A$ is a function. For $\sigma \in I^I$, σa is defined by $\sigma a(i) = a(\sigma(i))\ \forall i \in I$.

We have

Proposition 12)

B^{A^I} is a polyadic algebra

Proof

i) $\exists(\phi)\, p(a) = \sup_\phi\{p(a)\} = p(a)$

$$\therefore \exists(\phi) = id$$

ii) $\exists(J \cup M)\, p(a) = \sup_{J\cup M}\{p(a)\} = \sup_J\{p(a)\}\sup_M\{p(a)\}$

$$= \exists(J)\, p(a)\exists(M)\, p(a)$$

$$\therefore \exists(J \cup M) = \exists(J)\exists(M)$$

iii) $S(id)\, p(a(i)) = p(a(id(i))) = p(a(i))$

$$\therefore S(id) = id$$

iv) $S(\sigma\tau)\, p(a(i)) = p(a\sigma(\tau(i))) = p(\sigma a(\tau(i)))$

$$= S(\sigma)\, p(\tau a(i)) = S(\sigma)S(\tau)\, p(a(i))$$

$$\therefore S(\sigma\tau) = S(\sigma)S(\tau)$$

v) $S(\sigma)\exists(J)\, p(a(i)) = S(\sigma)\sup_J\{p(a(i))\} = \sup_J\{p(a\sigma(i))\}$

$$= \sup_J\{p(a\tau(i))\}$$

if $\sigma\big|_{I-J} = \tau\big|_{I-J}$

$$= S(\tau)\sup_J\{p(a(i))\} = S(\tau)\exists(J)\, p(a(i))$$

$$\therefore S(\sigma)\exists(J) = S(\tau)\exists(J)$$

vi) $\exists(J)S(\tau)\, p(a(i)) = \exists(J)\, p(a\tau(i)) = \sup_J\{p(a\tau(i))\}$

$= \sup_{\tau^{-1}(J)}\{p(a\tau(i))\}$ if τ is 1-1 on $\tau^{-1}(J)$

$$= \sup_{\tau^{-1}(J)}\{S(\tau)\, p(a(i))\} = S(\tau)\sup_{\tau^{-1}(J)}\{p(a(i))\}$$

$$= S(\tau)\exists(\tau^{-1}(J))\, p(a(i))$$

$$\therefore \exists(J)S(\tau) = S(\tau)\exists(\tau^{-1}(J))$$

Natural inference rules of polyadic logic are now transferred into certain algebraic rules in B^A governing the algebraic deduction in $F(\Gamma)$ where $\Gamma \subseteq B^A$. These are given as follows:

Theorem 13)

Let $\Gamma \subseteq B^A$.Then

i) $\Gamma \vdash 1$ and $\Gamma \nvdash 0$.

ii) $\Gamma \vdash p \wedge q$ iff $\Gamma \vdash p$ and $\Gamma \vdash q$.

iii) $\Gamma \vdash p$ iff $\Gamma \nvdash p'$.

iv) $\Gamma \vdash p \vee q$ iff $\Gamma \vdash p$ or $\Gamma \vdash q$.

v) $\Gamma \vdash p$ iff $\Gamma \vdash \forall(J)p$.

vi) $F(\Gamma) \vdash p(a_0)$ iff $F(\Gamma) \vdash \exists(J)p(a)$.

Proof.

i) By proposition 6(i)

ii) Follows from definition of filter and proposition 1

iii) This is so because if both $\Gamma \vdash p$ and $\Gamma \vdash p'$ we get $p(a)$ and $(p(a))'$ in $F(\Gamma)$. Thus we get 0 ,which is a contradiction.

iv)Let $\Gamma \vdash p \vee q$

$\therefore p(a) \vee q(a) \in F(\Gamma)$

$\therefore (p(a) \vee q(a))' \notin F(\Gamma)$

$(p(a))' \wedge (q(a))' \notin F(\Gamma)$

$\therefore (p(a))' \notin F(\Gamma)$ or $(q(a))' \notin F(\Gamma)$

$\therefore p(a) \in F(\Gamma)$ or $q(a) \in F(\Gamma)$.

Thus $\Gamma \vdash p$ or $\Gamma \vdash q$.

Let $\Gamma \vdash p$ or $\Gamma \vdash q$

$\therefore p(a) \in F(\Gamma)$ or $q(a) \in F(\Gamma)$

$\therefore p(a) \vee q(a) \geq p(a)$

$\therefore p(a) \vee q(a) \in F(\Gamma)$. Therefore $\Gamma \vdash p \vee q$.

v)This is by $\forall p \leq p$ and $\forall(F(\Gamma)) \subseteq F(\Gamma)$.

vi)This is so by the definition
$\exists(J)p(a) = \sup_J\{p(a) : a \in A\}$.

If we alter the definition of deduction to $\Gamma \vdash p$ if $p \in UF(\Gamma)$ we get the following dichotomy by proposition 5

Theorem 14)
Either $\Gamma \vdash p$ or $\Gamma \vdash p'$.

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