On the Structure of Some Groups Containing $PSL(2,7)\,wr\,M_{11}$

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Abstract—**In this paper, we will generate the wreath product $PSL(2,7)\,wr\,M_{11}$ using only two permutations. Also, we will show the structure of some groups containing the wreath product $PSL(2,7)\,wr\,M_{11}$. The structure of the groups founded is determined in terms of wreath product $(PSL(2,7)\,wrM_{11})\,wr\,C_k$. Some related cases are also included. Also, we will show that S_{77k+1} and A_{77K+1} can be generated using the wreath product $(PSL(2,7)\,wrM_{11})\,wr\,C_k$ and a transposition in S_{77k+1} and an element of order 3 in A_{77K+1}. We will also show that S_{77k+1} and A_{77K+1} can be generated using the wreath product $PSL(2,7)\,wr\,M_{11}$ and an element of order $k+1$.**

Keywords— **Group presentation, group generated by n-cycle, Wreath product, Mathieu group,.**

I. Introduction

Hammas and Al-Amri [1], have shown that A_{2n+1} of degree $2n+1$ can be generated using a copy of S_n and an element of order 3 in A_{2n+1}. They also gave the symmetric generating set of Groups A_{kn+1} and S_{kn+1} using S_n [5] .

Shafee [2] showed that the groups A_{kn+1} and S_{kn+1} can be generated using the wreath product A_m wr S_a and an element of order *k*+1. Also she showed how to generate S_{kn+1} and A_{kn+1} symmetrically using *n* elements each of order *k*+1.

In [3], Shafee and Al-Amri have shown that the groups A_{110k+1} and S_{110k+1} can be generated using the wreath product $L_2(9)wrM_{11}$ and an element of order *k*+1.

$PSL(2,7)$ and M_{11} are two groups of the well known simple groups. In [6], they are fully described. In a matter of fact, they can be faintly presented in different ways. They have presentations in [6] as follows :

$$PSL(2,7) =< X,Y \mid X^7 = (X^4Y)^4 = (XY)^3 = Y^2 = 1 >$$

$$M_{11} =< X,Y,Z \mid X^{11} = Y^5 = (XZ)^3 = 1, X^Y = X^4 = Y^Z = Y^2 >$$

$PSL(2,7)$ can be generated using two permutations, the first is of order 7 and an involution as follows : $PSL(2,7) =< (1,2,3,4,5,6,7)(1,2)(3,5) >$.

M_{11} can be generated using two permutations, the first is of order 11 and two cycles each of order 4 as follow:

$M_{11} =< (1,2,...,11)(1,2,3,7,6)(4,8,5,9,10) >$

In this paper, we will generate the wreath product $(PSL(2,7)\,wrM_{11})$ using only two permutations. Also, we show the structure of some groups containing the wreath product $(PSL(2,7)\,wrM_{11})$. The structure of the groups founded is determined in terms of wreath product $(PSL(2,7)\,wrM_{11})\,wr\,C_k$. Some related cases are also included. Also, we will show that S_{77k+1} and A_{77K+1} can be generated using the wreath product $(PSL(2,7)\,wrM_{11})\,wr\,C_k$ and a transposition in S_{77k+1} and an element of order 3 in A_{77K+1}. We will also show that S_{77k+1} and A_{77K+1} can be generated using the wreath product $(PSL(2,7)\,wrM_{11})$ and an element of order $k+1$.

II. Preliminary Results

DEFINITION 2.1.[6] *The general linear group $GL_n(q)$ consists of all the $n \times n$ matrices that have non-zero determinant over the field F_q with q-elements. The special linear group $SL_n(q)$ is the subgroup of $GL_n(q)$ which consists of all matrices of determinant one. The projective general linear group $PGL_n(q)$ and projective special general linear group $PSL_n(q)$ are the groups optained from $GL_n(q)$ and $SL_n(q)$ .The projective special general linear group $PSL_n(q)$ is also denoted by $L_n(q)$ .The orders of these groups are*

$$| GL_n(q) | = (q-1)N, | SL_n(q) | = | PGL_n(q) | = N, | PSL_n(q) | = | L_n(q) | = \frac{N}{d},$$

where

$$N = q^{\frac{1}{2}n(n-1)} (q^n - 1)(q^{n-1} - 1)...(q^2 - 1)\,and\; d = (q-1, n)$$

DEFINITION 2.2. Let *A* and *B* be groups of permutations on non empty sets Ω_1 and Ω_2 respectively.

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The wreath product of A and B is denote by A wr B and defined as A wr $B = A^{\Omega_2} \times_\theta B$, i.e., the direct product of $|\Omega_2|$ copies of A and a mapping θ

THEOREM 2.3 (Jorden-Moore) [7] *The group* $PSL_n(q)$ *is simple if and only if* $q \geq 3$.

THEOREM 2.4 [4] *Let* G *be the group generated by the* n-*cycle (1, 2, ..., n) and the 2-cycle (n, a). If* $1 < a < n$ *is an integer with* n = am , *then* $G \cong S_m$ wr C_a.

THEOREM 2.5 [4] Let $1 \leq a \neq b < n$ be any integers. Let n be an odd integer and let G be the group generated by the n- cycle (1,2,…,n) and the 3-cycle (n, a, b). If the $hcf(n,a,b)$=1, then $G = A_n$. While if n can be an even then $G = S_n$.

THEOREM 2.6 [4] Let $1 \leq a < n$ be any integer. Let $G = \langle (1,2,...,n), (n,a) \rangle$. If $h.c.f.(n,a) = 1$, then $G = S_n$.

THEOREM 2.7 [4] Let $1 \leq a \neq b < n$ be any integers. Let n be an even integer and let G be the group generated by the (n-1)-cycle $(1, 2, ..., n-1)$ and 3-cycle (n,a,b). Then $G = A_n$.

III. THE RESULTS

THEOREM 3.1 The wreath product $(PSL(2,7)\, wrM_{11})\, wr\, C_k$ can be generated using two permutations, the first is of order 252 and the second is of order 4.

Proof : Let $G = \langle X,Y \rangle$, where: X=(1, 2, 3, 4, …, 77), which is a cycle of order 252, Y=(1, 9)(2, 6)(4, 5)(7, 8)(12, 20, 23, 31)(13, 17)(15, 16)(18, 19)(24, 28)(26, 27) (29, 30)(34, 42, 56, 64)(35, 39)(37, 38)(40, 41)(45, 53)(46, 50)(48, 49)(51, 52)(57, 61)(59, 60)(62, 63)(67, 75)(68, 72)(70, 71)(73, 74), which is the product of two cycles each of order 4 and twenty four transpositions. Let $\alpha_1 = \left((XY)^6 [X,Y]^5\right)^{18}$. Then

$$\alpha_1 = (17, 22, 33, 44, 55, 66, 77),$$

which is a cycle of order 7. Let $\alpha_2 = \alpha_1^{-1} X$. It is easy to show that

$$\alpha_2 = (1, 2, 3, \ldots, 11)(12, 13, 14, \ldots, 22) \ldots (67, 68, 69, \ldots, 77),$$

which is the product of seven cycles each of order 11. Let: $\beta_1 = (Y^2)^{(XY)^{18}}$ =(9, 20)(12, 23)(31, 53)(34, 56), $\beta_2 = \beta_1 Y^{-1}$ =(1, 9, 12, 20)(2, 6)(4, 5) (7, 8)(13, 17)(15, 16)(18, 19)(23, 31, 45, 53)(24, 28)(26, 27)(29, 30)(34, 42)(35, 39)(37, 38)(40, 41)(46, 50)(48, 49)(51, 52)(56, 64)(57, 61)(59, 60)(62, 63)(67, 75)(68, 72)(70, 71) (73, 74), $\beta_3 = (Y^3 \beta_2)^2$ =(1, 45)(12, 23), $\beta_4 = \beta_3^{(\alpha_2^{-1}\alpha_1^3)}$ =(11, 44)(55, 66) and $\beta_5 = \beta_4^{\beta_3^{\alpha_2^{-1}}}$ = (11, 66)(44, 55). Let $\alpha_3 = \beta_5^{\beta_3^{(\alpha_2^{-1}\alpha_1)}}$. Hence

$$\alpha_3 = (11, 22)(33, 55).$$

Let $\alpha_4 = YX^{-1}\alpha_3^{-1}X$. We can conclude that

$$\alpha_4 = (1,9)(2,6)(4,5)(7,8)(12,20)(13,17)(15,16)(18,19)(23,31)(24,28)(26,27)(29,30)(34,42)(35,39)(37,38)(40,41)(45,53)(46,50)(48,49)(51,52)(56,64)(57,61)(59,60)(62,63)(67,75)(68,72)(70,71)(73,74),$$

which is the product of twenty eight transpositions. Let $K = \langle \alpha_2, \alpha_4 \rangle$. Let $\theta : K \to L_2(11)$ be the mapping defined by

$\theta(11i+j) = j \quad \forall\ 0 \leq i \leq 6,\ \forall\ 1 \leq j \leq 11$ Since $\theta(\alpha_2) = (1, 2, \ldots, 11)$ and $\theta(\alpha_4) = (1, 9)(2, 6)(4, 5)(7, 8)$, then $K \cong \theta(K) = M_{11}$. Let $H_0 = \langle \alpha_1, \alpha_3 \rangle$. Then $H_0 \cong PSL\ (2,7)$. Moreover, K conjugates H_0 into H_1, H_1 into H_2 and so it conjugates H_{10} into H_0 , where $H_i = \langle (i, 11+i, 22+i, 33+i, 44+i, 55+i, 66+i)(i, 11+i)(22+i, 44+i) \rangle$ $\forall\ 1 \leq i \leq 10$. Hence we get $PSL(2,7)wrM_{11} \subseteq G$.

On the other hand, Since X= $\alpha_1\alpha_2$ and Y= $\alpha_4\alpha_3^X$, then $G \subseteq PSL(2,7)wrM_{11}$. Hence $G = PSL(2,7)wrM_{11}$ ◊

THEOREM 3.2 The wreath product $(PSL(2,7)\, wrM_{11})\, wr\, C_k$ can be generated using two permutations, the first is of order 77k and an involution, for all integers k ≥ 1.

Proof : Let σ = (1, 2, …, 77k) and τ =(k, 9k)(2k, 6k)(4k, 5k)(7k, 8k)(12k, 20k, 23k, 31k)(13k, 17k)(15k, 16k)(18k, 19k)(24k, 28k)(26k, 27k)(29k, 30k)(34k, 42k, 56k, 64k)(35k, 39k)(37k, 38k)(40k, 41k)(45k, 53k)(46k, 50k)(48k, 49k)(51k, 52k)(57k, 61k) (59k, 60k) (62k, 63k)(67k, 75k)(68k, 72k)(70k, 71k). If k=1, then we get the group $PSL\ (2,7)wrM_{11}$ which can be considered as the trivial wreath product $PSL\ (2,7)wrM_{11}$ wr<id>. Assume that $k > 1$. Let $\alpha = \prod_{i=0}^{11} \tau^{\sigma^{ik}}$, we get an element $\delta = \alpha^{45}$ =(k, 2k, 3k, …, 77k). Let $G_i = \langle \delta^{\sigma^i}, \tau^{\sigma^i} \rangle$, be the groups acts on the sets Γ_i ={ i, k+i, 2k+i,…, 76k+i }, for all $1 \leq i \leq k$. Since $\bigcap_{i=1}^{k} \Gamma_i = \varphi$, then we get the direct product $G_1 \times G_2 \times \ldots \times G_k$, where, by theorem 3.1 each $G_i \cong PSL(2,7)wrM_{11}$. Let $\beta = \delta^{-1}\sigma$ = (1, 2, …, k)(k+1, k+2, …, 2k) … (76k+1, 76k+2, …, 77k). Let $H = \langle \beta \rangle \cong C_k$. H conjugates G_1 into G_2, G_2 into G_3,…and G_k into G_1. Hence we get the wreath product $(PSL(2,7)wrM_{11})wrC_K \subseteq G$. On the other hand, since $\delta\,\beta$ = (1, 2, …, k, k+1, k+2, …, 2k, …,

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76k+1, 76k+2, …, 77k)=σ , then $\sigma \in (PSL(2,7)wrM_{11})wrC_K$. Hence $G=<\sigma,\tau>\cong (PSL\ (2,3)wrM_{11})wrC_K$.◊

THEOREM 3.3 The wreath product $(PSL(2,7)\,wr\,M_{11}))\,wr\,S_k$ can be generated using three permutations, the first is of order 77k, the second and the third are involutions, for all $k \geq 2$.

Proof : Let σ = (1, 2, …, 77k), τ =(k, 9k)(2k, 6k)(4k, 5k)(7k, 8k)(12k, 20k, 23k, 31k)(13k, 17k)(15k, 16k)(18k, 19k)(24k, 28k)(26k, 27k)(29k, 30k)(34k, 42k, 56k, 64k)(35k, 39k)(37k, 38k)(40k, 41k)(45k, 53k)(46k, 50k)(48k, 49k)(51k, 52k)(57k, 61k)(59k, 60k)(62k, 63k)(67k, 75k)(68k, 72k)(70k, 71k) and μ = (1, 2)(k+1, k+2)(2k+1, 2k+2) … (76k+1, 76k+2). Since by Theorem 3.2, $<\sigma,\tau>= (PSL(2,7)\,wrM_{11})\,wr\,C_k$ and (1, 2, …, k)(k+1, k+2, …, 2k) … (76k+1, …,77k)∈ $(PSL(2,7)\,wrM_{11})\,wr\,C_k$

then ⟨(1,…,k)(k+1,…,2k)…(76k+1,…,77k) , μ ⟩ $\cong S_k$. Hence $G=\langle\sigma,\ \tau,\ \mu\rangle$ $(PSL(2,7)\,wr\,M_{11}))\,wr\,S_k$. ◊

COROLLARY 3.4 The wreath product $(PSL(2,7)\,wr\,M_{11}))\,wr\,A_k$ can be generated using three permutations, the first is of order 77k, the second is an involution and the third is of order 3, for all odd integers $k \geq$ 3.

THEOREM 3.5 The wreath product $(PSL(2,7)\,wr\,M_{11})\,wr(S_m\ wr\ C_a)$ can be generated using three permutations, the first is of order 252k, the second and the third are involutions, where $k=am$ be any integer with $1<a<k$.

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Proof : Let σ = (1, 2, …, 77k), τ =(k, 9k)(2k, 6k)(4k, 5k)(7k, 8k)(12k, 20k, 23k, 31k)(13k, 17k)(15k, 16k)(18k, 19k)(24k, 28k)(26k, 27k)(29k, 30k)(34k, 42k, 56k, 64k)(35k, 39k)(37k, 38k)(40k, 41k)(45k, 53k)(46k, 50k)(48k, 49k)(51k, 52k)(57k, 61k)(59k, 60k)(62k, 63k)(67k, 75k)(68k, 72k)(70k, 71k) and μ = (k, a)(2k, k+a)(3k, 2k+a) … (77k, 76k+a). Since by Theorem 3.2, $<\sigma,\tau>\cong (PSL(2,7)\,wrM_{11})\,wr\,C_k$ and (1, …, k)(k+1, …, 2k) … (251k+1, … ,77k)∈ $(PSL(2,7)\,wrM_{11})\,wr\,C_k$ then ⟨(1, …, k)(k+1, …, 2k) …(76k+1, …, 77k, μ ⟩$\cong$ $(S_m \mathrm{wr}\ C_a)$. Hence $G=\langle\sigma,\ \tau,\ \mu\rangle \cong (PSL(2,7)wrM_{11})wr(S_m wrC_a)$.◊

THEOREM 3.6 S_{77k+1} and A_{77k+1} can be generated using the wreath product $(PSL(2,7)\,wrM_{11})\,wr\,C_k$ and a transposition in S_{77k+1} for all integers $k>1$ and an element of order 3 in A_{77k+1} for all odd integers $k>1$.

Proof: Let σ = (1, 2, …, 77k), τ =(k, 9k)(2k, 6k)(4k, 5k)(7k, 8k)(12k, 20k, 23k, 31k)(13k, 17k)(15k, 16k)(18k, 19k)(24k, 28k)(26k, 27k)(29k, 30k)(34k, 42k, 56k, 64k)(35k, 39k)(37k, 38k)(40k, 41k)(45k, 53k)(46k, 50k)(48k, 49k)(51k, 52k)(57k, 61k)(59k, 60k)(62k, 63k)(67k, 75k)(68k, 72k)(70k, 71k), μ =(252k+1,1) and μ' =(1,k, 252k+1) be four permutations, of order 77k, 2, 2 and 3 respectively. Let $H=\langle\sigma,\ \tau\rangle$. By theorem 3.2 $H \cong (PSL(2,7)\,wrM_{11})\,wr\,C_k$.

Case 1: Let $G=\langle\sigma,\ \tau,\ \mu\rangle$. Let $\alpha=\sigma\mu$, then $\alpha=(1,2,...,77k,77k+1)$ which is a cycle of order $77k+1$. By theorem 2.6 $G<\sigma,\tau,\mu'>\cong<\alpha,\mu>\cong\ S_{77k+1}$.

Case 2: Let . By theorem 2.7 $<\sigma,\mu'>\cong A_{77k+1}$. Since $G=\langle\sigma,\ \tau,\mu'\rangle$ τ is an even permutation, then $G\cong A_{77k+1}$.

THEOREM 3.7 S_{77k+1} and A_{77k+1} can be generated using the wreath product $PSL(2,7)\,wr\,M_{11}$ and an element of order $k+1$ in S_{77k+1} and A_{77k+1} for all integers $k\geq 1$.

Proof: Let G=$\langle\sigma,\ \tau\ ,\mu\rangle$, where, σ =(1, 2, 3, 77)(77(k-(k-1))+1, …, 77(k-(k-1))+77) … (77(k-1)+1, …, 77(k-1)+77), τ =(1, 9)(2, 6)(4, 5)(7, 8)(12, 20, 23, 31)(13, 17)(15, 16)(18, 19)(24, 28)(26, 27)(29, 30)(34, 42, 56, 64)(35, 39)(37, 38)(40, 41)(45, 53)(46, 50)(48, 49)(51, 52)(57, 61)(59, 60)(62, 63)(67, 75)(68, 72)(70, 71)(73, 74) … (77(k-1)+1, 77(k-1)+9) … (77(k-1)+73, 77(k-1)+74), and μ =(77, 154, …,77k, 77k+1), where k-i >0, be three permutations of order 77, 4 and k+1 respectively. Let $H=\langle\sigma,\ \tau\rangle$. Define the mapping θ as follows ;

$$\theta(11(k-\ i)+j)=j\ \ \forall\ 1\leq i\leq k\ ,\ \forall\ \ 1\leq j\leq 11$$

Hence $H=<\sigma,\tau>\cong PSL(2,7)\,wr\,M_{11}$. Let $\alpha=\mu\sigma$ it is easy to show that $\alpha=(1,2,3,...,77k+1)$, which is a cycle of order $77k+1$. Let $\mu'=\mu^{\sigma}=(1,78,...,77(k-1)+1,77k+1)$ and $\beta=[\mu,\ \mu']=(1,77,77k+1)$. Since $h.c.f(1,77,77k+1)$, then by theorem 2.5 $G=\langle\sigma,\tau,\mu\rangle\cong\langle\alpha,\beta\rangle\ S_{77k+1}$ or A_{77k+1} depending on whether k is an odd or an even integer respectively. ◊

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