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A General Iterative Scheme for Variational Inequality Problems and Fixed Point Problems

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Abstract—**We introduce a general iterative scheme for finding a common of the set solutions of variational inequality problems for an inverse-strongly monotone mapping and the set of common fixed points of a countable family of nonexpansive mappings in a real Hilbert space. We show that the sequence converges strongly to a common element of the above two sets under some parameters controlling conditions. The results presented in this paper improve and extend the corresponding results announced by many others.**

***Index Terms*—Fixed point, variational inequality, optimization problem, nonexpansive mapping**

I. Introduction

Let H be a real Hilbert space with inner product and norm, are denoted by $\langle \cdot,\cdot \rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H, and let $B: C \to H$ be a nonlinear map. The classical variational inequality which is denoted by VI(C,B) is to find $v \in C$ such that $\langle Bv, u-v \rangle \geq 0, \forall u \in C.$ The variational inequality has been extensively studied in literature. See, for example, [6], [7], [9], and the references therein. A mapping A of C into H is called $\alpha-$ inverse-strongly monotone, see [12]-[13], if there exists a positive real number α such that $\langle Au - Av \rangle \geq \alpha \|Au - Av\|^2$, $\forall u, v \in C$. A mapping T of C into itself is called nonexpansive if $\|Tu - Tv\| \leq \|u - v\|, \forall u, v \in C.$ We denoted by F(T) the set of fixed points of T. A mapping $f: C \to C$ is said to be contractive with coefficient $\alpha \in (0,1)$, if $\|f(u) - f(v)\| \leq \alpha \|u - v\|, \forall u, v \in C.$ Let G be a strongly positive bounded linear operator on H: that is, there is a constant $\bar{\gamma} > 0$ with property $\langle Gx, x \rangle \geq \bar{\gamma} \|x\|^2, \forall x \in H.$ Recently, many authors proposed some new iterative schemes for finding element in $F(S) \cap VI(C,B)$, see [1]-[3], [5], [8], [13], and reference therein. Moreover, Jung [4] introduced the following iterative scheme as the following. Let C a nonempty closed convex subset of a real Hilbert space H such that $C \pm C \subset C$. Let A be an $\alpha-$ inverse-strongly monotone mapping of C into H and S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C,A) \neq \emptyset$. Let $u \in C$ and let B be a strong positive bounded linear operator on C with constant $\bar{\gamma} \in (0,1)$ and f be a contractive of C into itself with constant $k \in (0,1)$. Assume that $\mu > 0$ and $0 < \gamma < (1+\mu)\bar{\gamma}/k$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$,

$$y_n = \alpha_n(u + \gamma f(x_n)) + (I - \alpha_n(I + \mu B))SP_C(x_n - \lambda_n Ax_n),$$
$$x_{n+1} = (1-\beta_n)y_n + \beta_n SP_C(y_n - \lambda_n Ay_n), n \geq 1.$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$, $\{\lambda_n\}$, and $\{\beta_n\}$ of parameters, then the sequence $\{x_n\}$ converges strongly to $q \in F(S) \cap VI(C,A)$, which is a solution of the optimization problem:

$$\min_{x \in F(S) \cap VI(C,A)} \tfrac{\mu}{2}\langle Bx, x \rangle + \tfrac{1}{2}\|x - u\|^2 - h(x),$$

where h is a potential function for γf. In this paper motivated by the iterative scheme proposed by Jung [4], we will introduce a general iterative for a common element of the set solution of variational inequality problem for an inverse-strongly monotone mapping and the set of common fixed points of a countable family of nonexpansive mappings which will present in the main result.

II. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. It well known that H satisfies the Opial's condition, that is, for any sequence $\{x_n\}$ with $\{x_n\}$ converges weakly to x (denote by $x_n \xrightarrow{w} x$), the inequality: $\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$ holds for every $y \in H$ with $y \neq x$. For every point $x \in H$, there exist a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. P_C is called the metric projection of H onto C. It well known that P_C is a nonexpansive mapping of H onto C and satisfies $\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2$, $\forall x, y \in H.$ Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and $\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C y\|^2$, $x \in H, y \in C$. It is easy to see that $u \in VI(C,A) \Leftrightarrow u = P_C(u - \lambda Au),\ \lambda > 0.$

Proposition 2.1 (See [4].) Let C b e a bounded nonempty closed convex subset of a real Hilbert space H and let B be an $\alpha-$ inverse-strongly monotone mapping of C into H. Then, VI(C,B) is nonempty.

A set-valued mapping $M: H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Mx$ and $g \in My$ imply $\langle x - y, f - g \rangle \geq 0.$

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A monotone mapping $M: H \to 2^H$ is maximal if the graph G(T) of T is not properly contained in the graph of any other monotone mapping. It well known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H, \langle x - y, f - g \rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let B be an inverse-strongly monotone mapping C into H and let $N_C v$ be the normal cone to C at v, that is, $N_C v = \{w \in H : \langle v - u, w \rangle \geq 0,$ for all $u \in C\}$, and define

$$Tv = \begin{cases} Bv + N_C v, v \in C, \\ \emptyset, v \neq C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, B)$ (see [9], [10], [12]).

The following Lemmas will be useful for proving our theorem in the next section.

Lemma 2.1 (See [9].) Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that

$a_{n+1} \leq (1 - \alpha_n) a_n + \delta_n, \ n \geq 0$

where $\{\alpha_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence in R such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2 (See [11].) Let K be a nonempty closed convex subset of a Banach space and let $\{T_n\}$ be a sequence of mappings of K into itself. Suppose that $\sum_{n=1}^{\infty} \sup\{\|T_{n+1} z - T_n z : z \in K\|\} < \infty$. Then, for each $y \in K$, $\{T_n y\}$ converges strongly to some point of K. Moreover, let T be a mapping of K into itself defined by $Ty = \lim_{n\to\infty} T_n y$ for all $y \in K$. Then $\lim_{n\to\infty} \sup\{\|Tz - T_n z\| : z \in K\} = 0$.

Lemma 2.3 (See [4].) In a real Hilbert space H, there holds the inequality

$$\|z + y\|^2 \leq \|z\|^2 + 2\langle y, z + y \rangle.$$

Lemma 2.4 (See [4].) Let C be a bounded nonempty closed convex subset of a real Hilbert space H, and let $g : C \to R \cup \{\infty\}$ be a proper lower semicontinuous differentiable convex function. If x^* is a solution to the minimization problem $g(x^*) = \inf_{x\in C} g(x)$, then $\langle g'(x), x - x^* \rangle \geq 0, x \in C$. In particular, if x^* solves the optimization problem

$$\min_{x\in C} \tfrac{\mu}{2}\langle Bx, x \rangle + \tfrac{1}{2}\|x - u\|^2 - h(x),$$

then $\langle u + (\gamma f - (I + \mu B)) x^*, x - x^* \rangle \leq 0, x \in C,$ where h is a potential function for γf.

Lemma 2.5 (See [9].) Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.

III. Main Result

In this section, we prove a strong convergence theorem.

Theorem 3.1. Let C be a nonempty closed convex subset of a real Hilbert space H such that $C \pm C \subset C$. Let B be an $\alpha-$ inverse-strongly monotone mapping of C into H and $\{T_n\}$ be a sequence of nonexpansive mappings of C into itself such that $\Omega := \bigcap_{n=1}^{\infty} F(T_n) \cap VI(C, B) \neq \emptyset$. Let $u \in C$ and let A be a strongly positive bounded linear operator on C with constant $\bar{\gamma} \in (0,1)$ and f be a contractive of C into itself with constant $\alpha \in (0,1)$. Assume that $\mu > 0$ and $0 < \gamma < (1+\mu)\bar{\gamma}/\alpha$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$,

$$\begin{aligned} y_n &= \alpha_n (u + \gamma f(x_n)) + (I - \alpha_n (I + \mu A)) T_n P_C (x_n - \lambda_n B x_n), \\ x_{n+1} &= (1 - \beta_n) y_n + \beta_n T_n P_C (y_n - \lambda_n B y_n), n \geq 1, \end{aligned} \tag{3.1}$$

where $\{\alpha_n\} \subset [0,1), \{\lambda_n\} \subset [0, 2\alpha]$, and $\{\beta_n\} \subset [0,1]$ satisfy the following conditions:

i) $\lim_{n\to\infty} \alpha_n = 0$; $\sum_{n=1}^{\infty} \alpha_n = \infty$;

ii) $\beta_n \in [0, b)$ for all $n \geq 0$ and for some $b \in (0,1)$;

iii) $\lambda_n \in [r, s]$ for all $n \geq 0$ and for some r, s with $0 < r < s < 2\alpha$;

iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty,$ and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.$

Suppose that $\sum_{n=1}^{\infty} \sup\{\|T_{n+1} z - T_n z\| : z \in D\} < \infty$ for any bounded subset D of C. Let T be mapping of H into itself defined by $Tx = \lim_{n\to\infty} T_n x$, for all $x \in C$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Then, $\{x_n\}$ converges strongly to $\omega \in F(T) \cap VI(C, B)$, which is a solution of the optimization problem

$$\min_{x\in F(T)\cap VI(C,B)} \tfrac{\mu}{2}\langle Ax, x \rangle + \tfrac{1}{2}\|x - u\|^2 - h(x), \tag{3.2}$$

where h is a potential function for γf.

Proof. From the condition i), we may assume that $\alpha_n \leq (1 + \mu\|A\|)^{-1}$. Applying Lemma 2.5 and by the same argument as that in the proof of Jung ([4], Theorem 3.1, pp. 6-7),we have that $\langle (I - \alpha_n (I + \mu A)) u, u \rangle \geq 1 - \alpha_n - \alpha_n \mu \langle Au, u \rangle \geq 0$, $\|I - \alpha_n (I + \mu A))\| < 1 - \alpha_n (1+\mu)\bar{\gamma}$, $\|t_n - v\| \leq \|x_n - v\|$, and $\|v_n - v\| \leq \|y_n - v\|$, where $v \in \Omega$, $t_n = P_C(x_n - \lambda_n B x_n)$, and $v_n = P_C(y_n - \lambda_n B y_n)$. Let $\bar{\kappa} = (I + \mu A)$. It follows that

$$\begin{aligned} \|y_n - v\| &= \|\alpha_n u + \alpha_n (\gamma f(x_n) - \bar{\kappa} v) + (I - \alpha_n \bar{\kappa})(T_n - v)\| \\ &\leq (1 - (1+\mu)\bar{\gamma}\alpha_n)\|t_n - v\| + \alpha_n \|u\| + \alpha_n \gamma\alpha \|x_n - v\| \\ &\quad + \alpha_n \|\gamma f(v) - \bar{\kappa} v\| + ((1+\mu)\bar{\gamma} - \gamma\alpha)\alpha_n \tfrac{\|\gamma f(v) - \bar{\kappa} v\| + \|u\|}{(1+\mu)\bar{\gamma} - \gamma\alpha}, \end{aligned}$$

and

$$\|x_{n+1} - v\| = \|(1 - \beta_n)(y_n - v) + \beta_n (T_n v_n - T_n v)\|$$

$$\leq \max\left\{\|x_n - v\|, \tfrac{\|\gamma f(v)-\bar{\kappa} v\|+\|u\|}{(1+\mu)\bar{\gamma}-\gamma\alpha}\right\}.$$

By induction that $\|x_n - v\| \leq \max\left\{\|x_1 - v\|, \tfrac{\|\gamma f(v)-\bar{\kappa} v\|+\|u\|}{(1+\mu)\bar{\gamma}-\gamma\alpha}\right\}, n \geq 1.$ Hence $\{x_n\}$ is bounded, so are $\{y_n\},\{t_n\},\{v_n\}$, $\{f(x_n)\}$, $\{By_n\},\{Bx_n\}$, $\{T_n t_n\},\{\bar{\kappa}T_n t_n\}$, and $\{T_n v_n\}$. Moreover, we observe that

$$\|t_{n+1} - t_n\| \leq \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\|Bx_n\| \tag{3.3}$$

and

$$\|v_{n+1} - v_n\| \leq \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\|Bx_n\|. \tag{3.4}$$

It follows from the assumption and using (3.1), (3.3), and (3.4), we have

$$\begin{aligned}\|y_{n+1} - y_n\| \leq\ & |\alpha_{n+1} - \alpha_n|(\|u\| + \|\gamma f(x_n)\| + \|\bar{\kappa}\|\|T_{n+1}t_n\|)\\ &+\alpha_{n+1}\gamma\alpha\|x_{n+1} - x_n\|\\ &+(1-(1+\mu)\bar{\gamma})\alpha_{n+1})\|x_{n+1} - x_n\|\\ &+(1-(1+\mu)\bar{\gamma}\alpha_n)|\lambda_{n+1} - \lambda_n|\|Bx_n\|\\ &+\alpha_n\bar{\kappa}\sup\{\|T_{n+1}z - T_n z\| : z \in \{t_n\}\}.\end{aligned} \tag{3.5}$$

Then, we obtain

$$\begin{aligned}\|x_{n+2} - x_{n+1}\| \leq\ & (1-(1+\mu)\bar{\gamma} - \gamma\alpha)\alpha_{n+1}\|x_{n+1} - x_n\|\\ &+G_1|\alpha_{n+1} - \alpha_n| + G_2|\lambda_{n+1} - \lambda_n| + G_3,\end{aligned} \tag{3.6}$$

where $G_1 = \sup\left\{\|u\| + \gamma\|f(x_n)\| + \|\bar{\kappa}\|\|T_{n+1}t_n\| : n \in N\right\}$, $G_2 = \sup\left\{\|Bx_n\| + \|By_n\| : n \in N\right\}$, and $G_3 = \sup\left\{\|T_n v_n\| + \|y_n\| : n \in N\right\} + \bar{\kappa}\sup\{\|T_{n+1}z - T_n z\| : z \in \{v_n\}\}$. Applying lemma 2.1 to (3.6), we have

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.7}$$

By using (3.3) and (3.5), we have

$$\lim_{n\to\infty}\|t_{n+1} - t_n\| = 0 \tag{3.8}$$

and

$$\lim_{n\to\infty}\|y_{n+1} - y_n\| = 0. \tag{3.9}$$

From (3.1), we note that

$$\|y_n - T_n t_n\| \leq \alpha_n\|(u + \gamma f(x_n) - \bar{\kappa}T_n t_n\| \to 0\ n \to \infty, \tag{3.10}$$

and

$$\|v_n - t_n\| \leq \|y_n - x_n\|. \tag{3.11}$$

Moreover, by (3.1), (3.11), and the condition ii), we have

$$\|x_{n+1} - y_n\| \leq \tfrac{b}{(1-b)}\left[\|x_{n+1} - x_n\| + \|T_n t_n - y_n\|\right]. \tag{3.12}$$

From (3.12) and using (3.7) and (3.10), we obtain

$$\|x_{n+1} - y_n\| \to 0 \text{ as } n \to \infty. \tag{3.13}$$

We apply that

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \to 0 \text{ as } n \to \infty. \tag{3.14}$$

Let $p \in \Omega$. By the same argument as in [4] (Theorem 3.1, pp. 11-12), we can show that

$$\begin{aligned}\|y_n - p\|^2 \leq\ & \alpha_n\|u + \gamma f(x_n) - \bar{\kappa}p\|^2 + \|x_n - p\|^2\\ &+2\alpha_n\|u + \gamma f(x_n) - \bar{\kappa}\|\|t_n - p\|\\ &+(1-\alpha_n(1+\mu)\bar{\gamma})r(s-2\alpha)\|Bx_n - Bp\|^2.\end{aligned} \tag{3.15}$$

Then, we obtain

$$\begin{aligned}&-(1-\alpha_n(1+\mu)\bar{\gamma})r(s-2\alpha)\|Bx_n - Bp\|^2\\ &\leq \alpha_n\|\gamma u + f(x_n) - \bar{\kappa}p\|^2 + (\|x_n - p\| + \|y_n - p\|)\|x_n - y_n\|\\ &\leq \alpha_n\|\gamma u + f(x_n) - \bar{\kappa}p\|^2 + (\|x_n - p\| + \|y_n - p\|)\|x_n - y_n\|\\ &\quad+2\alpha_n\|\gamma u + f(x_n) - \bar{\kappa}p\|\|t_n - p\|.\end{aligned} \tag{3.16}$$

It follows from the condition i), we have

$$\|Bx_n - Bp\| \to 0 \text{ as } n \to \infty \tag{3.17}$$

Similarly, we can show that

$$\begin{aligned}\|t_n - p\|^2 \leq\ & \|x_n - p\|^2 - \|x_n - t_n\|^2 + 2\lambda_n\langle x_n - t_n, Bx_n\\ &-Bp\rangle - \lambda_n{}^2\|Bx_n - Bp\|^2.\end{aligned} \tag{3.18}$$

Then, we obtain

$$\begin{aligned}&(1-\alpha_n(1+\mu)\bar{\gamma})\|x_n - t_n\|^2\\ &\leq \alpha_n\|u + \gamma f(x_n) - \bar{\kappa}p\|^2 + (\|x_n - p\| + \|y_n - p\|)\|x_n - y_n\|\\ &\quad+2(1-\alpha_n(1+\mu)\bar{\gamma})\lambda_n\langle x_n - t_n, Bx_n - Bp\rangle\\ &\quad-(1-\alpha_n(1+\mu)\bar{\gamma})\lambda_n{}^2\|Bx_n - Bp\|^2\\ &\quad+2\alpha_n\|u + \gamma f(x_n) - \bar{\kappa}p\|\|t_n - p\|.\end{aligned} \tag{3.19}$$

It follows from (3.14), (3.17), and the condition i), we obtain

$$\lim_{n\to\infty}\|x_n - t_n\| = 0 \tag{3.20}$$

and so

$$\|y_n - t_n\| \leq \|y_n - x_n\| + \|x_n - t_n\| \to 0 \text{ as } n \to \infty. \tag{3.21}$$

For $p \in \Omega$, we define a subset D of H by $D = \left\{y \in C : \|y - p\| \leq K\right\}$, where $K = \max\left\{\|p - x\|, \tfrac{\|\gamma f(p)-\bar{\kappa}p\|+\|u\|}{(1+\mu)\bar{\gamma}-\gamma\alpha}\right\}$. Clearly, D is

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bounded, closed convex subset of H, $T(D) \subseteq D$ and $\{t_n\} \subseteq D$. By our assumption, where $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in D\}$ and Lemma 2.2, we have $\lim_{n\to\infty} \sup\{\|Tz - T_n z\| : z \in D\} = 0$. Then we have $\lim_{n\to\infty} \sup\{\|Tz - T_n z\| : z \in \{t_n\}\} \leq \lim_{n\to\infty} \sup\{\|Tz - T_n z\| : z \in D\} = 0$. This implies that

$$\lim_{n\to\infty} \|Tz - T_n z\| = 0. \tag{3.22}$$

From (3.1), the condition i), and using (3.10), (3.19), we note that

$$\|T_n t_n - t_n\| \leq \|T_n t_n - y_n\| + \|y_n - t_n\| \to 0 \tag{3.23}$$

and

$$\|T_n t_n - x_{n+1}\| \leq \alpha_n \|(u + \gamma f(x_n) - \bar{\kappa} T_n t_n\| + \beta_n \|y_n - T_n t_n\| \to 0 \text{ as } n \to \infty. \tag{3.24}$$

Using (3.13), and (3.19), we have $\|x_{n+1} - t_n\| \to 0$, as $n \to \infty$. Then we have

$$\|Tt_n - t_n\| \leq \|Tt_n - T_n t_n\| + \|T_n t_n - x_{n+1}\| + \|x_{n+1} - t_n\| \to 0. \tag{3.25}$$

Then, from (3.21) and (3.25), we obtain

$$\|y_n - Tt_n\| \leq \|y_n - t_n\| + \|t_n - T_n t_n\| \to 0 \text{ as } n \to \infty. \tag{3.26}$$

Next we show that $\limsup_{n\to\infty} \langle u + (\gamma f - \bar{\kappa})\omega, y_n - \omega \rangle \leq 0$, where ω is a solution of the optimization (3.2). First we prove that $\limsup_{n\to\infty} \langle u + (\gamma f - \bar{\kappa})\omega, Tt_n - \omega \rangle \leq 0$. Since $\{t_n\}$ is bounded, we can choose a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that

$$\begin{aligned} &\limsup_{n\to\infty} \langle u + (\gamma f - \bar{\kappa})\omega, T_n t_n - \omega \rangle \\ &= \lim_{i\to\infty} \left\langle u + (\gamma f - \bar{\kappa})\omega, T_n t_{n_i} - \omega \right\rangle. \end{aligned} \tag{3.27}$$

Without loss of generality, we may assume that $t_{n_i} \xrightarrow{w} z$, where $z \in C$. We will show that $z \in \Omega$. First, let us show $z \in F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Assume that $z \notin F(T)$. Since $t_{n_i} \xrightarrow{w} z$, $z \neq T_n z$, and (2.23), it follows by the Opial's condition that

$$\begin{aligned} \liminf_{i\to\infty} \|t_{n_i} - z\| &< \liminf_{i\to\infty} \|t_{n_i} - Tz\| \\ &\leq \liminf_{i\to\infty} \left(\|t_{n_i} - Tt_{n_i}\| + \|Tt_{n_i} - Tz\| \right) \\ &= \liminf_{i\to\infty} \|Tt_{n_i} - Tz\| \\ &\leq \liminf_{i\to\infty} \|t_{n_i} - z\|. \end{aligned} \tag{3.28}$$

This is a contradiction. Hence $z \in F(T)$. From the property of the maximal monotone, B is an $\alpha-$ inverse-strongly monotone, and (3.20), we obtain $z \in VI(C, B)$. Therefore, $z \in \Omega$. By Lemma 2.4 and (3.25), we have

$$\limsup_{n\to\infty} \langle u + (\gamma f - \bar{\kappa})\omega, Tt_n - \omega \rangle \leq 0. \tag{3.29}$$

Hence, by (3.26) and (3.29), we obtain

$$\begin{aligned} &\limsup_{n\to\infty} \langle u + (\gamma f - \bar{\kappa})\omega, y_n - \omega \rangle \\ &\leq \limsup_{n\to\infty} \langle u + (\gamma f - \bar{\kappa})\omega, y_n - Tt_n \rangle \\ &\quad + \limsup_{n\to\infty} \langle u + (\gamma f - \bar{\kappa})\omega, Tt_n - \omega \rangle \\ &\leq \limsup_{n\to\infty} \|u + (\gamma f - \bar{\kappa})\omega\| \|y_n - Tt_n\| \\ &\quad + \limsup_{n\to\infty} \langle u + (\gamma f - \bar{\kappa})\omega, Tt_n - \omega \rangle \\ &\leq 0. \end{aligned}$$

Finally, we prove that $\lim_{n\to\infty} \|x_n - \omega\| = 0$, where ω is a solution of (3.2). We observe that

$$\begin{aligned} \|x_{n+1} - \omega\|^2 \leq\ & (1 - 2((1+\mu)\bar{\gamma} - \gamma\alpha)\alpha_n) \|x_n - \omega\|^2 \\ & + \alpha_n^2 ((1+\mu)\bar{\gamma})^2 \|x_n - \omega\|^2 \\ & + 2\alpha_n \gamma\alpha \|x_n - \omega\| \|y_n - x_n\| \\ & + 2\alpha_n \langle u + (\gamma f - \bar{\kappa})\omega, y_n - \omega \rangle \end{aligned} \tag{3.30}$$

and applying Lemma 2.1, 2.3, and 2.4 to (3.30), we have $\lim_{n\to\infty} \|x_n - \omega\| = 0$. This completes the proof.

IV. Conclusion

We introduced an iterative scheme for finding a common element of the set solutions of variational inequality problems and the set of common fixed point of a countable family of nonexpansive. Then, we proved that the sequence of the proposed iterative scheme converges strongly to a common element of the above two sets, which is a solution of a certain optimization problems. Theorem 3.1 improve and extends Theorem 3.1 of Jung [4] and reference therein in the sense that our iterative scheme and convergence theorem are for the more general class of nonexpansive mappings.

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