A General Iterative Scheme for Variational Inequality Problems and Fixed Point Problems

Wichan Khongtham

Abstract—We introduce a general iterative scheme for finding a common of the set solutions of variational inequality problems for an inverse-strongly monotone mapping and the set of common fixed points of a countable family of nonexpansive mappings in a real Hilbert space. We show that the sequence converges strongly to a common element of the above two sets under some parameters controlling conditions. The results presented in this paper improve and extend the corresponding results announced by many others.

Index Terms—Fixed point, variational inequality, optimization problem, nonexpansive mapping

I. INTRODUCTION

Let $H$ be a real Hilbert space with inner product and norm, are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let $C$ be a nonempty closed convex subset of $H$, and let $B: C \to H$ be a nonlinear map. The classical variational inequality which is denoted by VI$(C,B)$ is to find $v \in C$ such that $\langle Bv, u - v \rangle \geq 0, \forall u \in C$. The variational inequality has been extensively studied in literature. See, for example, [6], [7], [9], and the references therein. A mapping $A$ of $C$ into $H$ is called $\alpha$–inverse-strongly monotone, see [12]-[13], if there exists a positive real number $\alpha$ such that $\langle Av - Av', v - v' \rangle \geq \alpha \| Av - Av' \|^2, \forall v, v' \in C$. A mapping $T$ of $C$ into itself is called nonexpansive if $\| Tx - Ty \| \leq \| x - y \|, \forall x, y \in C$. We denoted by $F(T)$ the set of fixed points of $T$. A mapping $f: C \to C$ is said to be contractive with coefficient $\alpha \in (0,1)$, if $\| f(x) - f(y) \| \leq \alpha \| x - y \|, \forall x, y \in C$. Let $G$ be a strongly positive bounded linear operator on $H$: that is, there is a constant $\gamma > 0$ with property $\langle Gx, x \rangle \geq \gamma \| x \|^2, \forall x \in H$. Recently, many authors proposed some new iterative schemes for finding element in $F(S) \cap VI(C, B)$, see [1]-[3], [5], [8], [13], and reference therein. Moreover, Jung [4] introduced the following iterative scheme as the following. Let $C$ a nonempty closed convex subset of a real Hilbert space $H$ such that $C \subset C$. Let $A$ be an $\alpha$–inverse-strongly monotone mapping of $C$ into $H$ and $S$ be a nonexpansive mapping of $C$ into itself such that $F(S) \cap VI(C, A) \neq \emptyset$. Let $u \in C$ and let $B$ be a strong positive bounded linear operator on $C$ with constant $\gamma \in (0, 1)$ and $f$ be a contractive of $C$ into itself with constant $\alpha \in (0,1)$. Assume that $\mu > 0$ and $0 < \gamma < (1 + \mu) \gamma / k$. Let $\{ x_n \}$ be a sequence generated by $x_n = u + \gamma f(x_n) + (I - \alpha_n (1 + \mu B))SP_{C_n}(x_n - \lambda_n Ax_n)$, $x_{n+1} = (1 - \beta_n) y_n + \beta_n SP_{C_n}(y_n - \lambda_n A y_n), n \geq 1$. They proved that under certain appropriate conditions imposed on $\{ \alpha_n \}, \{ \beta_n \}$, and $\{ \gamma \}$ of parameters, then the sequence $\{ x_n \}$ converges strongly to $q \in F(S) \cap VI(C, A)$, which is a solution of the optimization problem:

$$
\min_{x \in F(S) \cap VI(C, A)} \frac{\gamma}{2} \langle Bx, x \rangle + \frac{1}{2} \| x - u \|^2 - h(x),
$$

where $h$ is a potential function for $\gamma f$. In this paper motivated by the iterative scheme proposed by Jung [4], we will introduce a general iterative for a common element of the set solution of variational inequality problem for an inverse-strongly monotone mapping and the set of common fixed points of a countable family of nonexpansive mappings which will present in the main result.

II. PRELIMINARIES

Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. It well known that $H$ satisfies the Opial’s condition, that is, for any sequence $\{ x_n \}$ with $\{ x_n \}$ converging weakly to $x$ (denote by $x_n \rightharpoonup x$), the inequality: $\liminf_{n \to \infty} \| x_n - x \|^2 \leq \liminf_{n \to \infty} \| x_n - y \|^2$ holds for every $y \in H$ with $y \neq x$. For every point $x \in H$, there exist a unique nearest point in $C$, denoted by $P_C x$, such that $\| x - P_C x \|^2 \leq \| x - y \|^2$ for all $y \in C$. $P_C$ is called the metric projection of $H$ onto $C$. It well known that $P_C$ is a nonexpansive mapping of $H$ onto $C$ and satisfies $\langle x - P_C x, P_C y - P_C x \rangle \geq \| P_C x - P_C y \|^2, \forall x, y \in H$. Moreover, $P_C$ is characterized by the following properties: $P_C x \in C$ and $\| x - P_C x \|^2 \geq \| x - P_C y \|^2 + \| P_C y - y \|^2, \forall x \in C$. It is easy to see that $u \in VI(C, A) \iff u = P_C (u - \lambda A u), \lambda > 0$.

Proposition 2.1 (See [4].) Let $C$ be a bounded nonempty closed convex subset of a real Hilbert space $H$ and let $B$ be an $\alpha$–inverse-strongly monotone mapping of $C$ into $H$. Then, $VI(C, B)$ is nonempty.

A set-valued mapping $M: H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Mx$ and $g \in My$ imply $\langle x - y, f - g \rangle \geq 0$.  

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W. Khongtham is with Faculty of Science, Maejo University, Chiang Mai, Thailand, 5029 (PHONE: 66-5387-3519, Fax: 66-5387-8225; e-mail: wichan_k@mju.ac.th).
A monotone mapping \( M : H \to 2^H \) is maximal if the graph \( G(T) \) of \( T \) is not properly contained in the graph of any other monotone mapping. It is well known that a monotone mapping \( T \) is maximal if and only if for \( (x, f) \in H \times H, (x - y, f - g) \geq 0 \) for every \((y, g) \in G(T)\) implies \( f \in T x \). Let \( B \) be an inverse-strongly monotone mapping \( C \) into \( H \) and let \( N_C v \) be the normal cone to \( C \) at \( v \), that is, \( N_C v = \{ w \in H : \langle v - u, w \rangle \geq 0, \text{ for all } u \in C \} \), and define

\[
T v = \begin{cases} 
B v + N_C v, & v \in C, \\
\phi, & v \notin C.
\end{cases}
\]

Then \( T \) is maximal monotone and \( 0 \in T v \) if and only if \( v \in V I(C, B) \) (see [9], [10], [12]).

The following Lemmas will be useful for proving our theorem in the next section.

**Lemma 2.1** (See [9].) Assume \( \{a_n\} \) is a sequence of nonnegative real numbers such that \( a_{n+1} \leq (1 - \alpha_n) a_n + \delta_n, \ n \geq 0 \)
where \( \{\alpha_n\} \) is a sequence in \( (0,1) \) and \( \{\delta_n\} \) is a sequence in \( R \) such that

1. \( \sum_{n=1}^{\infty} \alpha_n = \infty \);
2. \( \limsup_{n \to \infty} \delta_n \leq 0 \) or \( \sum_{n=1}^{\infty} \delta_n < \infty \).

Then \( \lim_{n \to \infty} a_n = 0 \).

**Lemma 2.2** (See [11].) Let \( K \) be a nonempty closed convex subset of a Banach space \( X \) and let \( \{T_n\} \) be a sequence of mappings of \( K \) into itself. Suppose that \( \sum_{n=1}^{\infty} \sup\|T_{n+1} z - T_n z : z \in K\| < \infty \). Then, for each \( y \in K \), \( \{T_n y\} \) converges strongly to some point of \( K \). Moreover, let \( T \) be a mapping of \( K \) into itself defined by \( Ty = \lim_{n \to \infty} T_n y \) for all \( y \in K \). Then \( \lim_{n \to \infty} \sup \|T z - T_n z : z \in K\| = 0 \).

**Lemma 2.3** (See [4].) In a real Hilbert space \( H \), there holds the inequality

\[
\|z + y\|^2 \leq \|z\|^2 + 2 \langle y, z + y \rangle.
\]

**Lemma 2.4** (See [4].) Let \( C \) be a bounded nonempty closed convex subset of a real Hilbert space \( H \), and let \( g : C \to R \cup \{\infty\} \) be a proper lower semicontinuous differentiable convex function. If \( x^* \) is a solution to the minimization problem \( g(x^*) = \inf_{x \in C} g(x) \), then \( g'(x^*), x - x^* \geq 0, x \in C \). In particular, if \( x^* \) solves the optimization problem

\[
\min_{x \in C} \frac{1}{2} \langle B x, x \rangle + \frac{1}{2} \|x - u\|^2 - h(x),
\]

then \( (u + (1/\mu)B)x^*, x - x^* \leq 0, x \in C \), where \( h \) is a potential function for \( g' \).

**Lemma 2.5** (See [9].) Assume \( A \) is a strongly positive linear bounded operator on a Hilbert space \( H \) with coefficient \( \gamma > 0 \) and \( 0 < \rho \leq \|A\|^{-1} \). Then \( \|I - \rho A\| \leq 1 - \rho \gamma \).

### III. MAIN RESULT

In this section, we prove a strong convergence theorem.

**Theorem 3.1.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \) such that \( C \subset C \subset C \). Let \( B \) be an \( \alpha - \) inverse-strongly monotone mapping of \( C \) into \( H \) and \( \{T_n\} \) be a sequence of nonexpansive mappings of \( C \) into itself such that 
\[
\Omega := \cap_{n=1}^{\infty} F(T_n) \cap \text{VI}(C, B) \neq \emptyset.
\]
Let \( u \in C \) and let \( A \) be a strongly positive bounded linear operator on \( C \) with constant \( \gamma \in (0,1) \) and \( f \) be a contractive of \( C \) into itself with constant \( \alpha \in (0,1) \). Assume that \( \mu > 0 \) and \( 0 < \gamma < (1 + \mu)^{-1} \), where \( \{\alpha_n\} \subset [0,1) \), \( \{\lambda_n\} \subset [0,2\alpha) \), and \( \{\beta_n\} \subset [0,1] \) satisfy the following conditions:

i) \( \lim \alpha_n = 0 \);
ii) \( \sum \alpha_n = \infty \);
iii) \( \lambda_n \in [r,s] \) for all \( n \geq 0 \) and some \( r < 0 < s < 2\alpha \);
iv) \( \sum \alpha_n - \alpha_n < \infty \), \( \sum \beta_n - \beta_n < \infty \), and \( \sum \beta_n - \lambda_n < \infty \).

Suppose that \( \sum \sup \|T_{n+1} z - T_n z : z \in D\| < \infty \) for any bounded subset \( D \) of \( C \). Let \( T \) be mapping of \( H \) into itself defined by \( Ty = \lim_{n \to \infty} T_n y \), all \( x \in C \) and suppose that \( F(T) = \cap_{n=1}^{\infty} F(T_n) \). Then, \( \{x_n\} \) converges strongly to \( \omega \in F(T) \cap \text{VI}(C, B) \), which is a solution of the optimization problem

\[
\min_{x \in F(T) \cap \text{VI}(C, B)} \frac{1}{2} \langle A x, x \rangle + \frac{1}{2} \|x - u\|^2 - h(x),
\]

where \( h \) is a potential function for \( g' \).

**Proof.** From the condition i), we may assume that \( \alpha_n \leq (1 + \mu)\|A\|^{-1} \). Applying Lemma 2.5 and by the same argument as that in the proof of Jung ([4], Theorem 3.1, pp. 6-7), we have that \( \langle (1 - \alpha_n (1 + \mu A) u, u) \rangle \geq 1 - \alpha_n - \alpha_n \mu \langle A, u \rangle \geq 0 \), \( \langle I - \alpha_n (1 + \mu A) \rangle \langle I - \alpha_n (1 + \mu A) \rangle \geq 1 - \alpha_n - \alpha_n \mu \langle A, u \rangle \), \( \langle I - \alpha_n (1 + \mu A) \rangle \langle I - \alpha_n (1 + \mu A) \rangle \geq 1 - \alpha_n - \alpha_n \mu \langle A, u \rangle \), and \( \langle I - \alpha_n (1 + \mu A) \rangle \langle I - \alpha_n (1 + \mu A) \rangle \geq 1 - \alpha_n - \alpha_n \mu \langle A, u \rangle \). It follows that

\[
\|y_n - v\| \leq \alpha_n \|A u + \alpha_n \gamma \langle y_n - \bar{v}, v \rangle + (1 - \alpha_n \gamma \langle T_n v - v \rangle) \|
\]

and

\[
\|x_{n+1} - v\| \leq \|1 - \beta_n (y_n - v) + \beta_n (T_n v - T_n u)\|. 
\]
we define a subset $D$ of $H$ by

By the same argument as in [4] (Theorem 3.1, (3.5)),

and

It follows from the assumption and using (3.1), (3.3), and (3.4), we have

Then, we obtain

where $G_1 = \sup \{ ||u|| + \gamma f(x_n) + \|t_n\| : n \in N \}$,

and

where $G_3 = \sup \{ ||T v_n|| : n \in N \}$, and $G_4 = \sup \{ ||T v_n|| : n \in N \} - \sup \{ ||T v_n|| : z \in \{ v_n \} \}$.

Applying lemma 2.1 to (3.6), we have

By using (3.3) and (3.5), we have

and

From (3.1), we note that

and

Moreover, by (3.1), (3.11), and the condition ii), we have

From (3.12) and using (3.7) and (3.10), we obtain

We apply that

Let $p \in \Omega$. By the same argument as in [4] (Theorem 3.1, pp. 11-12), we can show that

Then, we obtain

It follows from the condition i), we have

Similarly, we can show that

Then, we obtain

It follows from (3.14), (3.17), and the condition i), we obtain

and so

For $p \in \Omega$, we define a subset $D$ of $H$ by $D = \{ y \in C : \| y - p \| \leq K \}$, where $K = \max \{ \| p - x \| : \| x \| \leq \lambda \}$. Clearly, $D$ is
bounded, closed convex subset of $H$, $T(D) \subseteq D$ and $\{t_n\} 
\subseteq D$. By our assumption, where \( \sum_{n=1}^{\infty} \sup \{\|T_n z - T z\|: z \in D\} \) and Lemma 2.2, we have \( \lim_{n \to \infty} \sup \{\|T z - T_n z\|: z \in D\} = 0 \). Then we have \( \lim_{n \to \infty} \sup \{\|T z - T_n z\|: z \in \{t_n\}\} \leq \lim_{n \to \infty} \sup \{\|T z - T_n z\|: z \in D\} = 0 \). This implies that

\[
\lim_{n \to \infty} \|T z - T_n z\| = 0. \tag{3.22}
\]

From (3.1), the condition i), and using (3.10), (3.19), we note that

\[
\|T_n z - t_n\| \leq \|T_n z - T z\| + \|T z - T_n z\| \to 0 \tag{3.23}
\]

and

\[
\|T_n z - x_{n+1}\| \leq \alpha_n \|u + \gamma f(x_n) - \bar{K} T_n z\| + \beta_n \|T z - T_n z\| \to 0 \text{ as } n \to \infty. \tag{3.24}
\]

Using (3.13), and (3.19), we have \( \|x_{n+1} - t_n\| \to 0 \), as \( n \to \infty \). Then we have

\[
\|T z - t_n\| \leq \|T z - T z\| + \|T z - T_n z\| + \|T_n z - x_{n+1}\| + \|x_{n+1} - t_n\| \to 0. \tag{3.25}
\]

Then, from (3.21) and (3.25), we obtain

\[
\|y_n - T z\| \leq \|y_n - t_n\| + \|T z - T_n z\| \to 0 \text{ as } n \to \infty. \tag{3.26}
\]

Next we show that \( \limsup_{n \to \infty} \{u + (\gamma f - \bar{K}) \alpha, y_n - \omega\} \leq 0 \), where \( \omega \) is a solution of the optimization (3.2). First we prove that \( \limsup_{n \to \infty} \{u + (\gamma f - \bar{K}) \alpha, T z - \omega\} \leq 0 \). Since \( \{t_n\} \) is bounded, we can choose a subsequence \( \{t_{n_i}\} \) of \( \{t_n\} \) such that

\[
\limsup_{n \to \infty} \{u + (\gamma f - \bar{K}) \alpha, T z - \omega\} = \lim_{i \to \infty} \{u + (\gamma f - \bar{K}) \alpha, T t_{n_i} - \omega\}. \tag{3.27}
\]

Without loss of generality, we may assume that \( t_{n_i} \xrightarrow{i \to \infty} z \), where \( z \in C \). We will show that \( z \in \Omega \). First, let us show \( z \notin F(T) = \bigcap_{i=1}^{\infty} F(T_i) \). Assume that \( z \notin F(T) \). Since \( t_{n_i} \xrightarrow{i \to \infty} z \), \( z \notin T z \), and (2.23), it follows by the Opial’s condition that

\[
\liminf_{i \to \infty} \|t_{n_i} - z\| < \liminf_{i \to \infty} \|t_{n_i} - T z\| \leq \liminf_{i \to \infty} \left( \|t_{n_i} - T z\| + \|T t_{n_i} - T z\| \right) \tag{3.28}
\]

\[
= \liminf_{i \to \infty} \|T t_{n_i} - T z\|.
\]

This is a contradiction. Hence \( z \in F(T) \). From the property of the maximal monotone, \( B \) is an \( \alpha \)- inverse-strongly monotone, and (3.20), we obtain \( z \in VI(C, B) \). Therefore, \( z \in \Omega \) by Lemma 2.4 and (3.25), we have

\[
\limsup_{n \to \infty} \{u + (\gamma f - \bar{K}) \alpha, T z - \omega\} \leq 0. \tag{3.29}
\]

Hence, by (3.26) and (3.29), we obtain

\[
\limsup_{n \to \infty} \{u + (\gamma f - \bar{K}) \alpha, y_n - \omega\} \leq \limsup_{n \to \infty} \{u + (\gamma f - \bar{K}) \alpha, y_n - T z\} + \limsup_{n \to \infty} \{u + (\gamma f - \bar{K}) \alpha, y_n - T z\} \leq 0.
\]

Finally, we prove that \( \lim_{n \to \infty} \|x_n - \omega\| = 0 \), where \( \omega \) is a solution of (3.2). We observe that

\[
\|x_{n+1} - \omega\|^2 \leq (1 - 2((1 + \mu)\gamma - \gamma \alpha)\alpha_n) \|x_n - \omega\|^2 + \alpha_n^2 (1 + \mu)\gamma \alpha_n^2 \|x_n - \omega\|^2 + 2\alpha_n \gamma \alpha_n \|x_n - \omega\|^2 + 2\alpha_n (u + (\gamma f - \bar{K}) \alpha, y_n - \omega) \tag{3.30}
\]

and applying Lemma 2.1, 2.3, and 2.4 to (3.30), we have \( \lim_{n \to \infty} \|x_n - \omega\| = 0 \). This completes the proof.

IV. CONCLUSION

We introduced an iterative scheme for finding a common element of the set solutions of variational inequality problems and the set of common fixed point of a countable family of nonexpansive. Then, we proved that the sequence of the proposed iterative scheme converges strongly to a common element of the above two sets, which is a solution of a certain optimization problems. Theorem 3.1 improve and extends Theorem 3.1 of Jung [4] and reference therein in the sense that our iterative scheme and convergence theorem are for the more general class of nonexpansive mappings.

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