Euler's Constant: New Insights by Means of Minus One Factorial

Okoh Ufuoma

***Abstract*—The great object of this paper is to furnish, in a concise and plain manner, new insights into that mysterious constant whose arithmetic nature was shrouded in obscurity for over 250 years, the famous Euler's constant. The cardinal instrument which generates these new insights is the unwonted infinite number, the minus one factorial, which in this paper is discussed in detail.**

***Index Terms*—Euler's constant, infinity, harmonic number, harmonic series, natural logarithm, minus one factorial**

I. INTRODUCTION

ONE of the most enchanting mathematical discoveries of the 18th century was the mysterious and celebrated constant, the Euler's constant γ, discovered by the immortal Swiss professor in mathematics in Berlin and St Petersburg, Leonhard Euler (1707-1783) [23], who, in his admirable paper titled *De progressionibus harmonics observationes* (1734/5) [10], defined the constant in a fascinating manner as

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}-\ln(n+1)\right)$$

and computed its numerical value to 6 decimal places as

$$\gamma = 0.577218.$$

In the 1st section of another paper entirely devoted to the investigation of γ and intriguingly entitled *De numero memorabili in summatione progressionis harmonicae naturalis occurrente* [11] [20], Euler painstakingly computed the value of the constant to 16 decimal places as

$$\gamma = 0.5772156649015325.$$

The constant is sometimes called Euler–Mascheroni constant [22] to honour together with Euler the Italian mathematician Lorenzo Mascheroni (1750–1800) [6], probably the second to intensively investigate the constant. In 1790 Mascheroni published his *Adnotationes ad calculum integralem Euleri* [6], [14] and there computed γ up to 32 decimal places, denoting it with the letter A. A few years later, in 1809, a German mathematician, Johann Georg von Soldner (1766–1833) found a value of the constant which was in harmony with only the first 19 decimal places of Mascheroni's computed value. It was in 1812 that a young prodigious mathematician Friedrich Bernhard Gottfried Nicolai (1793–1846), under the supervision of the famous German mathematician Johann Carl Friedrich Gauss (1777–1855), evaluated γ up to 40 correct decimal places in accordance with Soldner's value [6], [15], [32].

The constant whose numerical value is $\gamma = 0.577215664\ldots$ [14], [23], a real number, forms a triple constant with two famous and important real constants, $\pi = 3.141592653\ldots$ and $e = 2.718281828\ldots$ as it is often ranked the third most important real constant after π and e [6], [14], not based on its position in respect of magnitude, for example, $\pi > e > \gamma$, but depending on its position with regard to its ubiquity in mathematics, revealed by its frequent appearance in equations. Of the triple constant it is only γ whose arithmetic nature was an enigma to mathematicians as its irrationality and transcendence were unfathomable for many years [14] . The irrationality of π and e was proved by a Swiss mathematician Johann Heinrich Lambert (1728–1777) in 1761 and that of e^{π} by a Soviet mathematician Alexander Osipovich Gelfond (1906–1968) in 1929 [16].

The main aim of this paper is to provide new insights into the character of γ by means of the minus one factorial $(-1)!$ together with other relevant results which we shall also consider in this paper. We shall here employ $(-1)!$ to derive, in a systematic manner, a new and beautiful expression for γ.

The rest of the paper is structured into seven sections. Section II provides, for the reader to evince enthusiastic interest in the paper, a brief review of Euler's quest for the arithmetic nature of γ. Section III introduces the new and unfamiliar constant $(-1)!$ while Section IV elegantly treats its arithmetic nature. Section V discusses precise details of the harmonic number and the natural logarithm, the two concomitants responsible for the creation of the Euler's constant. Section VI introduces for the first time the natural logarithm of $(-1)!$ as the sum of the divergent harmonic series. That the reader may see the great beauty and force of the new constant $(-1)!$, Section VII treats the use of the natural logarithm of $(-1)!$ for deriving a familiar formula that links the Euler's constant with the famous Riemann zeta constants. Finally, Section VIII provides for the first time a new and interesting identity that forever links γ with $(-1)!$.

II. EULER'S QUEST FOR THE ARITHMETIC NATURE OF γ

The discovery of the Euler's constant, γ, originated from a famous problem, the Basel problem, posed in the 17th century in a 1650 book *Novae Quadraturea Arithmeticae* by a professor of mechanics in Bologna, Pietro Mengoli $(1625 - 1686)$, an Italian mathematician and clergyman [1]. The Basel problem asked for the exact sum of the series

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \lim_{n\to\infty}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}\right)$$

together with a proof that the sum was precise [1], [14].

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The eminent English mathematician and professor at Oxford John Wallis (1616–1703) had communicated in his 1655 book *Arithmetica Infinitorum* that he had got the sum to three decimal places as 1.645, but he was unable to say anything more concrete [1]. The problem became notable because it was also raised by one of the eight prominent mathematicians produced by a remarkable Swiss family in three generations, Jakob Bernoulli $(1654-1705)$, a professor of mathematics at the University in Basel. He attempted the problem without success, and in his posthumous book, the illustrious *Ars Conjectandi*, published in 1713, he appealed [4], [20]:

> ... it is more difficult than one would have expected, which is noteworthy. If someone should succeed in finding what till now withstood our efforts and communicate it to us, we would be much obliged to them.

This problem prompted Euler's research and he spent much effort evaluating the sum of reciprocals of powers which in modern notation is written as

$$\zeta(k)=\sum_{n=1}^{\infty}\frac{1}{n^k}, \qquad k\geq 1$$

where $\zeta(k)$ is a zeta constant. In 1735 he obtained the exact sum

$$\zeta(2)=\frac{\pi^2}{6}$$

and announced his noteworthy solution to the Basel problem [1], [14]. He also procured the exact values for higher even zeta constants $\zeta(2m)$, but did not succeed in finding the exact value for any odd zeta constant $\zeta(2m-1)$. In a letter to his tutor, the notable Johann Bernoulli (1667– 1748), the younger brother of Jakob, he wrote, "... the odd powers I cannot sum, and I don't believe that their sums depend on the quadrature of the circle $[\pi]$ "[14].

In an attempt to assign a value to the odd zeta constant $\zeta(1)$, also called the harmonic series, he discovered his favorite constant which we now call the Euler's constant γ, computed it to six decimal places and published his results in the paper [10], and there denoted his constant with the letter C, stating that it was "worthy of serious consideration"[17].

Now the birth of Euler's constant as Euler presented it in his paper [10] is as follows: We start with the Taylor series expansion of $\ln(1+x)$,

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\cdots,$$

which was discovered by a 17th-century German mathematician Nicholas Mercator (c.1620–1687). We rearrange this series as

$$x=\ln(1+x)+\frac{x^2}{2}-\frac{x^3}{3}+\frac{x^4}{4}-\frac{x^5}{5}+\cdots,$$

substitute consecutively $1,\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{n}$ in place of x, and obtain the following tabular form:

$$\begin{aligned}
1&=\ln(2)+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\cdots\\
\frac{1}{2}&=\ln\left(\frac{3}{2}\right)+\frac{1}{2\cdot 4}-\frac{1}{3\cdot 8}+\frac{1}{4\cdot 16}-\frac{1}{5\cdot 32}+\cdots\\
\frac{1}{3}&=\ln\left(\frac{4}{3}\right)+\frac{1}{2\cdot 9}-\frac{1}{3\cdot 27}+\frac{1}{4\cdot 81}-\frac{1}{5\cdot 243}+\cdots\\
&\vdots\\
\frac{1}{n}&=\ln\left(\frac{n+1}{n}\right)+\frac{1}{2\cdot n^2}-\frac{1}{3\cdot n^3}+\frac{1}{4\cdot n^4}-\frac{1}{5\cdot n^5}+\cdots.
\end{aligned}$$

Adding by columns the first n terms, we get

$$\begin{aligned}
1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}=\ln(n+1)&\\
+\frac{1}{2}\left(1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2}\right)&\\
-\frac{1}{3}\left(1+\frac{1}{8}+\frac{1}{27}+\cdots+\frac{1}{n^3}\right)&\\
+\frac{1}{4}\left(1+\frac{1}{16}+\frac{1}{81}+\cdots+\frac{1}{n^4}\right)&\\
-\frac{1}{5}\left(1+\frac{1}{32}+\frac{1}{243}+\cdots+\frac{1}{n^5}\right)&\\
+\cdots.&
\end{aligned}$$

If n tends to the infinitely large integer Ω so that $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$ becomes the harmonic series, we have

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{\Omega}=\ln(\Omega+1)+\gamma$$

where

$$\begin{aligned}
\gamma&=\frac{1}{2}\left(1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{\Omega^2}\right)\\
&-\frac{1}{3}\left(1+\frac{1}{8}+\frac{1}{27}+\cdots+\frac{1}{\Omega^3}\right)\\
&+\frac{1}{4}\left(1+\frac{1}{16}+\frac{1}{81}+\cdots+\frac{1}{\Omega^4}\right)\\
&-\frac{1}{5}\left(1+\frac{1}{32}+\frac{1}{243}+\cdots+\frac{1}{\Omega^5}\right)\\
&+\cdots\\
&=\frac{1}{2}\zeta(2)-\frac{1}{3}\zeta(3)+\frac{1}{4}\zeta(4)-\frac{1}{5}\zeta(5)+\cdots\\
&=\frac{1}{2}\left(\frac{\pi^2}{6}\right)-\frac{1}{3}\left(1.202056903\ldots\right)+\frac{1}{4}\left(\frac{\pi^4}{90}\right)-\cdots\\
&=0.577215664\ldots.
\end{aligned}$$

Euler, as was his custom, would never give up the search for anything he thought deserves serious consideration. He engaged himself with great ardor in a long search for the arithmetic nature of γ, desiring to know the sort of number it is, whether or not it is irrational, incapable of being written as a ratio or quotient of two integers, and transcendental, incapable of being expressed as a polynomial.

In the 1768 paper [8], mainly devoted to zeta constants and Bernoulli numbers, Euler obtained formulas for γ and used the letter O to denote the constant. In the 24th section he gave the following remark, touching γ [20]:

> This number seems also the more noteworthy because even though I have spent much effort in investigating it, I have not been able to reduce it to a known kind of quantity.

and in the 29th section he concluded:

> Therefore the question remains of great moment, of what character the number O is and among what species of quantities it can be classified.

In the 1776 paper [11] presented in 1781 and posthumously published in 1785, Euler intensively investigated γ and conjectured that γ was the logarithm of some other number $N = e^{\gamma}$ of significance but was unable to identify the character of any such number. In the 2nd section we read Euler, the master of us all [20]:

> And therefore, if x is taken to be an infinitely large number it will then be
>
> $$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{x} = C + \ln x.$$
>
> One may suspect from this that the number C is the hyperbolic logarithm of some notable number, which we put $= N$, so that $C = \ln N$ and the sum of the infinite series is equal to the logarithm of the number $N \cdot x$. Thus it will be worthwhile to inquire into the value of this number N, which indeed it suffices to have defined to five or six decimal figures, since then one will be able to judge without difficulty whether this agrees with any known number or not.

Here the hyperbolic logarithm is the natural logarithm or the Napierian logarithm named after the inventor of logarithm John Napier (1550—1617), the Baron of Merchiston in Scotland, who gave it to the world.

In another 1776 paper [12], not published until 1789, Euler employed the letter n to denote his constant and gave the following comment [20]:

> ... whose value I have been able in no way to reduce to already known transcendental measures; therefore, it will hardly be useless to try to resolve the formula in many different ways.

One remark on Euler's work is appropriate before we wind up this brief discussion of Euler's quest for the nature of γ. Euler, craving for the sort of number γ is, derived many formulas involving γ, among which are the familiar identities [10], [15], [29]:

$$\gamma = \sum_{n=2}^{\infty} \frac{(n-1)(\zeta(n)-1)}{n} \quad \text{and} \quad \gamma = 1 - \sum_{n=2}^{\infty} \frac{\zeta(n)-1}{n}.$$

We conclude this section by saying that Euler did not give up his quest for the nature of his beloved, γ. He investigated it over and over again until his demise in 1783 and abandoned the search for the solution to the later renowned problem of finding the nature of his darling constant, which had drilled his mind.

III. THE FACTORIAL OF MINUS ONE

The idea we will deal with in this section is that which has to do with the product of decreasing counting numbers, the factorial, a concept which occurs quite often in myriad branches of mathematics, especially in combinatorics and algebra. All such expressions as

$$\begin{aligned} &1 \\ &2 \cdot 1 \\ &3 \cdot 2 \cdot 1 \\ &4 \cdot 3 \cdot 2 \cdot 1 \end{aligned}$$

and so on, are factorials since they are the product of decreasing consecutive natural or counting numbers. We find that some useful space in paper can be wasted by continuing this multiplication and we overcome this by adopting a shorthand [7], the symbol $n!$, introduced in 1808 by the French mathematician Christian Kramp (1760 - 1826) who investigated mainly factorial; the letter n standing for any given natural number. Employing Kramp's symbol for the factorial of n, we write the first few factorial values:

$$\begin{aligned} 1! &= 1 \\ 2! &= 2 \cdot 1 \\ 3! &= 3 \cdot 2 \cdot 1 \\ 4! &= 4 \cdot 3 \cdot 2 \cdot 1 \end{aligned}$$

and in general

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1.$$

We thus give the definition of the factorial $n!$ as the product of all counting numbers from n down through 1.

It is wise to extend the factorial to zero, since, in many applications of the factorial, the zero factorial, with symbol 0!, occurs frequently in calculations. Zero factorial cannot be defined by the rule used in the preceding examples because zero is not a counting number. To investigate zero factorial, we introduce the familiar recurrence relation,

$$n! = n \cdot (n-1)! \tag{1}$$

which, upon putting 1, a counting number, in place of n, gives

$$\begin{aligned} 1! &= 1 \cdot 0! \\ \therefore 0! &= 1. \end{aligned}$$

So we define 0! as being equal to unity. Thus we redefine, for every nonnegative integer n, the factorial of n by means of the repeated application of the recurrence relation (1) as

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1 \cdot 0!$$

where $0! = 1$, as was remarked upon.

If we attempt to perform the operation already named–factorial–on any of the real numbers thus recognized, we find that there is one case in which the result of the operation cannot be expressed without the introduction of yet another type of numbers. The case referred to is that in which the operation of factorial, is applied to a negative integer, e.g. to find the factorial of -2, i.e. $(-2)!$. To express the results of this , we make use of a new number, the minus one factorial, $(-1)!$ and to find $(-1)!$ we need to extend the idea of the factorial one step further. This is accomplished by the recurrence relation which has just been introduced; that is $n! = n \cdot (n-1)!$. If we, as we did for 0, put 0 in place of n, we get an intriguing result

$$1 = 0 \cdot (-1)!$$

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or

$$0 \cdot (-1)! = 1. \tag{2}$$

What is $(-1)!$? It is the infinite product of all negative integers from -1 down to a minus infinity, that is

$$(-1)! = (-1) \cdot (-2) \cdot (-3) \cdots .$$

We shall call the relation (2), in which $(-1)!$ has for its multiplication with zero the positive number 1, the fundamental property of $(-1)!$ because it forms a basis for the investigations of other properties of $(-1)!$, though we will not refer to these properties in this paper.

It is often asserted that zero, the neutral integer, has no inverse, that is to say, the reciprocal of zero is undefined, but this would be true only when finite integers were considered. If, therefore, we considered all manner of integers, positive and negative, finite and infinite, then we can unquestionably say that the multiplicative inverse of zero is minus one factorial, so that we can properly write

$$\frac{1}{0} = (-1)!$$

or

$$\frac{1}{(-1)!} = 0.$$

This result will provide inspiration throughout this paper, as in calculations where it is impossible to work with zero such as in division by zero or the logarithm of zero, we will employ the property $\frac{1}{0} = (-1)!$ which links nothing with infinity.

It is needful to discuss two mathematical concepts usually embodied in the number series—zero with the symbol 0 and infinity with the symbol ∞. For these concepts unique rules of operation are required.

Now zero springs in the first place from subtracting a quantity from an equal quantity; thus, $x - x = x \cdot 0 = 0$. It implies in this sense the absence of quantity, nothing. It cannot, then, either operate upon a quantity or be operated upon ; for all operations imply the existence of the quantities concerned. Although the expressions

$$a \times 0, \qquad \frac{0}{a}, \qquad \frac{a}{0},$$

are meaningless, it is possible to give them conventional meanings, as follows : Take the three expressions

$$a \times x, \qquad \frac{x}{a}, \qquad \frac{a}{x},$$

and consider what happens when x is decreased constantly to zero. We need only elementary arithmetic to see that

$$a \times x \qquad \text{and} \qquad \frac{x}{a}$$

may each be made as small as we please by taking x infinitely small, while

$$\frac{a}{x}$$

becomes infinitely great as x decreases, and may be made greater than any quantity we may choose to name. We may express the first two results concisely by the formulas

$$a \times 0 = 0 \qquad \text{and} \qquad \frac{0}{a} = 0.$$

We can express the last result in a formula, however, only by introducing the concept of infinity denoted by the symbol ∞ and of which $(-1)!$ is the head. We may express our third result by the formula

$$\frac{a}{0} = \infty$$

which means that when the denominator of a fraction decreases constantly to zero, the value of the fraction increases and becomes greater than any quantity which can be named.

The expressions

$$a \times \infty, \qquad \frac{\infty}{a}, \qquad \frac{a}{\infty}$$

are also literally meaningless, but we can give a conventional meaning to them by writing

$$a \times x, \qquad \frac{x}{a}, \qquad \frac{a}{x},$$

and studying the effect of increasing x indefinitely. We obtain by elementary arithmetic the results expressed by the formulas

$$a \times \infty = \infty, \qquad \frac{\infty}{a} = \infty, \qquad \frac{a}{\infty} = 0.$$

To cope with certain problems in mathematics, especially those involving division by zero, we need to make use of $(-1)!$, a number at infinity. This number, though infinite and immeasurable or has no value in the ordinary sense, should be, in my opinion, admitted, like the imaginary unit $i = \sqrt{-1}$ which at first was thought to be a play of the imagination, into the great family of numbers. We should overcome any prejudice against $(-1)!$ and build up a distinct body of mathematics around it. As we will see in the rest of the paper, this number will prove to be very valuable as it will help us to derive a familiar formula that relates Euler's constant to zeta constants, and more importantly, derive a new identity for γ.

To round off this discussion, let me add something which certainly is more curious than useful. The fundamental property of $(-1)!$ asserts that

$$0 \cdot (-1)! = 1$$

and this implies that when 0 is added to itself $(-1)!$ times, the result is unity, that is,

$$\underbrace{0 + 0 + 0 + 0 + \cdots}_{(-1)!\text{times}} = 1.$$

From this, it is evident that

$$\underbrace{0 + 0 + 0 + 0 + \cdots}_{\frac{(-1)!}{a}\text{times}} = \frac{1}{a}$$

since

$$0 \cdot \frac{(-1)!}{a} = \frac{0 \cdot (-1)!}{a} = \frac{1}{a}.$$

Let us use this idea together with the assumption that there are $(-1)!$ terms in any infinite series consisting of equal terms to explain why $\frac{1}{2}$ could be the sum of the famous Grandi's series

$$\underbrace{1 - 1 + 1 - 1 + 1 - 1 + \cdots}_{(-1)!\,\text{ones}}$$

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which has caused intense and endless dispute among mathematicians and for which Euler's reputation was badly tarnished [19]. Bracketing the series as

$$(1-1)+(1-1)+\cdots$$

appears to furnish the infinite series with only zeros as its terms; the number of zeros being half the number of ones in the Grandi's series. Thus the sum of the Grandi's series is

$$\underbrace{0+0+0+0+\cdots}_{\frac{(-1)!}{2}\text{ zeros}}=0\cdot\frac{(-1)!}{2}=\frac{1}{2}.$$

Also, bracketing the series as

$$1-(1-1)-(1-1)-\cdots$$

which is

$$1-\underbrace{((1-1)+(1-1)+\cdots)}_{(-1)!-1\text{ ones}}$$

appears to furnish the infinite series

$$1-\underbrace{(0+0+0+\cdots)}_{\frac{(-1)!-1}{2}\text{ zeros}}$$

so that the sum of the Grandi's series is

$$\begin{aligned}1-\left(0\cdot\frac{(-1)!-1}{2}\right)&=1-\left(\frac{0\cdot(-1)!-0\cdot 1}{2}\right)\\&=1-\frac{1}{2}\\&=\frac{1}{2}.\end{aligned}$$

Let us also use this approach to investigate the series $1+0-1+1+0-1+\cdots$ which is often given as a counter–example as its sum is not $\frac{1}{2}$ but $\frac{2}{3}$ [19]. If we now follow the method of Gottfried Wilhelm Leibniz (1646–1716), Germany's marvelous prodigy, we see that the sequence corresponding to this series has, out of every three succeeding terms, once the value 0 and twice the value 1 [26]. If we group the series into two as

$$(1-1+1-\cdots)+(0+0+0+\cdots),$$

it will then be found that there are $\frac{2(-1)!}{3}$ ones in the first grouping $(1-1+1-\cdots)$ and $\frac{(-1)!}{3}$ zeros in the second grouping $(0+0+0+\cdots)$. Bracketing the first grouping as

$$(1-1)+(1-1)+\cdots$$

appears to furnish the series $(0+0+0+\cdots)$ with the number of zeros being half the number of ones in the first grouping, that is there are now $\frac{(-1)!}{3}$ zeros in the first grouping. Thus, there is a total of $\frac{2(-1)!}{3}$ zeros, combining the two groupings, and hence the sum of the series $1+0-1+1+0-1+\cdots$ is

$$0\cdot\frac{2(-1)!}{3}=\frac{0\cdot 2(-1)!}{3}=\frac{2}{3}.$$

We wish we could further pursue this subject that opens before us; for we want to show how this new number $(-1)!$ may be employed in settling all manner of disputes arising from the investigation of infinite series.

IV. ARITHMETIC NATURE OF $(-1)!$

It is not difficult to show that $(-1)!$ is an integral and infinite number as it is the product of all negative integers from -1 to negative infinity, that is,

$$(-1)!=(-1)\cdot(-2)\cdot(-3)\cdots.$$

If the infinite product is terminated somewhere so that the nth partial product P_n is

$$P_n=(-1)\cdot(-2)\cdot(-3)\cdots(-n),$$

we have the sequence of partial products of $(-1)!$ as follows:

$$\begin{aligned}P_1&=(-1)=-1!\\P_2&=(-1)\cdot(-2)=2!\\P_3&=(-1)\cdot(-2)\cdot(-3)=-3!\\P_4&=(-1)\cdot(-2)\cdot(-3)\cdot(-4)=4!\end{aligned}$$

and in general

$$P_n=(-1)\cdot(-2)\cdot(-3)\cdots(-n)=(-1)^n n!.$$

The limit of P_n as n tends to infinity is thus

$$\lim_{n\to\infty}(P_n)=\lim_{n\to\infty}((-1)^n n!)=(-1)!.$$

The above limit clearly shows that the number $(-1)!$ is an infinite integer.

Therefore, from our above illustrations, if n is made even, then the product is a positive number, but if, on the other hand, n is made odd, the product is a negative number. Now if, therefore, n is taken to infinity and consequently we cannot assert without proof that the infinite product $(-1)!$ is a positive or negative number. Thus, a beautiful question concerning the nature of $(-1)!$ as to whether it is positive or negative arises immediately. If $(-1)!$ is an infinite number, is it a positive or negative infinite number or a number oscillating between positive and negative as one may suppose considering the above limit? The answer to this fundamental question is found by considering the inequality

$$e^x>0$$

which is true for all x [2], whether positive or negative, small or large. If we find the natural logarithm of both sides of the inequality and evaluate both sides step by step, we get the following:

$$\begin{aligned}\ln e^x&>\ln 0\\-\ln 0&>-\ln e^x\\\ln\left(\frac{1}{0}\right)&>-x\\\ln(-1)!&>-x\\(-1)!&>e^{-x}\\\therefore(-1)!&>\frac{1}{e^x}.\end{aligned}$$

Since $\frac{1}{e^x}$ is positive for all manner of x it is evident that $(-1)!$ being greater than $\frac{1}{e^x}$ is a positive number.

Again, we demonstrate, in a simple and naive manner, that $(-1)!$ is actually positive and never alternating positive and negative. We start with the inequality with which we all are familiar,

$$1>0$$

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which, after employing the property $\frac{1}{(-1)!} = 0$, becomes the inequality

$$1 > \frac{1}{(-1)!}.$$

Therefore,

$$(-1)! > 1$$

which indisputably shows that $(-1)!$ being greater than +1 is a positive number.

Now that we have shown for the second time that $(-1)!$ is a positive number, no further investigation is needed concerning this matter and all we need do is to inquire further whether $(-1)!$ is actually an integer or not, a result necessary for describing the character of γ.

It is apparent that the algebraic fraction

$$\frac{1}{1-x},$$

on being expanded by the Taylor series expansion furnishes the infinite geometric series [26]

$$1 + x + x^2 + x^3 + \cdots + x^n + \cdots,$$

the general term of which is x^n; for this is usually called the general term, because from that all the numbers on being substituted in place of n successively give rise to all terms of the series. If we investigate the behavior of the denominator, $1 - x$, of the algebraic fraction at the right hand side of the equation

$$1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \frac{1}{1-x}$$

as x becomes closer and closer to 1, we will observe that $1-x$ approaches zero. For instance, if we let $x = 0.9, 0.99, 0.999$, we have the following respective results:

$$1 + 0.9 + 0.9^2 + 0.9^3 + \cdots = \frac{1}{1-0.9} = \frac{1}{0.1},$$
$$1 + 0.99 + 0.99^2 + 0.99^3 + \cdots = \frac{1}{1-0.99} = \frac{1}{0.01},$$
$$1 + 0.999 + 0.999^2 + 0.999^3 + \cdots = \frac{1}{1-0.999} = \frac{1}{0.001}.$$

Continuing our computation in this manner, we obtain, for any value of x with the digit 9 repeated n times, the value of $1 - x$ as 10^{-n}. If we let $x = 0.999\ldots$ where the ellipsis "$\ldots$" indicates that the digit 9 is repeated indefinitely, we get $1 - x = 0$ since 10^{-n} tends to zero as n tends to infinity. This is actually so because the value $x = 0.999\ldots$ can be expressed as the infinite geometrics series

$$\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$$

whose sum, when computed, furnishes exactly unity. Thus, instead of writing

$$1 + (0.999\ldots) + (0.999\ldots)^2 + \cdots = \frac{1}{1-(0.999\ldots)}$$

we can boldly write [13]

$$1 + 1 + 1 + \cdots = \frac{1}{1-1} = \frac{1}{0} = (-1)!.$$

This result may be obtained more easily by beginning with the recurrence relation $x! = x\,(x-1)!$ which may be rewritten as

$$(x-1)! = \frac{x!}{x}.$$

Letting $x = 1 - x$, we get

$$(-x)! = \frac{(1-x)!}{(1-x)}, \tag{3}$$

which, employing the Taylor series expansion of $\frac{1}{1-x}$, becomes

$$(-x)! = (1-x)!\left(1 + x + x^2 + x^3 + \cdots\right).$$

If we set $x = 1$, we obtain, again,

$$(-1)! = 1 + 1 + 1 + \cdots$$

which, for brevity, can also be expressed as

$$\sum_{k=1}^{\infty} 1 = (-1)!.$$

Now the partial sums of $\sum_{k=1}^{\infty} 1$ are

$$S_1 = \sum_{k=1}^{1} 1 = 1$$
$$S_2 = \sum_{k=1}^{2} 1 = 1 + 1 = 2$$
$$S_3 = \sum_{k=1}^{3} 1 = 1 + 1 + 1 = 3$$
$$S_4 = \sum_{k=1}^{4} 1 = 1 + 1 + 1 + 1 = 4$$

and in general, the mth partial sum is

$$S_m = \sum_{k=1}^{m} 1 = m$$

where m is a natural number or a positive integer. As $m \to \infty$, we have

$$\lim_{m\to\infty} (S_m) = \lim_{m\to\infty} (m) = (-1)!.$$

The limit $\lim_{m\to\infty} (m) = (-1)!$ which can also be written as

$$\lim_{m\to(-1)!} (m) = (-1)!$$

shows that $(-1)!$ is a positive integer. Thus, we have incontestably demonstrated that $(-1)!$ is an infinitely large positive integer.

Here the sceptic may remark that he sees no beauty or significance in the introduction of the constant $(-1)!$. At this stage of our investigation it would indeed be difficult to convince him that he is wrong. For the moment he is, in fact, right. But we shall see that just this new and strange concept leads to important applications in which $(-1)!$ appears to be essential.

It is, therefore, left for us to suppress any dubiety which may be entertained at this stage concerning the utility of

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$(-1)!$, which we have been discussing; for this number whose value is immeasurable because it is infinite is itself useful. So at this stage of our inquire, it would not be shocking if $(-1)!$ was considered entirely useless. This would be a great mistake for the computations involving $(-1)!$, many of which will not be demonstrated in this paper because they are better left for a text, are of the greatest importance.

V. HARMONIC SERIES AND γ

In this section we treat, in connection with the Euler's constant γ, one of the most celebrated series of all – the harmonic series, a divergent infinite series of the reciprocals of all the natural numbers or positive integers from 1 up to infinity, that is to say, the harmonic series is

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

with the terms of the series forming the harmonic sequence $1, 1/2, 1/3, \ldots$ which continues and matches into infinity. Any term of the sequence is less than 1 except the first term which is itself unity, but the further a term is in the sequence, the closer its value approaches zero. In the terminology of the German mathematician who with Riemann founded the modern theory of function, Karl Weierstrass (1815-1897) [23], the harmonic sequence $1, 1/2, 1/3 \ldots, 1/n, \ldots$ where n is a positive integer, approaches 0 as n tends to infinity, that is,

$$\lim_{n\to\infty} \left(\frac{1}{n}\right) = 0.$$

The earliest recorded appearance of the harmonic series seems to be in the work of the 14th century French mathematician Nicole Oresme (1323-1382) the Bishop of paris, probably the best mathematician of that century. He knew how to sum harmonic and arithmetico-geometric progressions as well as infinite geometric series, and was the first to prove that the harmonic series diverges by matching off to infinity [1]. His proof which relies on grouping the terms in the series furnishes the following inequality, sometimes called Oresme's inequality:

$$H_{2^m} > 1 + \frac{m}{2}, \qquad m \geq 0$$

which shows that the series is divergent [24].

Later, the divergence of the harmonic series was proved in the 17th century by Mengoli in his *Novae Quadraturea Arithmeticae* [1]. The divergence of the harmonic series was also proved independently by two of the talented Swiss mathematicians of the Bernoulli family, Jakob and Johann Bernoulli.

Let us now consider the nth partial sum of the harmonic series, called the harmonic number, H_n, and defined as

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

A careful analysis of Oresme's inequality, which requires a little calculus, shows that the harmonic number H_n increases at the same rate as the natural logarithm of n; this implies that the harmonic series has a logarithmic connection. Further analysis shows that the difference between the harmonic number H_n and the natural logarithm $\ln(n+1)$ decreases gradually as n increases and eventually converges to the Euler's constant as n tends to infinity. If we use the symbol γ_n to denote the nth difference between the concomitants H_n and $\ln(n+1)$ as they grow together, we write

$$\gamma_n = H_n - \ln(n+1)$$

which, being computed for $n = 1, 2, 3, \ldots$, gives values of γ_n that converge gradually to γ.

Suppose $\gamma_n = \gamma$ when $n = \Omega$, an infinitely large positive integer for which H_n becomes the harmonic series. We redefine Euler's constant as

$$\begin{aligned}\gamma &= \lim_{n\to\Omega} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \ln(n+1)\right)\\ &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{\Omega} - \ln(\Omega+1)\end{aligned}$$

or more concisely as

$$\gamma = H_\Omega - \ln(\Omega+1)$$

where H_Ω is the harmonic series and $\ln(\Omega+1)$ is the natural logarithm of $\Omega+1$.

It is essential, in order to understand the existence of the constant γ, to distinguish between the infinitely large numbers, H_Ω and $\ln(\Omega+1)$. It is evident, since γ is positive, that $H_\Omega > \ln(\Omega+1)$, but we will demonstrate this fact beginning with a familiar inequality [21], [28]:

$$\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} < \ln n < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}, \qquad n > 1$$

which can also be presented as

$$H_n - 1 < \ln n < H_{n-1}.$$

If we let $n = n+1$, then the inequality becomes

$$H_{n+1} - 1 < \ln(n+1) < H_n$$

which can be split into two inequalities:

$$H_{n+1} - \ln(n+1) < 1 \;\text{ or }\; H_n - \ln(n+1) < 1 - \frac{1}{n+1} \quad (4)$$

and

$$H_n > \ln(n+1) \;\text{ or }\; H_n - \ln(n+1) > 0. \quad (5)$$

The first inequality (4) shows that for every integer $n > 0, \gamma_n = H_n - \ln(n+1)$ is less than 1, a reason for the existence of $\gamma < 1$. The second inequality (5) shows that for every integer $n > 0, \gamma_n = H_n - \ln(n+1)$ is greater than 0. It follows from this that the divergent harmonic series H_Ω is greater than the divergent natural logarithm $\ln(\Omega+1)$, a reason for the existence of $\gamma > 0$. Thus γ exists and is between 0 and 1.

VI. $\ln(-1)!$ AS THE SUM OF THE HARMONIC SERIES

In the previous section, we defined the harmonic series, emphasizing on its nth partial sum, the nth harmonic number H_n, and its connection with the natural logarithm. We may now proceed to the logarithm of $(-1)!$ which will be needed in our treatment of the Euler's constant in the last two sections. In this section, however, we treat of the logarithm of $(-1)!$, a logarithmic infinity written as $\ln(-1)!$ and which, as we shall see, is the sum of the harmonic series.

We begin with the fundamental property of $(-1)!$:

$$0 \cdot (-1)! = 1. \quad (6)$$

We then find the natural logarithm of both sides of the equation and obtain the relation

$$\ln 0 + \ln(-1)! = 0 \tag{7}$$

which, on being rearranged, gives

$$\ln(-1)! = -\ln 0. \tag{8}$$

With the help of the relation (8), we will show that the sum of the divergent harmonic series H_Ω is $\ln(-1)!$, an infinitely large number less than $(-1)!$. Our aim at this point is, therefore, to demonstrate that $\ln(-1)!$ is the sum of the harmonic series, that is to say,

$$H_\Omega = 1 + \frac{1}{2} + \frac{1}{3} + \cdots = \ln(-1)!. \tag{9}$$

The possibility of such a relation as (9) is suggested by inspecting the Taylor series expansion of $\ln(1-x)$,

$$\ln(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right),$$

and letting $x = 1$. Accomplishing these, we obtain the following:

$$\ln 0 = -\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots\right)$$
$$-\ln 0 = 1 + \frac{1}{2} + \frac{1}{3} + \cdots.$$

Employing (8), we arrived at the required result

$$\ln(-1)! = 1 + \frac{1}{2} + \frac{1}{3} + \cdots.$$

This result can be obtained by another means; for if we start with the Taylor series expansion familiar to us, the expansion

$$\ln\left(\frac{1}{1-x}\right) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

and let

$$\frac{1}{1-x} = (-1)!,$$

so that

$$x = 1 - \frac{1}{(-1)!},$$

we get the interesting result:

$$\ln(-1)! = \left(1 - \frac{1}{(-1)!}\right) + \frac{1}{2}\left(1 - \frac{1}{(-1)!}\right)^2 + \cdots$$

which, upon applying the fundamental property of $(-1)!$, gives the result

$$\ln(-1)! = 1 + \frac{1}{2} + \frac{1}{3} + \cdots.$$

We now give a most convincing demonstration of the above result, and we begin with (3), that is

$$(-x)! = \frac{(1-x)!}{(1-x)}.$$

If we find the natural logarithm of both sides of the above formula, we shall obtain

$$\ln(-x)! = \ln(1-x)! + \ln\left(\frac{1}{1-x}\right)$$

which becomes

$$\ln(-x)! = \ln(1-x)! + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

which, setting $x = 1$, furnishes

$$\begin{aligned} \ln(-1)! &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots \\ &= \sum_{k=1}^{\infty} \frac{1}{k} \\ &= H_\Omega. \end{aligned} \tag{10}$$

Thus we see that the sum of the reciprocals of all the counting numbers or positive integers is equal to the natural logarithm of $(-1)!$. We can, therefore, explain the harmonic number in this manner. For all finite integer values of n greater than 1, the harmonic number is a finite number, but is a fractional type with denominator greater than 1. When n becomes the infinitely large integer Ω, the harmonic number is the logarithmic infinity, $\ln(-1)!$.

Let us take one step farther. If we make $(-1)!$ the subject of (10), we obtain

$$\begin{aligned} (-1)! &= e^{H_\Omega} \\ &= e^{1+\frac{1}{2}+\frac{1}{3}+\cdots} \\ &= e^1 \cdot e^{\frac{1}{2}} \cdot e^{\frac{1}{3}} \cdots \\ &= \prod_{k=1}^{\infty} e^{\frac{1}{k}} \end{aligned}$$

which clearly and satisfactorily shows that $(-1)!$ is positive and infinite.

It may seem at first that the natural logarithm of $(-1)!$, being the sum of the divergent harmonic series, has no use at all. As we shall see in the last two sections, $\ln(-1)!$ will be employed to derive a familiar formula for the Euler's constant γ and an interesting relation that connects γ to $(-1)!$.

VII. USE OF $\ln(-1)!$ IN DERIVING A FAMOUS FORMULA FOR γ

One famous and classical formula for γ which relates it to the Riemann zeta constants, $\zeta(k)$, should be mentioned, for soon we shall, employing the idea of $\ln(-1)!$, derive it:

$$1 - \gamma = \sum_{k=2}^{\infty} \frac{\zeta(k) - 1}{k}.$$

The above formula which can be used to compute γ was first found in the 1776 paper [11] by Euler and reappeared in several subsequent works by great and excellent mathematicians such as Glaisher [15], Johnson [18], Bromwich [5], Srivastava [29], Lagarias [20], and Barnes and Kaufman [3].

We now proceed to derive this formula which has fascinated the industry of such a great number of mathematicians and we shall begin from a familiar Maclaurin series expansion of the natural logarithm of $x!$. Euler derived the Maclaurin series expansion for $\ln(x!)$ [9], [30] which in modern notation reads

$$\ln(x!) = -\gamma x + \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k} x^k, \qquad |x| < 1.$$

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We shall here violate the stipulation that $|x| < 1$; for if we let $x = -1$ so that $|x| = 1$, an infringement of the proviso $|x| < 1$, then we obtain the result

$$\ln(-1)! = \gamma + \sum_{k=2}^{\infty} \frac{\zeta(k)}{k}.$$

We have already established in Section VI that the natural logarithm of $(-1)!$ is the sum of the harmonic series, $\sum_{k=1}^{\infty} \frac{1}{k}$. If we then replace $\ln(-1)!$ with the sum $\sum_{k=1}^{\infty} \frac{1}{k}$, we procure for ourselves

$$\sum_{k=1}^{\infty} \frac{1}{k} = \gamma + \sum_{k=2}^{\infty} \frac{\zeta(k)}{k}$$

which results in

$$1 + \sum_{k=2}^{\infty} \frac{1}{k} - \sum_{k=2}^{\infty} \frac{\zeta(k)}{k} = \gamma$$

which in turn furnishes our required formula

$$1 + \sum_{k=2}^{\infty} \frac{1 - \zeta(k)}{k} = \gamma$$

or

$$1 - \gamma = \sum_{k=2}^{\infty} \frac{\zeta(k) - 1}{k}.$$

Indeed, this application of $\ln(-1)!$ in deriving the above celebrated formula sounds strange to any one seeing it for the first time, and there is a reason to be particularly proud and pleased about it. This application of $\ln(-1)!$ is certainly satisfactory as the formula just derived is already existing and well known. From this, if our sum $\ln(-1)!$ for the divergent harmonic series should appear to some as not certain or reliable enough, a great confirmation comes to light here; thus there should not be any doubt about $\ln(-1)!$ as a replacement for the harmonic series H_Ω in the relation

$$\gamma = H_\Omega - \ln(\Omega + 1).$$

VIII. A NEW FORMULA CONNECTING γ TO $(-1)!$

We are now ready to discuss one of the most interesting applications of $(-1)!$, namely, the derivation of a new formula that connects γ to $(-1)!$. If we begin with the mystery of γ, in which Euler has beautifully mingled the harmonic series with the natural logarithm, that is the admirable relation

$$\gamma = \lim_{n \to \Omega} (H_n - \ln(n + 1))$$

where Ω, as already discussed in Section V, is an infinitely large positive integer for which the precise value of γ is obtainable, then the constant γ is also expressible as

$$\gamma = H_\Omega - \ln(\Omega + 1). \tag{11}$$

We have already seen that H_Ω, an infinitely large number less than Ω, is equal to the natural logarithm of $(-1)!$, that is

$$H_\Omega = \ln(-1)!,$$

but we must take Euler's relation (11) one step further and assume a simple relation between $(-1)!$ and $\Omega + 1$. We remember how indispensable it was for an understanding of the existence of Euler's constant to distinguish between H_Ω and $\ln(\Omega + 1)$ in Section V. It is equally important to distinguish between $(-1)!$ and $\Omega + 1$. The difference between these two infinitely great integers will soon be cleared up.

We have shown in Section V by means of the inequality

$$H_n - 1 < \ln n < H_{n-1}$$

that $H_\Omega > \ln(\Omega+1)$, and taking this, that is $H_\Omega > \ln(\Omega+1)$, as our starting point, we have, replacing H_Ω with $\ln(-1)!$,

$$\ln(-1)! > \ln(\Omega + 1)$$
$$\therefore (-1)! > \Omega + 1.$$

We thus have seen that, though both $(-1)!$ and $\Omega + 1$ are infinitely great integers, $(-1)!$ is greater than $\Omega + 1$.

We consider another distinction between $(-1)!$ and $\Omega + 1$. We know that, from the fundamental property of $(-1)!$, $\frac{1}{(-1)!} = 0$. Is $\frac{1}{\Omega+1} = 0$, since both $(-1)!$ and $\Omega + 1$ are infinitely great integers? The answer to this question comes when we first consider the series expansion

$$\ln\left(\frac{1}{1-x}\right) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots \tag{12}$$

and then let

$$\frac{1}{1-x} = \Omega + 1,$$

so that

$$x = 1 - \frac{1}{\Omega + 1}.$$

Substituting this into (12) furnishes the interesting result

$$\ln(\Omega + 1) = \left(1 - \frac{1}{\Omega+1}\right) + \frac{1}{2}\left(1 - \frac{1}{\Omega+1}\right)^2 + \cdots.$$

If $\frac{1}{\Omega+1} = 0$, then

$$\ln(\Omega + 1) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots = H_\Omega.$$

This is a great contradiction since we have already shown that $H_\Omega > \ln(\Omega + 1)$. Thus, $\frac{1}{\Omega+1}$ cannot be equal to 0. It will rather be infinitely less than 1 but infinitesimally greater than 0, that is

$$0 < \frac{1}{\Omega + 1} \ll 1.$$

This inequality implies that none of the terms of the harmonic series

$$H_\Omega = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{\Omega}$$

is exactly equal to the unit zero, $0 = 1 \times 0$, though each term approaches it as the series proceed onwards. This appears to be a reason for the divergence of the harmonic series. Moreover, since Ω is an infinitely great number, $\frac{1}{\Omega+1}$, the reciprocal of $\Omega + 1$, will then be an infinitely small number.

We now come to the derivation of the new formula for the Euler's constant. We can now express the constant γ in terms

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of $(-1)!$ and $\Omega + 1$, and the derivation runs thus:

$$\begin{aligned}\gamma &= H_\Omega - \ln(\Omega+1)\\ &= \ln(-1)! - \ln(\Omega+1)\\ &= \ln\left(\frac{(-1)!}{\Omega+1}\right)\\ \therefore e^\gamma &= \frac{(-1)!}{\Omega+1}\\ &= \frac{\lim_{m\to(-1)!}(m)}{\lim_{n\to\Omega}(n+1)}\\ &= \lim_{\substack{m\to(-1)!\\ n\to\Omega}}\left(\frac{m}{n+1}\right)\end{aligned}$$

where $\frac{m}{n+1}$ is a fraction, the ratio of two positive integers, m and $n+1$. We thus have obtained a most beautiful result; e^γ, being the ratio of the infinitely large integers, $(-1)!$ and $\Omega+1$, is a rational number.

It has been proved and is well known that e^y is irrational for every rational $y \neq 0$ [16], [25], [27]. If, therefore, $\gamma \neq 0$ were rational, then e^γ would be irrational, a contradiction, since e^γ as we have seen, is rational. Thus γ is an irrational number, incapable of being written as a ratio of two integers.

Let us conclude this paper by inquiring whether or not γ is transcendental. It is a well-known fact that all rationals are algebraic. For this reason e^γ, being a rational number, is algebraic. If γ were algebraic and then by the Lindermann–Weierstrass theorem [25], [31], e^γ would be transcendental, a contradiction. Thus, we conclude that γ which has just been proved to be irrational, is also transcendental.

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