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On the Inversion of Vandermonde Matrices

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Abstract **— A new approach for computing the inverses of Vandermonde matrices by using the cover-up technique of partial fraction expansion is presented. It provides insight of the relationship between the Vandermonde matrix and the coefficients of the associated partial fraction expansion. The method is suitable for either hand calculation or computer programming.**

Index Terms **— Vandermonde matrix, inverse of matrix, cover-up technique, partial fraction expansion.**

I. Introduction

The Vandermonde matrix plays an important role in several branches of applied mathematics, such as polynomial interpolation, numerical analysis, signal processing, statistics, geometry of curves and control theory. One can refer to [1-3] and the references therein for more details.

Among the different research topics related to Vandermonde matrix, the search for new and efficient approach for computation of its inverse is still a fundamental and important topic. In this paper, we will present a new approach for computing the inverse of Vandermonde matrix by using the cover-up technique of partial fraction expansion. It can provide us insight of the relationship between the Vandermonde matrix and the coefficients of the associated partial fraction expansion. The method is simple and suitable for either hand calculation or computer programming.

The whole paper is organized like this. The theoretical background is described in section 2. Some examples are provided in section 3. Then, a few concluding remarks are given in section 4.

II. Theoretical background

Consider a rational function of the following form:

$$f(x)=\frac{b_1x^{n-1}+b_2x^{n-2}+\cdots+b_n}{x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_n}$$

$$=\frac{b_1x^{n-1}+b_2x^{n-2}+\cdots+b_n}{(x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)}$$

where a_i, b_i, λ_i are given constants. By using the cover-up technique described in [6-9], we can obtain the partial fraction expansion of $f(x)$ as follows:

$$f(x)=\frac{k_1}{x-\lambda_1}+\frac{k_2}{x-\lambda_2}+\cdots+\frac{k_n}{x-\lambda_n}$$

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where the partial fraction coefficients k_i can be easily computed by using the following formula:

$$k_i=f(x)\cdot(x-\lambda_i)\big|_{x=\lambda_i}=\frac{b_1\lambda_i^{\,n-1}+b_2\lambda_i^{\,n-2}+\cdots+b_n}{\prod_{j\neq i}(\lambda_i-\lambda_j)}$$

We can re-write it as the product of a row vector and a column vector as follows:

$$k_i=\left(\frac{\lambda_i^{\,n-1}}{\prod_{j\neq i}(\lambda_i-\lambda_j)},\frac{\lambda_i^{\,n-2}}{\prod_{j\neq i}(\lambda_i-\lambda_j)},\cdots,\frac{1}{\prod_{j\neq i}(\lambda_i-\lambda_j)}\right)\cdot\begin{pmatrix}b_1\\ b_2\\ \vdots\\ b_n\end{pmatrix}$$

Now, let us define a $n\times n$ square matrix W below.

$$W=\begin{pmatrix}\frac{\lambda_1^{\,n-1}}{\prod_{j\neq 1}(\lambda_1-\lambda_j)} & \frac{\lambda_1^{\,n-2}}{\prod_{j\neq 1}(\lambda_1-\lambda_j)} & \cdots & \frac{1}{\prod_{j\neq 1}(\lambda_1-\lambda_j)}\\ \frac{\lambda_2^{\,n-1}}{\prod_{j\neq 2}(\lambda_2-\lambda_j)} & \frac{\lambda_2^{\,n-2}}{\prod_{j\neq 2}(\lambda_2-\lambda_j)} & \cdots & \frac{1}{\prod_{j\neq 2}(\lambda_2-\lambda_j)}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\lambda_n^{\,n-1}}{\prod_{j\neq n}(\lambda_n-\lambda_j)} & \frac{\lambda_n^{\,n-2}}{\prod_{j\neq n}(\lambda_n-\lambda_j)} & \cdots & \frac{1}{\prod_{j\neq n}(\lambda_n-\lambda_j)}\end{pmatrix}$$

So, the solutions of k_i can be expressed as $K=W\times B$, where K is a column vector with entries k_i and B is a column vector with entries b_i.

Referring to [5], a formula was proposed for evaluating k_i, namely

$$K=V^{-1}\times A^{-1}\times B$$

where V is the Vandermonde matrix such that

$$V=\begin{pmatrix}1 & 1 & \cdots & 1\\ \lambda_1 & \lambda_2 & \cdots & \lambda_n\\ \lambda_1^{\,2} & \lambda_2^{\,2} & \cdots & \lambda_n^{\,2}\\ \vdots & \vdots & \ddots & \vdots\\ \lambda_1^{\,n-1} & \lambda_2^{\,n-1} & \cdots & \lambda_n^{\,n-1}\end{pmatrix}$$

and A is a lower triangular matrix such that

$$A=\begin{pmatrix}1 & 0 & 0 & \cdots & 0\\ a_1 & 1 & 0 & \cdots & 0\\ a_2 & a_1 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & 1\end{pmatrix}.$$

By comparing the above two formulas for K, we get:

$$W=V^{-1}\times A^{-1}.$$

Hence, $V^{-1}=W\times A$. This result describes the relationship between the Vandermonde matrix V and the coefficients of the partial fraction expansion of $f(x)$. It also provides us a simple way to compute the inverse of V.

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Since the above formula only depends on the coefficients of the denominator of $f(x)$, we can define an associated partial fraction expansion with unit numerator for V as follows to ease the computations involved.

$$f(x)=\frac{1}{x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_n}$$
$$=\frac{1}{(x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)}$$

where the coefficients a_i can be computed by using the well-known formulas below:

$$a_1=-\sum\lambda_j\ ,\ a_2=\sum_{j\neq m}\lambda_j\lambda_m\,, a_3=-\sum_{j\neq m\neq s}\lambda_j\lambda_m\lambda_s\ , \ldots,$$
$$a_n=(-1)^n\prod\lambda_j\ .$$

III. Illustrative Examples

We now illustrate how to use the method described in section 2 to solve problems which involve finding the inverse of a Vandermonde matrix and its applications.

Example1. Find the inverse of the following Vandermonde matrix.

$$V=\begin{pmatrix}1&1&1\\0&-1&-2\\0&1&4\end{pmatrix}.$$

Solution. Let us define an associated partial fraction expansion of V as follows:

$$f(x)=\frac{1}{x^3+a_1x^2+a_2x+a_3}=\frac{1}{x(x+1)(x+2)}$$

So, $\lambda_1=0,\ \lambda_2=-1,\ \lambda_3=-2,\ a_1=3,\ a_2=2.$

Hence,

$$V^{-1}=\begin{pmatrix}0&0&1/2\\-1&1&-1\\2&-1&1/2\end{pmatrix}\begin{pmatrix}1&0&0\\3&1&0\\2&3&1\end{pmatrix}$$
$$=\begin{pmatrix}1&3/2&1/2\\0&-2&-1\\0&1/2&1/2\end{pmatrix}.$$

It is trivial to check that $V\times V^{-1}$ and $V^{-1}\times V$ are indeed equal to the identity matrix.

Example2. Derive the formula of the sum of the first n square numbers by the method of polynomial interpolation.

Solution. Let the formula required be in the form:

$$s(x)=a_1+a_2x+a_3x^2+a_4x^3.$$

Since $s(1)=1$, $s(2)=5$, $s(3)=14$, $s(4)=30$, we can express these data in matrix form below:

$$\begin{pmatrix}1\\5\\14\\30\end{pmatrix}=\begin{pmatrix}1&1&1&1\\1&2&4&8\\1&3&9&27\\1&4&16&64\end{pmatrix}\begin{pmatrix}a_1\\a_2\\a_3\\a_4\end{pmatrix}.$$

Hence,

$$\begin{pmatrix}a_1\\a_2\\a_3\\a_4\end{pmatrix}=\begin{pmatrix}1&1&1&1\\1&2&4&16\\1&3&9&27\\1&4&16&64\end{pmatrix}^{-1}\begin{pmatrix}1\\5\\14\\30\end{pmatrix}.$$

The matrix on the right is equal to the inverse of the transpose of the following Vandermonde matrix V

$$V=\begin{pmatrix}1&1&1&1\\1&2&3&4\\1&4&9&16\\1&8&27&64\end{pmatrix}.$$

We define an associated partial fraction expansion of V as follows:

$$f(x)=\frac{1}{x^4+a_1x^3+a_2x^2+a_3x+a_4}$$
$$=\frac{1}{(x-1)(x-2)(x-3)(x-4)}$$

So,

$\lambda_1=1,\ \lambda_2=2,\ \lambda_3=3,\lambda_4=4,a_1=-10,a_2=35,a_3=-50.$

Hence,

$$V^{-1}=\begin{pmatrix}\frac{-1}{6}&\frac{-1}{6}&\frac{-1}{6}&\frac{-1}{6}\\4&2&1&\frac{1}{2}\\\frac{-27}{2}&\frac{-9}{2}&\frac{-3}{2}&\frac{-1}{2}\\\frac{32}{3}&\frac{8}{3}&\frac{2}{3}&\frac{1}{6}\end{pmatrix}\begin{pmatrix}1&0&0&0\\-10&1&0&0\\35&-10&1&0\\-50&35&-10&1\end{pmatrix}$$
$$=\begin{pmatrix}4&-13/3&3/2&-1/6\\-6&19/2&-4&1/2\\4&-7&7/2&-1/2\\-1&11/6&-1&1/6\end{pmatrix}$$

Thus,

$$\begin{pmatrix}a_1\\a_2\\a_3\\a_4\end{pmatrix}=\begin{pmatrix}4&-6&4&-1\\-13/3&19/2&-7&11/6\\3/2&-4&7/2&-1\\-1/6&1/2&-1/2&1/6\end{pmatrix}\begin{pmatrix}1\\5\\14\\30\end{pmatrix}=\begin{pmatrix}0\\1/6\\1/2\\1/3\end{pmatrix}.$$

$$\therefore\ s(x)=\frac{x}{6}+\frac{x^2}{2}+\frac{x^3}{3}=\frac{x(x+1)(2x+1)}{6}\ .$$

By replacing x by n, we have obtained the formula of the sum of the first n square numbers, namely $n(n+1)(2n+1)/6$.

IV. Concluding Remarks

In this paper, we have introduced a simple method for computing the inverse of the Vandermonde matrix via the cover-up technique of partial fraction expansion. This method will be found useful in areas such as polynomial interpolation, statistics, signal processing and cryptography, etc. Further study or development of this method to solve systems of polynomial equations or apply it to handle problems in the areas mentioned above will be a meaningful and interesting research topic to pursue.

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REFERENCES

[1] A. Eisinberg and G. Fedele, "On the inversion of the Vandermonde matrix", *Appl. Math. and Comput.*, vol. 174, pp. 1384-1397, 2006.

[2] M. Goldwurm and V. Lonati, "Pattern statistics and Vandermonde matrices", *Theor. Comput. Sci.*, vol. 356, pp. 153-169, 2006.

[3] H. Hakopian, K. Jetter and G. Zimmermann, "Vandermonde matrices for intersection points of curves", *J. Approx.*, vol. 1, 67-81, 2009.

[4] J. J. Rushanan, "On the Vandermonde matrix", Amer. Math. Monthly, vol. 96, pp, 921-924, 1989.

[5] C.F. Chen and K. K. Leung, "A new look at partial fraction expansion from a high-level language viewpoint", *Comput. Math. Appl.*, vol. 7, pp. 361-367, 1981.

[6] Y. K. Man, "A simple algorithm for computing partial fraction expansions with multiple poles," *Int. J. of Math. Educ. in Sci. and Technol.*, vol. 38, pp. 247–251, 2007.

[7] Y. K. Man, "An improved Heaviside approach to partial fraction decomposition and its applications," in *Proceedings of WCE 2008*, London: IAENG, 2008, pp. 978–980.

[8] Y. K. Man, "Partial fraction decomposition by synthetic division and its applications," in *Current Themes in Engineering Science 2008*, A. M. Korsunsky, Ed. New York: AIP, 2009, pp. 71–82.

[9] Y. K. Man, "A cover-up approach to partial fractions with linear or irreducible quadratic factors in the denominators," *Appl. Math. and Comput.*, vol. 219, pp. 3855–3862, 2012.