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Wide Band Noises: Invariant Results

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Abstract—**In applications wide band noises are observed by autocovariance functions. At the same time there are infinitely many wide band noises having the same autocovariance function. Each of these wide band noises creates its own best estimate and optimal control. Therefore, it is practically important to obtain filtering and control results which depend on autocovariance function and independent on respective wide band noises. Shortly, we call such results as invariant results. In this paper we present two such results related to linear filtering and LQG problems.**

Index Terms—**wide band noise, white noise, Kalman filtering, LQG problem.**

I. Introduction

ESTIMATION and stochastic control theories are heavily based on the white noise model of disturbing noise processes. The most significant result in estimation theory, the Kalman filter, which originates from the works [1] and [2] and found further developments in the literature (see, for example, [3] and [4]), has been discovered for partially observable linear systems with a white noise disturbance. The same can be said about the optimal control law in LQG (linear quadratic Gaussian) control problem as well.

Although this and other results for white noise driven systems found wide applications in engineering problems (see, [5]), the noises in reality are not white indeed. At most, they are approximately white and, in general, far from being white. Fleming and Rishel [6] wrote that the real noises are wide band and white noises are the ideal case of wide band noises. When the parameters of white and wide band noises are sufficiently close to each other, white noises take place of wide band noises to make mathematical models simpler. Respectively, for more adequate estimation and control results, a mathematical method of handling and working with wide band noises is required.

According to our knowledge, there are just two major approaches to wide band noises. One of them is based on approximations and nearly optimal concept. This approach developed in a series of works [7]–[11] etc. The other approach is based on the so-called integral representation and initiated in [12]. This approach presents wide band noises as a distributed delay of white noises and reduces a wide band noise driven system to a white noise driven system. It has many common points with the approach to coloured noises, initiated in [13].

In this paper we discuss estimation and control problems by the second of these approaches. In the early papers [12], [14]–[16] this approach was used to study linear filtering problems for wide band noise driven systems via Wiener-Hopf equation. An essential progress has been achieved in [17] and [18], where by use of integral representation a wide band noise driven system was reduced to a white noise driven system. Using this reduction, in [19] linear filtering and LQG results were obtained for wide band noises with a special integral representation. The reduction technique became useful to obtain other results for wide band noise driven systems such as controllability [20], [21], stochastic maximum principle [22] and filtering with pointwise delayed white noises [23]–[26].

An important problem was raised in [27]. One of the difficulties of working with wide band noises is that in applications they are measured by autocovariance function. At the same time, different wide band noises may have the same autocovariance function. Denote by $W(\Lambda)$ the collection of all wide band noises with the autocovariance function Λ. Normally, different wide band noises from $W(\Lambda)$ create different best estimates and also different optimal controls in the estimation and stochastic control problems under consideration. It also may happen that they are independent on wide band noises from $W(\Lambda)$ and just depend on Λ. So, the following problems can be raised:

(a) Whether the autocovariance function Λ can be sufficient for construction of best estimates and optimal controls independently on wide band noises from $W(\Lambda)$?
(b) If yes, then under what conditions?
(c) In the case if Λ is not sufficient, how the best one of the best estimates and the optimal one of the optimal controls should be selected?
(d) If answering the above questions for $W(\Lambda)$ is difficult, is there a subset $W_0(\Lambda) \subseteq W(\Lambda)$ for which the above questions can be addressed?
(e) How reasonable is $W_0(\Lambda)$?

These problems were partially formulated in [27] and included to the list of unsolved problems in mathematical systems and control theory. In particular, for the item (c), a minimization of the error of estimation and the cost functional over $W(\Lambda)$ or $W_0(\Lambda)$ was suggested as a criteria of selection.

A partial contribution to the solution of the above set of problems has been done in [28] on the set $W_0(\Lambda)$ consisting of wide band noises with the integral representation. It is not yet investigated how wide is $W_0(\Lambda)$ in $W(\Lambda)$. But there are several theoretical and applied arguments supporting $W_0(\Lambda)$:

- The set $W_0(\Lambda)$ is rather wide. In [17] and [18] it is shown that $W_0(\Lambda)$ contains infinitely many wide band noise processes.
- The wide band noises from $W_0(\Lambda)$ have a natural interpretation: according to [29] and [30] they are distributed delays of white noises. So, the presence of wide band noises in real systems can be simply explained as an aftereffect of white noises.
- The wide band noises from $W_0(\Lambda)$ are manageable. They can be expressed via stochastic linear differential

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delay equations.

- Finally, we strongly believe that the wide band noises disturbing real systems are of the $W_0(\Lambda)$-sort for the following reason. Let w_t be a Wiener process and consider the ratio $(w_{t+\varepsilon} - w_t)/\varepsilon$. This ratio is a wide band noise of the $W_0(\Lambda)$-sort. The limit of this ratio as $\varepsilon \to 0$ does not exist in the ordinary sense, but we force it to converge and call the result as a white noise. It is evident that a substitution of wide band noises by white noises in real systems produces more or less acceptable filtering and control results. This allows to argue that in fact the above ratio, as an "uncompleted derivative" of a Wiener process for some sufficiently small $\varepsilon > 0$, acts as a noise process in real systems.

In this paper we consider a linear filtering and LQG problems for linear signal and observation systems disturbed by independent wide band and white noises, respectively. We present a complete set of formulae for the best estimate and optimal control for such systems in terms of just system parameters, parameters of the cost functional and autocovariance function of disturbing wide band noise, demonstrating the independence of these results on $\varphi \in W_0(\Lambda)$, just dependent on Λ. We call such results as invariant results pointing out the fact that the knowledge of autocovariance function is sufficient for them and there is no need for distinction the wide band noises in $W_0(\Lambda)$.

II. Wide band noises

A vector-valued random process φ is said to be a wide band noise if

$$\operatorname{cov}(\varphi_{t+\sigma}, \varphi_t) = \begin{cases} 0, & \sigma \geq \varepsilon, \\ \Lambda_{t,\sigma}, & 0 \leq \sigma < \varepsilon, \end{cases}$$

where $\operatorname{cov}(\cdot,\cdot)$ is a covariance matrix, $\varepsilon > 0$, and Λ is a matrix-valued nonzero function. Note that in this paper we do not specify the dimensions of vector-valued processes and matrix-valued functions assuming that they are consistent. In the case when φ_t has zero mean and Λ depends on just its second argument, the wide band noise φ is said to be stationary (in the wide sense).

One can verify that the random process

$$\varphi_t = \int_{\max(0,t-\varepsilon)}^{t} \Phi_{t,s-t}\, dw_s,\ t \geq 0, \tag{1}$$

where Φ is a matrix-valued function on $[0,\infty) \times [-\varepsilon, 0]$ and w is a vector-valued standard Wiener process, is a vector-valued wide band noise with

$$\operatorname{cov}(\varphi_{t+\sigma}, \varphi_t) = \int_{\max(0,t+\sigma-\varepsilon)}^{t} \Phi_{t+\sigma,s-t-\sigma}\Phi^*_{t,s-t}\, ds$$

if $0 \leq \sigma < \varepsilon$. Here Φ^* stands for the transpose of the matrix Φ. We will call Φ as a relaxing (damping) function and (1) as an integral representation for φ. If Φ depends only on its second argument, then

$$\operatorname{cov}(\varphi_{t+\sigma}, \varphi_t) = \int_{\max(-t,\sigma-\varepsilon)}^{0} \Phi_{s-\sigma}\Phi^*_s\, ds \tag{2}$$

if $0 \leq \sigma < \varepsilon$. Consequently, φ becomes stationary for $t \geq \varepsilon$.

In applications wide band noises are measured by aotocovariance function. Let φ be a stationary (since the instant ε) wide band noise with the autocovariance function Λ. By (2), φ has the integral representation

$$\varphi_t = \int_{\max(0,t-\varepsilon)}^{t} \Phi_{s-t}\, dw_s$$

if Φ is a solution of

$$\Lambda_\sigma = \int_{\sigma-\varepsilon}^{0} \Phi_{s-\sigma}\Phi_s\, ds.$$

This is a convolution equation. In [17], for one-dimensional case it is proved that if Λ is a positive definite function and some very general conditions hold, then this equation has an infinite number of solutions Φ in the space of square integrable functions. Notice that the positive definiteness is a defining property of autocovariance functions. This result seemingly extends to multidimensional and non-stationary cases. Therefore, given an autocovariance function Λ, there are infinitely many relaxing functions Φ and, respectively, infinitely many wide band noise processes in the form (1), which have the same autocovariance function Λ. We denote the collection of such wide band noise processes by $W_0(\Lambda)$.

Note that the "uncompleted derivative"

$$\eta_t = \frac{w_{t-\varepsilon} - w_t}{\varepsilon} = \int_{t-\varepsilon}^{t} \frac{1}{\varepsilon}\, dw_s,\ t \geq 0,$$

of the vector-valued Wiener process w for some $\varepsilon > 0$ belongs to $W_0(\Lambda)$ if

$$\Lambda_\sigma = \frac{I(\varepsilon - \sigma)}{\varepsilon^2},\ 0 \leq \sigma \leq \varepsilon,$$

where I is an identity matrix.

The noises of the form (1) have a universal nature, covering basic kinds of noises discussed in mathematical systems theory. The three kinds of such noise processes are as follows:

- *White noises.* These noises are mostly popular because of the well established Ito calculus, providing a strong mathematical tool for investigation of systems corrupted by white noises. Although such systems find wide applications, white noises are indeed ideal and do not exist in the nature. Therefore, modeling of real processes by white noise driven systems requires a compensation from adequacy.
- *Coloured noises.* These noises are more realistic taking place between white and real noises. They are outputs of linear systems with an additive white noise input.
- *Wide band noises.* These noises are indeed noises observed in reality. Unlike coloured noises, wide band noises are given by autocovariance functions.

Besides wide band noises, the noises of the form (1) cover white and coloured noises as well. Moreover, they cover pointwise delayed white noises as well. This is demonstrated below.

1. *White noises.* Keep ε in (1) small and let λ be a function of t, satisfying $t - \varepsilon \leq \lambda_t \leq t$. Choose

$$\Phi_{t,\theta} = F\delta_{\theta+t-\lambda_t},$$

where δ is Dirac delta-function. Then

$$\varphi_t = \int_{\max(0,t-\varepsilon)}^{t} F\delta_{s-\lambda_t}\, dw_s = Fw'_{\max(0,\lambda_t)}.$$

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If $\lambda_t = t$, then φ is a white noise without any delay. If $\lambda_t = t - \varepsilon$, then φ is a white noise with a single time-independent delay. Otherwise, φ is a white noise with time-dependent delay.

2. *Coloured noises.* Let ε be sufficiently large and choose Φ in (1) as

$$\Phi_{t,\theta} = e^{-A\theta} F_{t+\theta},$$

where e^{-At} is a transition matrix of $-A$. Then

$$\varphi_t = \int_0^t e^{A(t-s)} F_s \, dw_s, \ 0 \le t \le \varepsilon,$$

implying

$$d\varphi_t = A\varphi_t dt + F_t dw_t, \ \varphi_0 = 0, \ 0 < t \le \varepsilon.$$

Thus φ is a coloured noise.

Since in applied problems we are given just the autocovariance function of a wide band noise, the multivaluedness of $\Lambda \to W_0(\Lambda)$ is one of the difficulties for construction of best estimates and optimal controls under wide band noises. Therefore. it becomes important getting invariant results, which are independent on $\varphi \in W_0(\Lambda)$, just dependent on Λ. Below two such results are presented.

III. Discussion of main idea on Kalman filter

It is suitable to discuss the main idea of this paper on the example of Kalman filter. Consider the partially observable linear system

$$\begin{cases} dx_t = Ax_t \, dt + F \, dw_t, \ x_0 = \xi, \ t > 0, \\ dz_t = Cx_t \, dt + dv_t, \ z_0 = 0, \ t > 0, \end{cases}$$

under standard conditions assuming that w and v are correlated Wiener processes with $\operatorname{cov}(w_t, v_t) = Et$. The classic Kalman filtering result states that the best estimation process $\hat{x}$ exists and is a unique solution of

$$\begin{cases} d\hat{x}_t = A\hat{x}_t \, dt + (P_t C^* + FE)(dz_t - C\hat{x}_t \, dt), \\ \hat{x}_0 = 0, \ t > 0, \end{cases}$$

where P is a solution of the matrix Riccati equation

$$\begin{cases} P_t' = AP_t + P_t A^* + FF^* \\ \qquad -(P_t C^* + FE)(CP_t + E^* F^*), \\ P_0 = \operatorname{cov} \xi, \ t > 0. \end{cases}$$

One can see that if in the signal system the matrix F is unknown, and instead the matrix FF^* is known, then the matrix F can be recovered in different forms. For example, in two-dimensional case if $FF^* = I$, then each of the following matrices

$$F_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ F_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$F_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ F_4 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

can serve for F. Respectively, the Kalman filter produces the best estimate non-uniquely if F is unknown, just FF^* is known. But if $E = 0$ (independent white noises), then the best estimate depends just on FF^* and does not depend on the forms of F. In other words, the Kalman filter is invariant if $E = 0$.

In this regard, the advantage of Kalman filtering problem is that F is given. In the case of wide band noises, F is changed by the relaxing function Φ and FF^* by the autocovariance function Λ. We do not have the luxury of being known Φ, instead we know Λ. Therefore, it should be expected a similar invariant result for wide band noises.

IV. Wide band noise optimal filter

In this section we present the first invariant result related to linear filtering problem. Consider the partially observable linear system

$$\begin{cases} x_t' = Ax_t + \varphi_t, \ x_0 = \xi, \ t > 0, \\ dz_t = Cx_t \, dt + dv_t, \ z_0 = 0, \ t > 0, \end{cases} \tag{3}$$

where x and z are vector-valued signal and observation processes, A and C are matrices, $\varphi \in W_0(\Lambda)$ has an integral representation (1) for some square integrable relaxing function Φ, ξ is a Gaussian random variable with zero mean, w and v are Wiener processes, and ξ, w and v are independent.

Note that the signal system in (3) is given in terms of derivative while the observation system in terms of differential. By this, we stress on the fact that unlike white noises, which are generalized derivatives of Wiener processes and do not exist in the ordinary sense, wide band noises are well-defined random processes.

Under these conditions the best estimate (in the mean square sense) process $\hat{x}$ for the system (3) is uniquely determined as a solution of the system of equations

$$\begin{cases} d\hat{x}_t = (A\hat{x}_t + \psi_{t,0}) \, dt + P_t C^* (dz_t - C\hat{x}_t \, dt), \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}\right) \psi_{t,\theta} \, dt = Q_{t,\theta} C^* (dz_t - C\hat{x}_t \, dt), \\ \hat{x}_0 = 0, \ \psi_{0,\theta} = \psi_{t,-\varepsilon} = 0, \ -\varepsilon \le \theta \le 0, \ t > 0, \end{cases} \tag{4}$$

where P, Q and R are solutions of

$$\begin{cases} P_t' = AP_t + P_t A^* + Q_{t,0} + Q_{t,0}^* - P_t C^* C P_t, \\ P_0 = \operatorname{cov} \xi, \ t > 0, \end{cases} \tag{5}$$

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}\right) Q_{t,\theta} = Q_{t,\theta} A^* + \Lambda_{t,-\theta} - R_{t,\theta,0} \\ \qquad -Q_{t,\theta} C^* C P_t, \\ Q_{0,\theta} = Q_{t,-\varepsilon} = 0, \ -\varepsilon \le \theta \le 0, \ t > 0, \end{cases} \tag{6}$$

and

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \tau}\right) R_{t,\theta,\tau} = Q_{t,\theta} C^* C Q_{t,\tau}^*, \\ R_{0,\theta,\tau} = R_{t,-\varepsilon,\tau} = R_{t,\theta,-\varepsilon} = 0, \ -\varepsilon \le \theta, \tau \le 0, \ t > 0. \end{cases} \tag{7}$$

Moreover, the mean square error of estimation equals to

$$e_t = \mathbf{E} \|\hat{x}_t - x_t\|^2 = \operatorname{tr} P_t.$$

We call the filter, determined by equations (4)–(7) as a wide band noise filter. This filter with minor modifications is proved in [28].

The classic Kalman filter consists of two equations for $\hat{x}$ and P. But the wide band noise filter includes the associated equations for ψ, Q and R. What are the meanings of them within the filter? The solution of the equation for ψ (the second equation in (4)) has the representation

$$\psi_{t,\theta} = \int_{\max(0,t-\theta-\varepsilon)}^t Q_{s,s-t+\theta} C^* (dz_s - C\hat{x}_s ds),$$

implying

$$\psi_{t,0} = \int_{\max(0,t-\varepsilon)}^t Q_{s,s-t} C^* (dz_s - C\hat{x}_s ds).$$

Therefore, $\psi_{t,0}$ in the first equation in (4) acts as a wide band noise with the relaxing function $\Psi_{t,\theta} = Q_{t+\theta,\theta} C^*$ generated by the innovation process. Denote its autocovariance function

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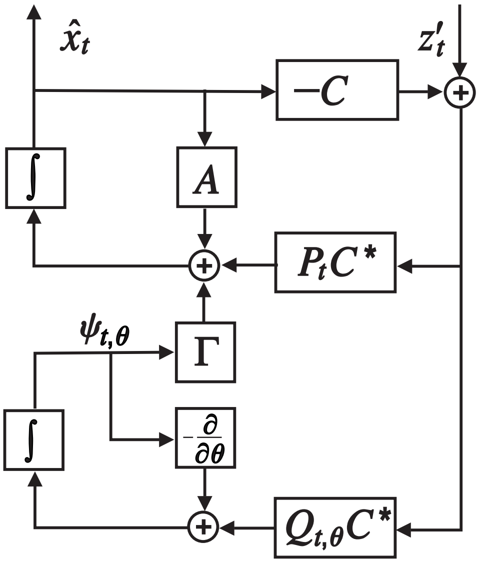

Fig. 1. The wide band noise filter (4)–(7).

by Σ. The function Q is an essential factor of the relaxing function of $\psi_{t,0}$. Q satisfies the equation (6), which includes $R_{t,\theta,0}$. R satisfies the equation (7) and has the representation

$$R_{t,\theta,\tau} = \int_{\max(0,t-\theta-\varepsilon,t-\tau-\varepsilon)}^{t} Q_{s,s-t+\theta} C^* C Q^*_{s,s-t+\tau}\, ds.$$

This implies

$$R_{t,\theta,0} = \int_{\max(0,t-\theta-\varepsilon)}^{t} Q_{s,s-t+\theta} C^* C Q^*_{s,s-t}\, ds.$$

One can see that in fact $R_{t,\theta,0} = \Sigma_{t,-\theta}$. Therefore, the wide band noise filter (4)–(7) works in the following form:

(a) The equation (7) produces the covariance function of the wide band noise $\psi_{t,0}$.
(b) The equation (6) produces an essential factor of the relaxing function of the wide band noise $\psi_{t,0}$.
(c) The equation (5) is a modified Riccati equation from classic Kalman filter.
(d) The second equation in (4) produces the wide band noise $\psi_{t,0}$.
(e) All these make the first equation in (4) to be driven by the sum of white and wide band noises with clear autocovariance and relaxing functions. This equation produces the best estimate process $\hat{x}$.

The equations (5)–(7) are deterministic and can be solved beforehand independently on (4) and the values of P and Q stored somewhere in computer. For the solution of (5)–(7), simple numerical methods for solution of partial differential equations can be used. Then the wide band noise filter (4)–(7) acts as in Figure 1, in which Γ stands for an operator, transforming a function h, on $[-\varepsilon, 0]$ to its value h_0.

V. Application to LQG problem.

Application of the wide band noise filter (4)–(7) to LQG problem can be done by use of control result from [19]. Consider LQG problem of minimizing the cost functional

$$J(u) = \mathbf{E}\left(\langle x_T, H x_T\rangle + \int_0^T (\langle x_t, M x_t\rangle + \langle u_t, N u_t\rangle)\, dt\right) \quad (8)$$

over the partially observable system

$$\begin{cases} x'_t = A x_t + B u_t + \varphi_t, \ x_0 = \xi, \ 0 < t \le T, \\ dz_t = C x_t\, dt + dv_t, \ z_0 = 0, \ 0 < t \le T, \end{cases} \quad (9)$$

where $\mathbf{E}$ stands for expectation. Assume that the conditions of the previous section hold and, additionally, B, H, M and N are matrices of respective dimensions in which H and M are nonnegative and N is positive. Then the optimal control u^* in the LQG problem (8)–(9) is uniquely determined by

$$u^*_t = -G^{-1} B^* \left(V_t \hat{x}^*_t + \int_t^{\min(T,t+\varepsilon)} \mathcal{Y}^*_{s,t} V_s \psi_{t,t-s}\, ds \right), \quad (10)$$

where $\hat{x}^*_t$ is the best estimate of the signal x^*_t, defined by (9) and corresponding to the optimal control $u = u^*$, ψ is the associated process, both satisfying

$$\begin{cases} d\hat{x}^*_t = (A\hat{x}^*_t + \psi_{t,0} + B u^*_t)\, dt \\ \qquad + P_t C^* (dz^*_t - C\hat{x}^*_t\, dt), \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}\right)\psi_{t,\theta}\, dt = Q^*_{t,\theta} C^* (dz^*_t - C\hat{x}^*_t\, dt), \\ \hat{x}^*_0 = 0, \ \psi_{0,\theta} = \psi_{t,-\varepsilon} = 0, \ -\varepsilon \le \theta \le 0, \ 0 < t \le T, \end{cases} \quad (11)$$

z^* is the observation process, defined by (9) and corresponding to the optimal control $u = u^*$, V is a solution of the Riccati equation

$$\begin{cases} V'_t + V_t A + A^* V_t + M - V_t B N^{-1} B^* V_t = 0, \\ V_T = H, \ 0 \le t < T, \end{cases} \quad (12)$$

P, Q and R are solutions of (5)–(7), and $\mathcal{Y}$ is a bounded perturbation of the transition matrix e^{At} of A by $-BN^{-1}B^* V_t$.

This result is proved in [19], pp. 224–225, for relaxing functions of special form. The filter from the previous section makes it valid under conditions of this section.

Similar to the wide band noise filter from previous section, the optimal control law (10)–(12) is also independent of $\varphi \in W_0(\Lambda)$, just depends on Λ. In other words, this is also an invariant result. Another notable feature of this result is that it does not fall into the frame of classical separation principle since the observations z_s, $0 \le s \le t$, are dependent on x_τ for $t \le \tau \le t + \varepsilon$, that is, (9) is an acausal system. Indeed, this result falls into extended separation principle (see [19], pp. 135–139).

VI. Concluding remarks

Two invariant results are presented in this paper. These results provide a complete sets of equations for designing the optimal filter and optimal control in the linear filtering and LQG problems on the base of the system and cost functional parameters A, B, C, M, N, H and the autocovariance function Λ of the wide band noise φ. Just ordinary differential equations in the case of classic theory are modified to systems of equations which include ordinary and partial differential equations.

To point out another implicit advantage from theorems of this paper, assume that a study of some real process requires estimation of a linear system disturbed by a wide band noise φ. To simplify the model, replace φ by a white noise, which is more or less close to φ. Then the error of estimation by white noise Kalman filter will deviate from the real error. One of the reasons for this deviation is the replacement of φ by a white noise. Therefore, the error of estimation of the wide band noise filter from Section IV is more adequate (precise) than the one of classic Kalman filter. This does not means

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that the error of wide band noise filter is always smaller than the error of the classic Kalman filter. If it is smaller, this is a consequence from the improvement of adequacy of the model. On the contrary, if it is greater, then this can be explained as an inappropriate replacement of φ by a white noise. This issue should be of great importance in tracking of satellites, in particular, for getting preciseness of GPS. In this way, it is remarkable numerical calculations from [25], where it was detected that a replacement of wide band noise (in the form of pointwise delayed white noise) by a white noise produces a lost of preciseness which is asymptotically (as time increases) nonrecoverable.

Another issue is a study of filtering and control problems for wide band noise driven observation systems. At this stage of developments, the author expects an interesting interpretation in terms of relaxing functions for the non-degeneracy of the observation white noise in the Kalman's filtering model.

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