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New Iterative Method for Variational Inclusion and Fixed Point Problems

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***Abstract*—We introduce an iterative method for finding a common element of the set solutions of equilibrium problems, the set of solutions of a variational inclusion problems for an inverse-strongly monotone mapping and set-valued maximal monotone mapping, and the set of fixed points of a nonexpansive mapping in a real Hilbert space. Then, we prove a strong convergence theorem of the proposed method with suitable control conditions.**

***Index Terms*—Fixed point, variational inequality, optimization problem, nonexpansive mapping**

I. Introduction

THROUGHOUT this paper, we always assume that H be a real Hilbert space with inner product and norm, are denoted by $\langle \cdot,\cdot \rangle$ and $\|\cdot\|$, respectively and let K be a nonempty closed convex subset of H. Let G be a bifunction of $K \times K \to R,$ where R is the set of real numbers. The equilibrium problem for a bifunction $G: K \times K \to R$ is to find $u \in K$ such that

$$G(u,v) \geq 0, \forall\, v \in K. \qquad (1.1)$$

The set of solutions of (1.1) is denoted by EP(G). Numerous problems in Physics, optimization, and economics reduce to find a solution of (1.1). let $A: K \to H$ be a nonlinear map. The classical variational inequality which is denoted by VI(K,A) is to find $u \in K$ such that $\langle Au, v-u \rangle \geq 0, \forall v \in K.$ We have known from Blum and Oettli [1] that the equilibrium problem contains the fixed point problem, optimization problem, saddle point problem, variational inequality problem and Nash equilibrium problem as its special case. Given a mapping $T: K \to H$, Let $G(u,v) =$ $\langle Tu, v-u \rangle, \forall u, v \in K.$ Then $z \in EP(G)$ if and only if $\langle Tz,$ $v - z \rangle \geq 0, \forall\, v \in K,$ i.e., z is a solution of the variational inequality. A mapping S of K into itself is called nonexpansive if $\|Su - Sv\| \leq \|u - v\|, \forall u, v \in K.$ We denoted by F(S) the set of fixed points of S (see [4], [5]). A mapping A of K into H is called $\alpha-$ inverse-strongly monotone (see [3], [8]) if there exists a positive real number α such that $\langle Au - Av \rangle \geq \alpha \|Au - Av\|^2, \forall u,\ v \in K.$ Recall that a mapping $f: K \to K$ is said to be contractive with coefficient $\beta \in (0,1),$ if $\|f(u) - f(v)\| \leq \beta\, \|u - v\|, \forall u, v \in K.$ Let B be a strongly positive bounded linear operator on H: that is, there is a constant $\bar{\gamma} > 0$ with property $\langle Bx, x \rangle \geq \bar{\gamma} \|x\|^2, \forall x \in H.$ Let $A: H \to H$ be a single-valued nonlinear mapping and let $M: H \to 2^H$ be a set-valued mapping. We consider the variational inclusion, which is to find $u \in H$ such that

$$\theta \in A(u) + M(u), \qquad (1.2)$$

where θ is the zero vector in H. The set of solution of problem (1.2) is denote by I(A,M). It is known that (1.2) provides a convenient in the framework for the unified study of optimal solutions in many optimization related areas including mathematical programming, complementarity, variational inequalities, optimal control, mathematical economics, equilibria, and game theory (see [8] and the reference therein). If $M = \partial\delta_K,$ where K is a nonempty closed convex subset of H and $\delta_K : H \to [0,\infty]$ is the indicator function of K, then the variational inclusion problem (1.2) is equivalent to variational inequality problem. Recall the resolvent operator $J_{M,\varepsilon}$ associated with M and ε as $J_{M,\varepsilon}(u) = (I + \varepsilon M)^{-1}(u), \forall u \in H,$ where M is maximal monotone mapping and ε is a positive number. The resolvent operator $J_{M,\varepsilon}$ is single-valued, monotone and 1-inverse-strongly monotone, and that a solution of problem (1.2) is a fixed point of the operator $J_{M,\varepsilon}(I - \varepsilon A)$ for all $\varepsilon > 0,$ see for example [8]. Some methods have been proposed to solve the equilibrium problem, variational inequality and fixed point problem of nonexpansive mapping (see [2]-[4], [7], [9], [10], and the reference therein). Very recently, Jung [3] introduced a new general composite iterative scheme for finding a common point of the set of solutions of the variational inequality problem and the set of fixed point of a nonexpansive mapping in Hilbert space. Starting with $x_1 = x \in K,$

$$\begin{aligned} y_n &= \alpha_n(u + \gamma f(x_n)) + (I - \alpha_n(I + \mu B))SP_K(x_n - \lambda_n A x_n), \\ x_{n+1} &= (1-\beta_n)y_n + \beta_n SP_K(y_n - \lambda_n A y_n), n \geq 1. \end{aligned} \qquad (1.3)$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}, \{\lambda_n\},$ and $\{\beta_n\}$ of parameters, then the sequence $\{x_n\}$ converges strongly to $q \in F(S) \cap VI(K,A),$ which is a solution of the optimization problem:

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$$\min_{x\in F(S)\cap VI(K,A)} \tfrac{\mu}{2}\langle Bx,x\rangle+\tfrac{1}{2}\|x-u\|^2-h(x), \qquad (1.4)$$

where h is a potential function for γf.

In this paper motivated by the iterative scheme that proposed by Jung [3]. We will introduce a new iterative method for a common element of the set solution of equilibrium problem, variational inclusion and the set of fixed point of a nonexpansive mapping which will present in the main result section.

II. Preliminaries

Let K be a nonempty closed convex subset of a real Hilbert space H. It well known that H satisfies the Opial's condition (see [6]), that is, for any sequence $\{x_n\}$ with $\{x_n\}$ converges weakly to x (denote by $x_n \xrightarrow{w} x$), the inequality: $\liminf_{n\to\infty}\|x_n-x\|<\liminf_{n\to\infty}\|x_n-y\|$ holds for every $y\in H$ with $y\neq x$.

The following lemmas are useful for proving our theorem.

Lemma 2.1 (See [3].) In a real Hilbert space H, there holds the inequality

$$\|x+y\|^2\le\|x\|^2+2\langle y,x+y\rangle, \forall x,y\in H.$$

Lemma 2.2 (See [7].) Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma}>0$ and $0<\rho\le\|A\|^{-1}$. Then $\|I-\rho A\|\le 1-\rho\bar{\gamma}$.

Lemma 2.3 (See [4].) Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that

$a_{n+1}\le(1-\alpha_n)a_n+\delta_n,\ n\ge 0,$

where $\{\alpha_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence in R such that

(1) $\sum_{n=1}^{\infty}\alpha_n=\infty$;

(2) $\limsup_{n\to\infty}\frac{\delta_n}{\alpha_n}\le 0$ or $\sum_{n=1}^{\infty}|\delta_n|<\infty$.

Then $\lim_{n\to\infty}a_n=0$.

For solving the equilibrium problem for a bifunction $G:K\times K\to R$ where R is the set of real numbers, let us assume that G satisfies the following conditions:

(A1) $G(x,x)=0$ for all $x\in K$;

(A2) G is monotone, that is, $G(x,y)+G(y,x)\le 0$ for all $x,y\in K$;

(A3) for each $x,y,z\in K$, $\lim_{t\to 0}G(tz+(1-t)x,y)\le G(x,y)$;

(A4) for each $x\in K$, $y\mapsto G(x,y)$ is convex and lower semicontinuous.

Lemma 2.4 (see [8].) Let K be a convex closed subset of a Hilbert spaces H. Let $G:K\times K\to R$, is a bifunction satisfying (A1)-(A4). Let $\lambda>0$ and $x\in H$. Then. There exists $z\in K$ such that

$$G(z,y)+\tfrac{1}{\lambda}\langle y-z,z-x\rangle\ge 0,\forall y\in K.$$

Moreover, let $F_\lambda:H\to K$ be a mapping defined by

$$F_\lambda(x)=\left\{z\in K:G(z,y)+\tfrac{1}{\lambda}\langle y-z,z-x\rangle\ge 0,\forall y\in K\right\},$$

for all $x\in H$. Then, the following hold:

(1) F_λ is a single value;

(2) F_λ is firmly nonexpansive; that is, for any $x,y\in H$, $\|F_\lambda x-F_\lambda y\|^2\le\langle F_\lambda x-F_\lambda y,x-y\rangle$;

(3) $F(F_\lambda)=EP(G)$;

(4) EP(G) is closed and convex.

Lemma 2.5 (See [3].) Let C be a bounded nonempty closed convex subset of a real Hilbert space H, and let $g:C\to R\cup\{\infty\}$ be a proper lower semicontinuous differentiable convex function. If x^* is a solution to the minimization problem $g(x^*)=\inf_{x\in C}g(x)$, then $\langle g'(x),x-x^*\rangle\ge 0, x\in C$. Inparticular, if x^* solves the optimization problem

$$\min_{x\in C}\tfrac{\mu}{2}\langle Bx,x\rangle+\tfrac{1}{2}\|x-u\|^2-h(x),$$

then $\langle u+(\gamma f-(I+\mu B))x^*,x-x^*\rangle\le 0, x\in C$, where h is a potential function for γf.

III. Main Result

In this section, we prove a strong convergence theorem.

Theorem 3.1. Let K be a nonempty closed convex subset of a real Hilbert space H such that $K\pm K\subset K$, let $G:K\times K\to R$ is a bifunction satisfying (A1)-(A4), and $M:H\to 2^H$ be a maximal monotone mapping. Let A be an $\alpha-$ inverse-strongly monotone mapping of K into H and S a nonexpansive mappings of K into itself such that $\Omega:=F(S)\cap EP(G)\cap I(A,M)\neq\varnothing$. Let f be a contractive of K into itself with constant $\beta\in(0,1)$ and let B be a strongly positive bounded linear operator on K with constant $\bar{\gamma}\in(0,1)$. Assume that $\mu>0$ and $0<\gamma<(1+\mu)\bar{\gamma}/\beta$. Let $\{x_n\}$ be a sequence generated by $x_1=x\in K$,

$$\begin{aligned}
&F(u_n,y)+\tfrac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge 0,\forall y\in K,\\
&y_n=\alpha_n(u+\gamma f(x_n))+(I-\alpha_n(I+\mu B))SJ_{M,\varepsilon_n}(u_n-\varepsilon_n Au_n),\\
&x_{n+1}=(1-\beta_n)y_n+\beta_n SJ_{M,\varepsilon_n}(y_n-\varepsilon_n Ay_n), n\ge 1,
\end{aligned} \qquad (3.1)$$

where $u_n=F_{r_n}x_n$, $\{\alpha_n\}\subset[0,1),\{\varepsilon_n\}\subset[0,2\alpha],\{r_n\}\subset(r,\infty)$, $r>0$, and $\{\beta_n\}\subset[0,1]$ satisfy :

i) $\lim_{n\to\infty}\alpha_n=0$; $\sum_{n=1}^{\infty}\alpha_n=\infty$;

ii) $\beta_n\in[0,d)$ for all $n\ge 0$ and for some $d\in(0,1)$;

iii) $\varepsilon_n\in[a,b]$ for all $n\ge 0$ and for some a, b with $0<a<b<2\alpha$;

iv) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$, $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty$, $\sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty$, and $\sum_{n=1}^{\infty}|\varepsilon_{n+1}-\varepsilon_n|<\infty$.

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Then $\{x_n\}$ converges strongly to $z \in F(S) \cap EP(G) \cap I(A,M)$, which is a solution of the optimization problem

$$\min_{x \in F(S)\cap EP(G)\cap I(A,M)} \tfrac{\mu}{2}\langle Bx, x\rangle + \tfrac{1}{2}\|x-u\|^2 - h(x), \quad (3.2)$$

where h is a potential function for γf.

Proof. From the condition i), we may assume that $\alpha_n \le (1+\mu\|B\|)^{-1}$. Applying Lemma 2.2, we obtain $\|I-\alpha_n(I+\mu A))\| < 1-\alpha_n(1+\mu)\bar{\gamma}$. Let $v \in \Omega$. Since $u_n = F_{r_n}x_n$ we have $\|u_n - v\| = \|F_{r_n}x_n - F_{r_n}v\| \le \|x_n - v\|, \quad \forall n \in N$. Let $z_n = J_{M,\varepsilon_n}(u_n - \varepsilon_n u_n)$ and $v_n = J_{M,\varepsilon_n}(y_n - \varepsilon_n y_n), \ \forall n \in N$. As $I - \varepsilon_n A$ is nonexpansive and $v \in \Omega$, we have $\|z_n - v\| = \|J_{M,\varepsilon_n}(u_n - \varepsilon_n Au_n) - J_{M,\varepsilon_n}(v - \varepsilon_n Av)\| \le \|u_n - v\|, \forall\ n \in N$. Similarly, we have

$$\|v_n - v\| \le \|y_n - v\|, \forall n \in N. \quad (3.3)$$

Then we obtain

$$\|z_n - v\| \le \|x_n - v\|, \forall n \in N. \quad (3.4)$$

From the condition i) and (3.1), we have

$$\|y_n - Sz_n\| = \alpha_n\|u + \gamma f(x_n) - \varpi Sz_n\| \to 0, n \to \infty. \quad (3.5)$$

For $v \in \Omega$, and let $\varpi = (I + \mu B)$, we have

$$\begin{aligned}\|y_n - v\| &= \alpha_n\|u\| + \alpha_n\|\gamma f(x_n) - \varpi v\| + \|I - \alpha_n\varpi\|\|Sz_n - v\| \\ &\le (1-((1+\mu)\bar{\gamma} - \gamma\beta)\alpha_n)\|x_n - v\| \\ &\quad + ((1+\mu)\bar{\gamma} - \gamma\beta)\alpha_n \tfrac{\|\gamma f(v) - \varpi v\| + \|u\|}{(1+\mu)\bar{\gamma} - \gamma\beta}, \forall n \ge 1.\end{aligned}$$

Then we have

$$\|x_{n+1} - v\| \le \max\left\{\|x_n - v\|, \tfrac{\|\gamma f(v) - \varpi v\| + \|u\|}{(1+\mu)\bar{\gamma} - \gamma\beta}\right\}, \forall n \ge 1. \quad (3.6)$$

It follows from (3.6) and induction that $\|x_n - v\| \le \max\left\{\|x_1 - v\|, \tfrac{\|\gamma f(v) - \varpi v\| + \|u\|}{(1+\mu)\bar{\gamma} - \gamma\beta}\right\}, n \ge 1$. Hence $\{x_n\}$ is bounded, so are $\{u_n\}, \{y_n\}, \{f(x_n)\}, \{Sz_n\}, \{Sv_n\}, \{Ay_n\}, \{Au_n\}$, and $\{\varpi Sz_n\}$. Next we show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. We observe that

$$\begin{aligned}\|u_n - u_{n-1}\| &= \|F_{r_n}x_n - F_{r_{n-1}}x_{n-1}\| \\ &\le \|x_n - x_{n-1}\| + \tfrac{1}{r}|r_{n-1} - r_n|(\|u_{n-1}\| + \|x_{n-1}\|),\end{aligned} \quad (3.7)$$

$\exists r > 0, \forall n \in N$. Moreover, we can note that

$$\begin{aligned}&\|z_n - z_{n-1}\| \\ &= \|J_{M,\varepsilon_n}(u_n - \varepsilon_n Au_n) - J_{M,\varepsilon_{n-1}}(u_{n-1} - \varepsilon_{n-1}Au_{n-1})\| \\ &\le \|u_n - u_{n-1}\| + |\varepsilon_n - \varepsilon_{n-1}|\|Au_{n-1}\|.\end{aligned} \quad (3.8)$$

Similarly, we have

$$\|v_n - v_{n-1}\| \le \|y_n - y_{n-1}\| + |\varepsilon_n - \varepsilon_{n-1}|\|Ay_{n-1}\| \quad (3.9)$$

Using (3.7) and (3.8), we obtain

$$\begin{aligned}\|z_n - z_{n-1}\| &\le \|x_n - x_{n-1}\| + \tfrac{1}{r}|r_n - r_{n-1}|\left(\|u_{n-1}\| + \|x_{n-1}\|\right) \\ &\quad + |\varepsilon_n - \varepsilon_{n-1}|\|Au_{n-1}\|\end{aligned} \quad (3.10)$$

From (3.1) and (3.10), we have

$$\begin{aligned}\|y_n - y_{n-1}\| &\le |\alpha_n - \alpha_{n-1}|(\|u\| + \|\gamma f(x_{n-1})\| + \|\varpi\|\|Sz_{n-1}\|) \\ &\quad + (1-((1+\mu)\bar{\gamma} - \gamma\beta)\alpha_n)\|x_n - x_{n-1}\| \\ &\quad + \tfrac{1}{r}|r_n - r_{n-1}|\left(\|u_{n-1}\| + \|x_{n-1}\|\right) + |\varepsilon_{n+1} - \varepsilon_n|\|Au_{n-1}\|.\end{aligned} \quad (3.11)$$

It follows from (3.3)and (3.11), we obtain

$$\begin{aligned}\|x_{n+1} - x_n\| &\le (1-((1+\mu)\bar{\gamma} - \gamma\beta)\alpha_n)\|x_n - x_{n-1}\| \\ &\quad + G_1|\alpha_n - \alpha_{n-1}| + G_2|\varepsilon_n - \varepsilon_{n-1}| \\ &\quad + G_3|\beta_n - \beta_{n-1}| + G_4\tfrac{1}{r}|r_n - r_{n-1}|,\end{aligned} \quad (3.12)$$

where $G_1 = \sup\{\|u\| + \gamma\|f(x_n)\| + \|\varpi\|\|Sz_n\| : n \in N\}$, $G_2 = \sup\{\|Au_n\| + \|By_n\| : n \in N\}$, $G_3 = \sup\{\|Sv_n\| + \|y_n\| :\ n \in N\}$, and $G_4 = \sup\{\|u_n\| + \|x_n\| : n \in N\}$. Then, from the condition i) and iv), we have

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \quad (3.13)$$

By using the condition ii), we can show that

$$\|x_{n+1} - y_n\| \le \tfrac{d}{(1-d)}\left[\|x_{n+1} - x_n\| + \|Sz_n - y_n\|\right]. \quad (3.14)$$

Combining (3.5) and (3.13), we get the following

$$\lim_{n\to\infty}\|x_{n+1} - y_n\| = 0. \quad (3.15)$$

We can also get that

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \quad (3.16)$$

Next, we show $\lim_{n\to\infty}\|x_n - u_n\| = 0$. Since $\|u_n - v\|^2 = \tfrac{1}{2}\left(\|u_n - v\|^2 + \|x_n - v\|^2 - \|x_n - u_n\|^2\right)$, we have

$$\|u_n - v\|^2 \le \|x_n - v\|^2 - \|x_n - u_n\|^2. \quad (3.17)$$

It follows from (3.15) and using $\|z_n - v\| \le \|u_n - v\|, \forall n \in N$, we have

$$\|x_{n+1} - v\|^2 \le \alpha_n\|u + \gamma f(x_n) - \varpi v\|^2 + \|x_n - v\|^2$$

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$$-\left((1-\alpha_n(1+\mu)\bar{\gamma}\right)\|x_n-u_n\|^2 \\ +2\alpha_n\|u+\gamma f(x_n)-\varpi v\|\|z_n-v\|. \tag{3.18}$$

Then we also have

$$\begin{aligned}&\left((1-\alpha_n(1+\mu)\bar{\gamma}\right)\|x_n-u_n\|^2\\&\le \alpha_n\|u+\gamma f(x_n)-\varpi v\|^2+\|x_n-v\|^2\\&\quad-\|x_{n+1}-v\|^2+2\alpha_n\|u+\gamma f(x_n)-\varpi v\|\|z_n-v\|\\&\qquad\le \alpha_n\|u+\gamma f(x_n)-\varpi v\|^2\\&\qquad+\left(\|x_n-v\|+\|x_{n+1}-v\|\right)\|x_n-x_{n+1}\|\\&\qquad\quad+2\alpha_n\|u+\gamma f(x_n)-\varpi v\|\|z_n-v\|.\end{aligned} \tag{3.19}$$

By the condition i) and using (3.11), we have

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{3.20}$$

We note from (3.1) and the condition iii) that

$$\begin{aligned}\|y_n-p\|^2&\le \alpha_n\|u+\gamma f(x_n)-\varpi v\|^2+\|x_n-v\|^2\\&\quad+2\alpha_n\|u+\gamma f(x_n)-\varpi v\|\|z_n-v\|\\&\quad+(1-\alpha_n(1+\mu)\bar{\gamma})a(b-2\alpha)\|Au_n-Av\|^2.\end{aligned} \tag{3.21}$$

Hence, we obtain

$$\begin{aligned}&-(1-\alpha_n(1+\mu)\bar{\gamma})a(b-2\alpha)\|Au_n-Av\|^2\\&\le \alpha_n\|u+\gamma f(x_n)-\varpi v\|^2+\left(\|x_n-v\|+\|y_n-v\|\right)\\&\quad\|x_n-y_n\|+2\alpha_n\|u+\gamma f(x_n)-\varpi v\|\|z_n-v\|.\end{aligned} \tag{3.22}$$

Using (3.16), (3.22), and the condition i), we have

$$\lim_{n\to\infty}\|Au_n-Av\|=0. \tag{3.23}$$

Furthermore, applying Lemma 2.1, we obtain

$$\begin{aligned}\|z_n-p\|^2&\le\|x_n-p\|^2-\|u_n-z_n\|^2\\&\quad+2\lambda_n\langle u_n-z_n, Au_n-Av\rangle-\varepsilon_n{}^2\|Au_n-Av\|^2.\end{aligned} \tag{3.24}$$

Then we obtain

$$\begin{aligned}&(1-\alpha_n(1+\mu)\bar{\gamma})\|u_n-z_n\|^2\\&\le \alpha_n\|u+\gamma f(x_n)-\varpi v\|^2+\left(\|x_n-v\|+\|y_n-v\|\right)\|x_n-y_n\|\\&\quad+2\left(1-\alpha_n(1+\mu)\bar{\gamma}\right)\varepsilon_n\langle u_n-z_n, Au_n-Av\rangle\\&\quad-(1-\alpha_n(1+\mu)\bar{\gamma})c(d-2\alpha)\|Au_n-Av\|^2\\&\quad+2\alpha_n\|u+\gamma f(x_n)-\varpi v\|\|z_n-v\|.\end{aligned} \tag{3.25}$$

Using (3.16), (3.23), (3.25), and the condition i), we have

$$\lim_{n\to\infty}\|u_n-z_n\|=0. \tag{3.26}$$

Moreover, by the condition i) and (3.26), we have

$$\lim_{n\to\infty}\|y_n-Su_n\|=0. \tag{3.27}$$

It follows from $\|y_n-u_n\|\le\|y_n-x_n\|+\|x_n-u_n\|$ and using (3.16) and (3.20), we obtain

$$\lim_{n\to\infty}\|y_n-u_n\|=0. \tag{3.28}$$

Using (3.27), (3.28), and this inequality $\|Su_n-u_n\|\le \|Su_n-y_n\|+\|y_n-u_n\|$, we have

$$\lim_{n\to\infty}\|Su_n-u_n\|=0. \tag{3.29}$$

Next, we show that $\limsup_{n\to\infty}\langle u+(\gamma f-\varpi)\tilde{x}, y_n-\tilde{x}\rangle\le 0$, where $\tilde{x}$ is a solution of (3.2). To show this inequality, we first show that $\limsup_{n\to\infty}\langle u+(\gamma f-\varpi)\tilde{x}, Su_n-\tilde{x}\rangle\le 0$. Since $\{u_n\}$ is bounded, we choose a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $\limsup_{i\to\infty}\langle u+(\gamma f-\varpi)\tilde{x}, Su_{n_i}-\tilde{x}\rangle=\limsup_{i\to\infty}\langle u+(\gamma f-\varpi)\tilde{x}, Su_n-\tilde{x}\rangle$ Without loss of generality, we can assume that $u_{n_i}\xrightarrow{w} z$. From (3.24), we have $y_{n_i}\xrightarrow{w} z$. It follows by (3.1) and (A2) that $\left\langle y-u_{n_i}, \frac{u_{n_i}-u_{n_i}}{r_{n_i}}\right\rangle\ge G(y,u_{n_i})$.

Since $\frac{u_{n_i}-u_{n_i}}{r_{n_i}}\to 0$ (as $i\to\infty$) and $u_{n_i}\xrightarrow{w} z$, it follows by (A4) that $0\ge G(y,z)$ for all $y\in H$. For t with $0<t\le 1$ and $y\in H$, let $y_t=ty+(1-t)z$. Since $y\in H$ and $z\in H$, we have $y_t\in H$ and hence $G(y_t,z)\le 0$. From (A1) and (A4), we have $0=G(y_t,y_t)\le tG(y_t,y)+(1-t)G(y_t,z)\le t(y_t,y)$, and $0\le G(y_t,y)$. From (A3), we have $0\le G(z,y)$ for all $y\in H$ and Lemma 2.4, we have $z\in EP(G)$. By the same argument as in proof of Theorem 3.1 of Plubtieng and Sriprad [8], we have $z\in F(S)\cap I(A,M)$. Then we have $z\in\Omega$. It follows from Lemma 2.5 and (3.29) that $\limsup_{n\to\infty}\langle u+(\gamma f-\varpi)\tilde{x}, Su_n-\tilde{x}\rangle=\limsup_{i\to\infty}\langle u+(\gamma f-\varpi)\tilde{x}, Su_{n_i}-\tilde{x}\rangle=\langle u+(\gamma f-\varpi)\tilde{x}, u_{n_i}-\tilde{x}\rangle=\langle u+(\gamma f-\varpi)\tilde{x}, z-\tilde{x}\rangle\le 0$. We can note that $\limsup_{n\to\infty}\langle u+(\gamma f-\varpi)\tilde{x}, y_n-Su_n\rangle+\limsup_{n\to\infty}\langle u+(\gamma f-\varpi)\tilde{x}, Su_n-\tilde{x}\rangle+\limsup_{n\to\infty}\langle u+(\gamma f-\varpi)\tilde{x}, Su_n-\tilde{x}\rangle\le \limsup_{n\to\infty}\|u+(\gamma f-\varpi)\tilde{x}\|\|y_n-Su_n\|+\limsup_{n\to\infty}\langle u+(\gamma f-\varpi)\tilde{x}, Su_n-\tilde{x}\rangle$. It follows from (3.27) and (3.29), we obtain that $\limsup_{n\to\infty}\langle u+(\gamma f-(I+\mu B))\tilde{x}, y_n-\tilde{x}\rangle\le 0$. Finally, we show that $\lim_{n\to\infty}\|x_n-\tilde{x}\|=0$, where $\tilde{x}$ is a unique solution of (3.2). Using Lemma 2.1, we can note that

$$\|x_{n+1}-\tilde{x}\|^2\le(1-2\alpha_n((1+\mu)\bar{\gamma}-\gamma\beta))\|x_n-\tilde{x}\|^2$$

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$$
\begin{aligned}
&+\alpha_n{}^2((1+\mu)\bar{\gamma})^2\|x_n-\tilde{x}\|^2 \\
&+2\alpha_n\gamma\beta\|x_n-\tilde{x}\|\|y_n-x_n\| \\
&+2\alpha_n\langle u+(\gamma f-\varpi)\tilde{x}, y_n-\tilde{x}\rangle.
\end{aligned} \tag{3.30}
$$

Applying Lemma 2.3 to (3.30), we have $\lim_{n\to\infty}\|x_n-\tilde{x}\| = 0$, that is, $\{x_n\}$ converges strongly to $\tilde{x}$. This completes the proof.

IV. Conclusion

We proposed an iterative method and proved that the sequence of the proposed iterative method converges to a point of solutions of above three sets. This iterative method and convergence theorem are improved and extended from Theorem 3.1 of Jung [3].

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