Fekete-Szegö Inequality for Subclasses of a New Class of Analytic Functions

Gurmeet Singh, Member, IAENG, M. S. Saroa, Gagandeep Singh

Abstract: We introduce a new class of analytic functions and its subclasses and obtain sharp upper bounds of the functional $|a_3 - \mu a_2^2|$ for the analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, |z| < 1$ belonging to these classes.
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I. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disc $\mathbb{E} = \{z: |z| < 1|\}$. Let $\mathcal{S}$ be the class of functions of the form (1.1), which are analytic univalent in $\mathbb{E}$.

In 1916, Bieber Bach ([8], [9]) proved that $|a_2| \leq 2$ for the functions $f(z) \in \mathcal{S}$. In 1923, Löwner [7] proved that $|a_3| \leq 3$ for the functions $f(z) \in \mathcal{S}$..

With the known estimates $|a_2| \leq 2$ and $|a_3| \leq 3$, it was natural to seek some relation between a_3 and ${a_2}^2$ for the class $\mathcal{S}$, Fekete and Szegö [10] used Löwner's method to prove the following well known result for the class $\mathcal{S}$.

Let $f(z) \in \mathcal{S}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, \text{if } \mu \leq 0; \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right), \text{if } 0 \leq \mu \leq 1; \\ 4\mu - 3, \text{if} \mu \geq 1. \end{cases} \tag{1.2}$$

The inequality (1.2) plays a very important role in determining estimates of higher coefficients for some sub classes $\mathcal{S}$ (Chhichra[1], Babalola[7]).

Let us define some subclasses of $\mathcal{S}$.

We denote by S*, the class of univalent starlike functions $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$ satisfying the condition

$$\mathrm{Re}\left(\frac{zg(z)}{g(z)}\right) > 0, z \in \mathbb{E}. \tag{1.3}$$

We denote by $\mathcal{K}$, the class of univalent convex functions $h(z) = z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{A}$ satisfying the condition

$$\mathrm{Re}\frac{((zh'(z))}{h'(z)} > 0, z \in \mathbb{E}. \tag{1.4}$$

Gurmeet Singh, M.Phil, is with Khalsa College, Patiala (meetgur111@gmail.com, +91-9041404543), India
M. S. Saroa, Ph.D., is with Maharishi Markandeshawar University, Mullana, India.
Gagandeep Singh, Ph.D., is with MSK Girls College, Bharowal, Tarantaran, India.

Gurmeet Singh, Saroa M. S. and Mehrok, B. S. [4] have introduced the class of Inverse Starlike functions as the functions $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$ and satisfying the condition

$$\mathrm{Re}\left(\frac{zf(z)}{2\int_0^z f(z)dz}\right) > 0, z \in \mathbb{E} \text{ i.e. } \frac{zf(z)}{2\int_0^z f(z)dz} \prec \frac{1+z}{1-z} \tag{1.5}$$

[4] denoted this class by $(S^*)^{-1}$.

The subclass of $(S^*)^{-1}$ consisting of the functions

$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$ and satisfying the condition

$$\frac{zf(z)}{2\int_0^z f(z)dz} \prec \frac{1 + Az}{1 + Bz}; -1 \leq B \leq A \leq 1 \tag{1.6}$$

is denoted by $(S^*)^{-1}[A, B]$ (See [4]).

Symbol $\prec$ stands for subordination, which we define as follows:

p-valent functions: A function $f(z) \in \mathcal{A}_p$ is said to be a p-valent function in E if it assumes no value more than p times in E.

The class of functions $f(z) \in \mathcal{A}_p$ satisfying the condition $\frac{zf(z)}{(p+1)\int_0^z f(z)dz} \prec \frac{1+z}{1-z}$ is denoted by $\left({S_p}^*\right)^{-1}$.(See [4])

These functions were called p- valently inverse starlike functions. In this paper, We will deal with $\left({S_p}^*\right)^{-1}[A, B]$, the subclass of $\left({S_p}^*\right)^{-1}$ consisting of the functionsf$(z) \in \mathcal{A}_p$ and satisfying the condition

$$\frac{zf(z)}{(p+1)\int_0^z f(z)dz} \prec \frac{1+Az}{1+Bz}; -1 \leq B \leq A \leq 1.$$

We will also deal with $\left({S_p}^*\right)^{-1}[A, B; \delta]$, the subclass of $\left({S_p}^*\right)^{-1}[A, B]$ consisting of the functions $f(z) \in \mathcal{A}$ and satisfying the condition

$$\frac{zf(z)}{(p+1)\int_0^z f(z)dz} \prec \left(\frac{1 + Az}{1 + Bz}\right)^{\delta}; -1 \leq B \leq A \leq 1; \delta > 0.$$

We will establish Fekete-Szegö inequality for these classes.

Principle of subordination: Let f(z) and F(z) be two functions analytic in $\mathbb{E}$. Then f(z) is called subordinate to F(z)

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in $\mathbb{E}$ if there exists a function $w(z)$ analytic in $\mathbb{E}$ satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z))$; $z \in \mathbb{E}$ and we write $f(z) \prec F(z)$.

By $\mathcal{U}$, we denote the class of analytic bounded functions of the form $w(z) = \sum_{n=1}^{\infty} d_n z^n, w(0) = 0, |w(z)| < 1.$ (1.7)

It is known that $|d_1| \leq 1, |d_2| \leq 1 - |d_1|^2$.

II. Main Results

(I) Fekete–Sszegö problem for the functions belonging to the class $(S_p{}^*)^{-1}[A, B]$: In this section we obtain the Fekete–Szegö inequality for functions in a general class $(S_p{}^*)^{-1}[A, B]$ of functions, which we defined above in introduction part. Also we give applications of our results in certain corollaries of the theorem.

Theorem 2.1: If $f(z) \in (S_p{}^*)^{-1}[A, B]$, then

$$\frac{1}{A-B}\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \begin{cases} (p+2)^2(A-B)\left(\frac{(p+3)\{(p+1)(A-B)-B\}}{2(p+2)^2(A-B)} - \mu\right), & \text{if } \mu \leq \frac{(p+3)\{(p+1)(A-B)-B\}}{2(p+2)^2(A-B)} \quad (2.1) \\ \frac{p+3}{2}, & \text{if } \frac{(p+3)[\{(p+1)(A-B)-B\}-1]}{2(p+2)^2(A-B)} \leq \mu \leq \frac{p+3}{2}\frac{[\{(p+1)(A-B)-B\}+1]}{(p+2)^2(A-B)} \quad (2.2) \\ (p+2)^2(A-B)\left(\mu - \frac{(p+3)\{(p+1)(A-B)-B\}}{2(p+2)^2(A-B)}\right), & \text{if } \mu \geq \frac{p+3}{2}\frac{[\{(p+1)(A-B)-B\}+1]}{(p+2)^2(A-B)} \quad (2.3) \end{cases}$$

The results are sharp.

Proof: By definition of $(S_p{}^*)^{-1}[A, B]$, we have

$$\frac{zf(z)}{(p+1)\int_0^z f(z)dz} = \left(\frac{1+Aw(z)}{1+Bw(z)}\right); -1 \leq B \leq A \leq 1, \quad (2.4)$$

Expanding (2.4), we have

$$\left(1 + a_{p+1}z + a_{p+2}z^2 + ---\right) = \left(1 + \frac{p+1}{p+2}a_{p+1}z + \frac{p+1}{p+3}a_{p+2}z^2 + ---\right)\left(1 + (A-B)c_1 z + (A-B)(c_2 - Bc_1^2)z^2 + ---\right) \quad (2.5)$$

Identifying terms in (2.5), we get

$a_{p+1} = (p+2)(A-B)c_1$ and

$$a_{p+2} = \frac{(p+3)(A-B)}{2}\left[c_2 + \{(p+1)(A-B) - B\}c_1^2\right] \quad (2.6)$$

Using (2.5) and (2.6), we obtain

$$a_{p+2} - \mu a_{p+1}^2 = \frac{(p+3)(A-B)}{2} c_2 + (p+2)^2(A-B)^2\left[\frac{(p+3)\{(p+1)(A-B)-B\}}{2(p+2)^2(A-B)} - \mu\right]c_1^2$$

$$\frac{1}{A-B}\left(a_{p+2} - \mu a_{p+1}^2\right) = \frac{(p+3)}{2} c_2 + (p+2)^2(A-B)\left[\frac{(p+3)\{(p+1)(A-B)-B\}}{2(p+2)^2(A-B)} - \mu\right]c_1^2$$

This leads to

$$\frac{1}{A-B}\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{p+3}{2}|c_2| + (+2)^2(-B)\left|\frac{(p+3)\{(p+1)(A-B)-B\}}{2(p+2)^2(A-B)} - \mu\right||c_1|^2$$

$$\leq \frac{p+3}{2} + (p+2)^2(A-B)\left[\left|\frac{(p+3)\{(p+1)(A-B)-B\}}{2(p+2)^2(A-B)} - \mu\right| - \frac{p+3}{2(p+2)^2(A-B)}\right]|c_1|^2 \quad (2.7)$$

Case I: $\mu \leq \frac{(p+3)\{(p+1)(A-B)-B\}}{2(p+2)^2(A-B)}$, we get from (2.7)

$$\frac{1}{A-B}\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{p+3}{2} + (p+2)^2(A-B)\left[\frac{p+3}{2}\frac{[\{(p+1)(A-B)-B\}-1]}{(p+2)^2(A-B)} - \mu\right]|c_1|^2 \quad (2.8)$$

Subcase I(a): $\mu \leq \frac{(p+3)[\{(p+1)(A-B)-B\}-1]}{2(p+2)^2(A-B)}$. From equation (2.8), we get

$$\frac{1}{A-B}\left|a_{p+2} - \mu a_{p+1}^2\right| \leq (p+2)^2(A-B)\left(\frac{(p+3)\{(p+1)(A-B)-B\}}{2(p+2)^2(A-B)} - \mu\right). \quad (2.9)$$

Subcase I(b): $\mu \geq \frac{(p+3)[\{(p+1)(A-B)-B\}-1]}{2(p+2)^2(A-B)}$. From equation (2.8), we get $\frac{1}{A-B}\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{p+3}{2}$. (2.10)

Case II: $\mu \geq \frac{(p+3)\{(p+1)(A-B)-B\}}{2(p+2)^2(A-B)}$, we get from (2.7)

$$\frac{1}{A-B}\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{p+3}{2} + (p+2)^2(A-B)\left[\mu - \frac{p+3}{2}\frac{[\{(p+1)(A-B)-B\}+1]}{(p+2)^2(A-B)}\right]|c_1|^2 \quad (2.11)$$

Subcase II(a): $\mu \leq \frac{p+3}{2}\frac{[\{(p+1)(A-B)-B\}+1]}{(p+2)^2(A-B)}$. From equation (2.11), we get $\frac{1}{A-B}\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{p+3}{2}$. (2.12)

Combining subcase I(a) and subcase II(b), we get

$$\frac{1}{A-B}\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{p+3}{2}, \text{if } \frac{(p+3)[\{(p+1)(A-B)-B\}-1]}{2(p+2)^2(A-B)} \leq \mu \leq \frac{p+3}{2}\frac{[\{(p+1)(A-B)-B\}+1]}{(p+2)^2(A-B)}. \quad (2.13)$$

Subcase II(b): $\mu \geq \frac{p+3}{2}\frac{[\{(p+1)(A-B)-B\}+1]}{(p+2)^2(A-B)}$. From equation (2.11), we get $\frac{1}{A-B}\left|a_{p+2} - \mu a_{p+1}^2\right| \leq (p+2)^2(A-B)\left(\mu - \frac{(p+3)\{(p+1)(A-B)-B\}}{2(p+2)^2(A-B)}\right)$. (2.14)

This completes the theorem. The results are sharp.

Extremal function for first and third inequality is

$$f_1(z) = (p+1)z^p(1 + Az)(1 + Bz)^{\frac{(p+1)(A-B)-B}{B}}$$

Extremal function for second inequality is

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$f_2(z) = (p+1)z^p(1+Az^2)(1+Bz^2)^{\frac{(p+1)(A-B)-2B}{2B}}$

Corollary 2.2: Putting $A = 1, B = -1$ in Theorem 2.1, we get

$\frac{1}{2}|a_{p+2} - \mu a_{p+1}^2| \le$

$$\begin{cases} 2(p+2)^2\left(\frac{(p+3)(2p+3)}{4(p+2)^2} - \mu\right), \text{if } \mu \le \frac{(p+3)(2p+3)}{4(p+2)^2}; \\ \frac{p+3}{2}, \text{if } \frac{(p+3)(p+1)}{2(p+2)^2} \le \mu \le \frac{p+3}{2(p+2)}; \\ 2(p+2)^2\left(\mu - \frac{(p+3)(2p+3)}{4(p+2)^2}\right), \text{if } \mu \ge \frac{p+3}{2(p+2)} \end{cases}$$

, which are the required results for the class $(S_p{}^*)^{-1}$ (See [4].

Corollary 2.3: Putting $p = 1$ in Theorem 2.1, we get

$\frac{1}{A-B}|a_3 - \mu a_2^2| \le$

$$\begin{cases} 9(A-B)\left(\frac{2(2A-3B)}{9(A-B)} - \mu\right), \text{if } \mu \le \frac{2(2A-3B)}{9(A-B)}; \\ 2, \text{if } \frac{2(2A-3B)}{9(A-B)} \le \mu \le \frac{2(2A-3B+1)}{9(A-B)}; \\ 9(A-B)\left(\mu - \frac{2(2A-3B)}{9(A-B)}\right), \text{if } \mu \ge \frac{2(2A-3B+1)}{9(A-B)} \end{cases}$$

, which are the required results for the class $(S^*)^{-1}[A, B]$ (See [4].

Corollary 2.4: Putting $A = 1, B = -1, p = 1$ in Theorem 2.1, we get

$$\frac{1}{4}|a_3 - \mu a_2^2| \le \begin{cases} (5 - 9\mu), if\ \mu \le \frac{4}{9}; \\ 1, if\ \frac{4}{9} \le \mu \le \frac{2}{3}; \\ (9\mu - 5), if\ \mu \ge \frac{2}{3}. \end{cases}$$

, which are the required results for the class $(S^*)^{-1}$. (See [4]).

(II) Fekete–Sszegö problem for the functions in the class $(S_p{}^*)^{-1}[\delta]$: In this section we obtain the Fekete–Szegö inequality for functions in a general class $(S_p{}^*)^{-1}[\delta]$ of functions, which we defined above in introduction part. Also we give certain corollaries of the theorem.

Theorem 2.5: If $(S_p{}^*)^{-1}[\delta]$, then

$\frac{1}{2\delta}|a_{p+2} - \mu a_{p+1}^2| \le$

$$\begin{cases} \delta\left\{\frac{p+3}{2}(2p+3) - 2\mu(p+2)^2\right\}, \\ if\ \mu \le \frac{(p+1)(p+3)}{2(p+2)^2}, & (2.15) \\ \frac{p+3}{2}, \\ if\ \frac{(p+1)(p+3)}{2(p+2)^2} \le \mu \le \frac{(p+3)(2p\delta+3\delta-3)}{4\delta(p+2)^2}; & (2.16) \\ \delta\left\{2\mu(p+2)^2 - \frac{p+3}{2}(2p+3)\right\}, \\ if\ \mu \ge \frac{(p+3)(2p\delta+3\delta-3)}{4\delta(p+2)^2}. & (2.17) \end{cases}$$

The results are sharp.

Proof: We have

$$\frac{zf(z)}{(p+1)\int_0^z f(z)dz} = \left(\frac{1+w(z)}{1-w(z)}\right)^\delta; \delta > 0 \quad (2.18)$$

Expanding we have

$$(1 + a_{p+1}z + a_{p+2}z^2 + ---) = (1 + \frac{p+1}{p+2}a_{p+1}z + \frac{p+1}{p+3}a_{p+2}z^2 + ---)(1 + 2\delta c_1 z + 2\delta[c_2 + \delta c_1{}^2]z^2 --)$$

Identifying terms, we get

$a_{p+1} = 2\delta(p+2)c_1$ and

$$a_{p+2} = (p+3)\delta c_2 + (p+3)(2p+3)\delta^2 c_1^2 \quad (2.19)$$

Using (2.19), we obtain

$$a_{p+2} - \mu a_{p+1}^2 = (p+3)\delta c_2 + (p+3)(2p+3)\delta^2 c_1^2 - \mu(2\delta(p+2)c_1)^2$$

This gives

$$a_{p+2} - \mu a_{p+1}^2 = (p+3)\delta c_2 + \delta^2\{(p+3)(2p+3) - 4\mu(p+2)^2\}c_1^2$$

This gives

$$\frac{1}{2\delta}|a_{p+2} - \mu a_{p+1}^2| \le \frac{p+3}{2}|c_2| + \delta\left|\frac{p+3}{2}(2p+3) - 2\mu(p+2)^2\right| |c_1^2|$$

$$\le \frac{p+3}{2} + \left[\delta\left|\frac{p+3}{2}(2p+3) - 2\mu(p+2)^2\right| - \frac{p+3}{2}\right] |c_1^2| \quad (2.20)$$

Case I: $\mu \le \frac{\frac{p+3}{2}(2p+3)}{2(p+2)^2}$, we get from (2.20) that

$$\frac{1}{2\delta}|a_{p+2} - \mu a_{p+1}^2| \le \frac{p+3}{2} + \delta[(p+1)(p+3) - 2\mu(p+2)^2] |c_1^2| \quad (2.21)$$

Subcase I(a): $\mu \le \frac{(p+1)(p+3)}{2(p+2)^2}$, Using $|c_1^2| \le 1$, (2.21) takes the form

$$\frac{1}{2\delta}|a_{p+2} - \mu a_{p+1}^2| \le \delta\left\{\frac{p+3}{2}(2p+3) - 2\mu(p+2)^2\right\} \quad (2.22)$$

Subcase I(b): $\mu \ge \frac{(p+1)(p+3)}{2(p+2)^2}$. (2.21) takes the form

$$\frac{1}{2\delta}|a_{p+2} - \mu a_{p+1}^2| \le \frac{p+3}{2}. \quad (2.23)$$

Case II: $\mu \ge \frac{\frac{p+3}{2}(2p+3)}{2(p+2)^2}$, we get from (2.20) that

$$\frac{1}{2\delta}|a_{p+2} - \mu a_{p+1}^2| \le \frac{p+3}{2} + \left[2\delta(p+2)^2\mu - \frac{p+3}{2}(2p\delta + 3\delta - 3)\right] |c_1^2| \quad (2.24)$$

Subcase II (a): $\mu \le \frac{(p+3)(2p\delta+3\delta-3)}{4\delta(p+2)^2}$. (2.24) takes the form

$$\frac{1}{2\delta}|a_{p+2} - \mu a_{p+1}^2| \le \frac{p+3}{2}. \quad (2.25)$$

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Subcase I(b): $\mu \geq \frac{(p+3)(2p\delta+3\delta-3)}{4\delta(p+2)^2}$. Using $|c_1^2| \leq 1$, (2.24) takes the form

$$\frac{1}{2\delta}\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \delta\left\{2\mu(p+2)^2 - \frac{p+3}{2}\,(2p+3)\right\} \quad (2.26)$$

This completes the theorem. The results are sharp.

Corollary 2.6: Putting $\delta = 1$ in Theorem 2.5, we get

$$\frac{1}{4}\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \begin{cases} (p+2)^2\left(\frac{(p+3)(2p+3)}{4(p+2)^2} - \mu\right), if\ \mu \leq \frac{(p+3)(2p+3)}{4(p+2)^2}; \\ \frac{p+3}{4}, if\ \frac{(p+3)(p+1)}{2(p+2)^2} \leq \mu \leq \frac{p+3}{2(p+2)}; \\ (p+2)^2\left(\mu - \frac{(p+3)(2p+3)}{4(p+2)^2}\right), if\ \mu \geq \frac{p+3}{2(p+2)} \end{cases}$$

, which are the required results for the class $\left({S_p}^*\right)^{-1}$ (See [4].

Corollary 2.7: Putting $p = 1$, in Theorem 2.5, we get

$$\frac{1}{4\delta}\left|a_3 - \mu a_2^2\right| \leq \begin{cases} (5\delta+1) - 9\mu\delta, if\ \mu \leq \frac{(5\delta-1)}{9\delta}, \\ 2, if\ \frac{(5\delta-1)}{9\delta} \leq \mu \leq \frac{(5\delta-3)}{9\delta}; \\ 9\mu\delta - (5\delta+1), if\ \mu \geq \frac{(5\delta-3)}{9\delta}. \end{cases}$$

, which are the required results for the class $(S^*)^{-1}[\delta]$. (See [4])

Corollary 2.8: Putting $\delta = 1, p = 1$, in Theorem 2.5, we get

$$\frac{1}{4}\left|a_3 - \mu a_2^2\right| \leq \begin{cases} (5-9\mu), if\ \mu \leq \frac{4}{9}; \\ 1, if\ \frac{4}{9} \leq \mu \leq \frac{2}{3}; \\ (9\mu-5), if\ \mu \geq \frac{2}{3}. \end{cases}$$

, which are the required results for the class $(S^*)^{-1}$. (See [4])

(III) Fekete–Sszegö problem for the functions in the class $\left({S_p}^*\right)^{-1}[A,B;\delta]$: In this section we obtain the Fekete–Szegö inequality for functions in the class $\left({S_p}^*\right)^{-1}[A,B;\delta]$ of functions, which we defined above in introduction part. Also, we give certain corollaries of the theorem.

Theorem 2.9: If $\left({S_p}^*\right)^{-1}[A,B;\delta]$, then

$$\frac{1}{\delta(A-B)}\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \begin{cases} \frac{p+3}{2}\left\{\left(p\delta + \frac{3\delta}{2} - \frac{1}{2}\right)(A-B) - B\right\} - \mu\delta(p+2)^2(A-B), \\ \quad if\ \mu \leq \frac{\frac{p+3}{2}\left\{\left(p\delta+\frac{3\delta}{2}-\frac{1}{2}\right)(A-B)-(1+B)\right\}}{\delta(p+2)^2(A-B)} \quad (2.27), \\ \frac{p+3}{2}, \\ if\ \frac{\frac{p+3}{2}\left\{\left(p\delta+\frac{3\delta}{2}-\frac{1}{2}\right)(A-B)-(1+B)\right\}}{\delta(p+2)^2(A-B)} \leq \mu \leq \frac{\frac{p+3}{2}\left\{\left(p\delta+\frac{3\delta}{2}-\frac{1}{2}\right)(A-B)-(1-B)\right\}}{\delta(p+2)^2(A-B)}; \quad (2.28) \\ \mu\delta(p+2)^2(A-B) - \frac{p+3}{2}\left\{\left(p\delta + \frac{3\delta}{2} - \frac{1}{2}\right)(A-B) - B\right\}, \\ \quad if\ \mu \geq \frac{2\left\{\frac{5\delta-1}{2}(A-B)-(1-B)\right\}}{9\delta(A-B)} \quad (2.29) \end{cases}$$

The results are sharp.

Proof: We have

$$\frac{zf(z)}{(p+1)\int_0^z f(z)dz} = \left(\frac{1+Aw(z)}{1+Bw(z)}\right)^{\delta};\ \delta > 0 \qquad (2.30)$$

Expanding we have

$$\left(1 + a_{p+1}z + a_{p+2}z^2 + - - -\right) = \left(1 + \frac{p+1}{p+2}a_{p+1}z + \frac{p+1}{p+3}a_{p+2}z^2 + - - -\right)\left(1 + \delta(A-B)\,c_1 z + \delta(A-B)\left[c_2 + \left\{\frac{(\delta-1)}{2}(A-B) - B\right\}\,{c_1}^2\right]z^2 - -\right)$$

Identifying terms, we get

$$a_{p+1} = \delta(p+2)(A-B)c_1 \text{ and } a_{p+2} = \frac{p+3}{2}\,\delta(A-B)c_2 + \frac{p+3}{2}\,\delta(A-B)\left\{\left(p\delta + \frac{3\delta}{2} - \frac{1}{2}\right)(A-B) - B\right\}\,c_1^2 \qquad (2.31)$$

Using (2.31), we obtain

$$a_{p+2} - \mu a_{p+1}^2 = \frac{p+3}{2}\,\delta(A-B)c_2 + \frac{p+3}{2}\,\delta(A-B)\left\{\left(p\delta + \frac{3\delta}{2} - \frac{1}{2}\right)(A-B) - B\right\}\,c_1^2 - \mu(\delta(p+2)(A-B)c_1)^2$$

This gives

$$a_{p+2} - \mu a_{p+1}^2 = \frac{p+3}{2}\,\delta(A-B)c_2 + \delta(A-B)\left[\frac{p+3}{2}\left\{\left(p\delta + \frac{3\delta}{2} - \frac{1}{2}\right)(A-B) - B\right\} - \mu\delta(p+2)^2(A-B)\right]c_1^2$$

This gives

$$\frac{1}{\delta(A-B)}\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{p+3}{2}|c_2| + \left|\frac{p+3}{2}\left\{\left(p\delta + \frac{3\delta}{2} - \frac{1}{2}\right)(A-B) - B\right\} - \mu\delta(p+2)^2(A-B)\right|\,\left|c_1^2\right|$$

$$\leq \frac{p+3}{2} + \left[\left|\frac{p+3}{2}\left\{\left(p\delta + \frac{3\delta}{2} - \frac{1}{2}\right)(A-B) - B\right\} - \mu\delta(p+2)^2(A-B)\right| - \frac{p+3}{2}\right]\,\left|c_1^2\right| \qquad (2.32)$$

Case I: $\mu \leq \frac{\frac{p+3}{2}\left\{\left(p\delta+\frac{3\delta}{2}-\frac{1}{2}\right)(A-B)-B\right\}}{\delta(p+2)^2(A-B)}$, we get from (2.32) that

$$\frac{1}{\delta(A-B)}\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{p+3}{2} + \left[\frac{p+3}{2}\left\{\left(p\delta + \frac{3\delta}{2} - \frac{1}{2}\right)(A-B) - (1+B)\right\} - \mu\delta(p+2)^2(A-B)\right]\,\left|c_1^2\right| \qquad (2.33)$$

Subcase I(a): $\mu \leq \frac{\frac{p+3}{2}\left\{\left(p\delta+\frac{3\delta}{2}-\frac{1}{2}\right)(A-B)-(1+B)\right\}}{\delta(p+2)^2(A-B)}$, Using $|c_1^2| \leq 1$, (2.33) takes the form

$$\frac{1}{\delta(A-B)}\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{p+3}{2}\left\{\left(p\delta + \frac{3\delta}{2} - \frac{1}{2}\right)(A-B) - B\right\} - \mu\delta(p+2)^2(A-B) \qquad (2.34)$$

Subcase I(b): $\mu \geq \frac{\frac{p+3}{2}\left\{\left(p\delta+\frac{3\delta}{2}-\frac{1}{2}\right)(A-B)-(1+B)\right\}}{\delta(p+2)^2(A-B)}$. (2.33) takes the form

$$\frac{1}{\delta(A-B)}\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{p+3}{2}. \qquad (2.35)$$

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Case II: $\mu \geq \frac{\frac{p+3}{2}\left\{\left(p\delta+\frac{3\delta}{2}-\frac{1}{2}\right)(A-B)-B\right\}}{\delta(p+2)^2(A-B)}$, we get from (2.32) that

$$\frac{1}{\delta(A-B)}\left|a_{p+2}-\mu a_{p+1}^2\right| \leq \frac{p+3}{2}+\left[\delta(p+2)^2(A-B)\mu-\frac{p+3}{2}\left\{\left(p\delta+\frac{3\delta}{2}-\frac{1}{2}\right)(A-B)-(1-B)\right\}\right]\left|c_1^2\right| \quad (2.36)$$

Subcase II (a): $\mu \leq \frac{\frac{p+3}{2}\left\{\left(p\delta+\frac{3\delta}{2}-\frac{1}{2}\right)(A-B)-(1-B)\right\}}{\delta(p+2)^2(A-B)}$. $(\mathbf{2.36})$ takes the form $\frac{1}{\delta(A-B)}\left|a_{p+2}-\mu a_{p+1}^2\right| \leq \frac{p+3}{2}$. (2.37)

Subcase I(b): $\mu \geq \frac{\frac{p+3}{2}\left\{\left(p\delta+\frac{3\delta}{2}-\frac{1}{2}\right)(A-B)-(1-B)\right\}}{\delta(p+2)^2(A-B)}$.

Using $\left|c_1^2\right| \leq 1$, $(\mathbf{2.36})$ takes the form

$$\frac{1}{\delta(A-B)}\left|a_{p+2}-\mu a_{p+1}^2\right| \leq \mu\delta(p+2)^2(A-B)-\frac{p+3}{2}\left\{\left(p\delta+\frac{3\delta}{2}-\frac{1}{2}\right)(A-B)-B\right\} \quad (2.38)$$

This completes the theorem. The results are sharp.

Corollary 2.10: Putting $\boldsymbol{\delta=1}$ in Theorem 2.9, we get

$$\frac{1}{A-B}\left|a_{p+2}-\mu a_{p+1}^2\right| \leq \begin{cases} (p+2)^2(A-B)\left(\frac{(p+3)\{(p+1)(A-B)-B\}}{2(p+2)^2(A-B)}-\mu\right), \\ \quad if\ \mu \leq \frac{(p+3)\{(p+1)(A-B)-B\}}{2(p+2)^2(A-B)}; \\ \frac{p+3}{2}, \\ if\ \frac{(p+3)[\{(p+1)(A-B)-B\}-1]}{2(p+2)^2(A-B)} \leq \mu \leq \frac{p+3}{2}\frac{[\{(p+1)(A-B)-B\}+1]}{(p+2)^2(A-B)}; \\ (p+2)^2(A-B)\left(\mu-\frac{(p+3)\{(p+1)(A-B)-B\}}{2(p+2)^2(A-B)}\right), \\ \quad if\ \mu \geq \frac{p+3}{2}\frac{[\{(p+1)(A-B)-B\}+1]}{(p+2)^2(A-B)} \end{cases},$$

Which are the result of $\boldsymbol{f(z)} \in \left(\boldsymbol{S_p}^*\right)^{-1}[\boldsymbol{A},\boldsymbol{B}]$, as proved in Theorem 2.1.

Corollary 2.11: Putting $\boldsymbol{A=1, B=-1}$, in Theorem 2.9, we get

$$\frac{1}{2\delta}\left|a_{p+2}-\mu a_{p+1}^2\right| \leq \begin{cases} \delta\left\{\frac{p+3}{2}(2p+3)-2\mu(p+2)^2\right\}, if\ \mu \leq \frac{(p+1)(p+3)}{2(p+2)^2}, \\ \frac{p+3}{2}, if\ \frac{(p+1)(p+3)}{2(p+2)^2} \leq \mu \leq \frac{(p+3)(2p\delta+3\delta-3)}{4\delta(p+2)^2}; \\ \delta\left\{2\mu(p+2)^2-\frac{p+3}{2}(2p+3)\right\}, if\ \mu \geq \frac{(p+3)(2p\delta+3\delta-3)}{4\delta(p+2)^2}. \end{cases},$$

which are the required results for the class $\left(\boldsymbol{S_p}^*\right)^{-1}[\boldsymbol{\delta}]$ as proved in Theorem 2.5.

Corollary 2.12: Putting $\boldsymbol{p=1}$, in Theorem 2.9, we get

$$\frac{1}{\delta(A-B)}\left|a_3-\mu a_2^2\right| \leq \begin{cases} 2\left\{\left(\frac{5\delta-1}{2}\right)(A-B)-B\right\}-9\mu\delta(A-B), \\ \quad if\ \mu \leq \frac{2\left\{\left(\frac{5\delta-1}{2}\right)(A-B)-(1+B)\right\}}{9\delta(A-B)}, \\ 2, \\ if\ \frac{2\left\{\left(\frac{5\delta-1}{2}\right)(A-B)-(1+B)\right\}}{9\delta(A-B)} \leq \mu \leq \frac{2\left\{\left(\frac{5\delta-1}{2}\right)(A-B)-(1-B)\right\}}{9\delta(A-B)}; \\ 9\mu\delta(A-B)-2\left\{\left(\frac{5\delta-1}{2}\right)(A-B)-B\right\}, \\ \quad if\ \mu \geq \frac{2\left\{\left(\frac{5\delta-1}{2}\right)(A-B)-(1-B)\right\}}{9\delta(A-B)} \end{cases}, \text{ which}$$

are the required results for the class $(\boldsymbol{S}^*)^{-1}[\boldsymbol{A},\boldsymbol{B};\boldsymbol{\delta}]$ (See [4]).

Corollary 2.13: Putting $\boldsymbol{\delta=1, p=1}$ in Theorem 2.9, we get

$$\frac{1}{A-B}\left|a_3-\mu a_2^2\right| \leq \begin{cases} 9(A-B)\left(\frac{2(2A-3B)}{9(A-B)}-\mu\right), if\ \mu \leq \frac{2(2A-3B)}{9(A-B)}; \\ \frac{p+3}{2}, if\ \frac{2(2A-3B)}{9(A-B)} \leq \mu \leq \frac{2(2A-3B+1)}{9(A-B)}; \\ 9(A-B)\left(\mu-\frac{2(2A-3B)}{9(A-B)}\right), if\ \mu \geq \frac{2(2A-3B+1)}{9(A-B)} \end{cases}, \text{ Which are}$$

the result of $f(z) \in \left(S_p{}^*\right)^{-1}[A,B]$.

Corollary 2.14: Putting $\delta=1, A=1, B=-1$, in Theorem 2.9, we get

$$\frac{1}{4}\left|a_{p+2}-\mu a_{p+1}^2\right| \leq \begin{cases} (p+2)^2\left(\frac{(p+3)(2p+3)}{4(p+2)^2}-\mu\right), if\ \mu \leq \frac{(p+3)(2p+3)}{2(p+2)^2(A-B)}; \\ \frac{p+3}{4}, if\ \frac{(p+3)(2p+3)}{2(p+2)^2(A-B)} \leq \mu \leq \frac{(p+3)(p+2)}{2(p+2)^2}; \\ (p+2)^2\left(\mu-\frac{(p+3)(2p+3)}{4(p+2)^2}\right), if\ \mu \geq \frac{(p+3)(p+2)}{2(p+2)^2} \end{cases}, \text{ Which}$$

are the result of $\left(S_p{}^*\right)^{-1}$(See[4]).

Corollary 2.15: Putting $\delta=1, A=1, \mathrm{B}=-1, \mathrm{p}=1$ in Theorem 2.9, we get

$$\frac{1}{4}\left|\mathrm{a}_3-\mu \mathrm{a}_2^2\right| \leq \begin{cases} (5-9\mu), \text{if } \mu \leq \frac{4}{9}; \\ 1, \text{if } \frac{4}{9} \leq \mu \leq \frac{2}{3}; \\ (9\mu-5), \text{if } \mu \geq \frac{2}{3}. \end{cases}, \text{ which are the required}$$

results for the class $(\mathrm{S}^*)^{-1}$. (See [4])

(IV) Conclusion: Fekete-Szego inequality exist for the functions belonging to the classes $\left(\mathrm{S_p}^*\right)^{-1}[\mathrm{A},\mathrm{B}]$, $\left(S_p{}^*\right)^{-1}[\delta]$ and $\left(\mathrm{S_p}^*\right)^{-1}[\mathrm{A},\mathrm{B},\delta]$defined in the introduction part and extremal functions also exist for these inequalities, which are responsible for the sharpness of these results. Also, these results give suitable results, when suitable values are given to A, B, p and δ.

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