

A New Modified Gradient-Projection Algorithm for Solution of Constrained Convex Minimization Problem in Hilbert Spaces

Cyril Dennis Enyi and Mukiawa Edwin Soh

Abstract— In this paper, we present a new iterative scheme for finding a minimizer for a constrained convex minimization problem. We will also prove that the sequence generated by our new scheme converges strongly to a solution of the constrained convex minimization problem in real Hilbert spaces.

Index Terms —Average Mappings, Constrained Convex Minimization problem, Fixed point, Gradient Projection Algorithm, Nonexpansive mappings.

I. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$.

Definition 1.1 Let C be a nonempty closed and convex subset of a real Hilbert space H . A map $T: C \rightarrow H$ is said to be nonexpansive if for all $x, z \in C$ we have

$$\|Tx - Tz\| \leq \|x - z\|.$$

We denote the fixed point set of T by $\text{Fix}(T)$.

Definition 1.2 For any $x \in H$, we define the map $P_C: H \rightarrow C$

satisfying $\|x - P_C x\| \leq \|x - z\| \quad \forall z \in C$.

P_C is called the *metric projection* of H onto C . It is well known that P_C is nonexpansive and

$$\langle x - P_C x, P_C x - z \rangle \geq 0 \quad \forall z \in C. \quad (1)$$

Clearly, (1) is equivalent to

$$\|x - z\|^2 \geq \|x - P_C x\|^2 + \|z - P_C x\|^2 \quad \forall x \in H \text{ and } \forall z \in C$$

Furthermore, $P_C x$ is characterized by the property that $P_C x \in C$ and $\langle x - z, P_C x - P_C z \rangle \geq 0 \quad \forall x \in C. \quad (2)$

Definition 1.3 A mapping A of C into H is called monotone if $\langle Ax - Az, x - z \rangle \geq 0 \quad \forall x, z \in C$,

A is called β -strongly monotone if there exists $\beta > 0$ such that $\langle x - z, Ax - Az \rangle \geq \beta \|x - z\|^2 \quad \forall x, z \in C$,

A is called α -inverse-strongly monotone if there exists $\alpha > 0$ such that

$$\langle x - z, Ax - Az \rangle \geq \alpha \|Ax - Az\|^2 \quad \forall x, z \in C,$$

also A is L -Lipschitz-continuous if there exists

$$L > 0 \text{ such that for all } x, z \in C \quad \|Ax - Az\| \leq L \|x - z\|.$$

Definition 1.4 A mapping $T: H \rightarrow H$ is said to be firmly nonexpansive if $2T - I$ is nonexpansive, or equivalently $\langle x - z, Tx - Tz \rangle \geq \|Tx - Tz\|^2 \quad \forall x, z \in H$.

Alternatively, T is firmly nonexpansive if T can be expressed as $T = \frac{1}{2}(I + S)$, for some $S: H \rightarrow H$ nonexpansive. As an example, the projection mappings are firmly nonexpansive.

Definition 1.5 A mapping $T: H \rightarrow H$ is said to be an averaged mapping if it can be written as the average of the identity mapping I and a nonexpansive mappings; i.e.

$$T = (1 - \alpha)I + \alpha S, \quad (3)$$

where $\alpha \in (0, 1)$ and $S: H \rightarrow H$ is nonexpansive. More precisely, when (3) holds, we say that T is α -averaged. Therefore, firmly nonexpansive mappings (e.g. projections) are $\frac{1}{2}$ -averaged mappings.

The proposition below gives some properties of averaged mappings.

Proposition 1.1 (Bryne [3], Combettes [5]) For given operators $S, T, V: H \rightarrow H$:

(a) If $T = (1 - \alpha)S + \alpha V$ for some $\alpha \in (0, 1)$ and if S is averaged and V is nonexpansive, then T is averaged.

(b) T is firmly nonexpansive if and only if the complement $I - T$ is nonexpansive.

(c) If $T = (1 - \alpha)S + \alpha V$ for some $\alpha \in (0, 1)$ and if S is firmly nonexpansive and V is nonexpansive, then T is averaged.

Manuscript received January 01, 2014; revised March 28, 2014.

This work was supported in part by King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia.

Cyril Dennis Enyi is with King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia. (corresponding author: +966550782390; e-mail: cenyi@kfupm.edu.sa).

Mukiawa Edwin Soh is with King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia. (e-mail: sohedwin2013@gmail.com).

(d) The composition of finitely many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^N$ is averaged, so is the composition $T_1 T_2 \cdots T_N$. In particular, if T_1 is α_1 averaged and T_2 is α_2 averaged, where $\alpha_1, \alpha_2 \in (0,1)$, then the composition $T_1 T_2$ is α -averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1 \alpha_2$.

The following are immediately noticeable: (i) If T is nonexpansive, then $I - T$ is monotone; (ii) the projection mapping P_C is a 1-ism. The inverse strongly monotone (also known as co-coercive) operators have being widely used in solving practical problems in diverse fields, e.g., the traffic assignment problems; see, [1, 6] and the references therein. The proposition that follows gives some important relationships between average mappings and inverse strongly monotone operators.

Proposition 1.2 (Bryne [3], Combettes [5])

(a) T is nonexpansive if and only if the complement $I - T$ is $\frac{1}{2}$ -ism.

(b) If T is ν -ism, then for $\gamma > 0$, γT is $\frac{\nu}{\gamma}$ -ism.

(c) T is averaged if and only if the complement $I - T$ is τ -ism for some $\tau > \frac{1}{2}$. Indeed, for $\alpha \in (0,1)$, T is

α -averaged if and only if $I - T$ is $\frac{1}{2}$ -ism.

Given the following constrained convex minimization problem:

$$\text{minimize } \{f(x) : x \in C\}, \quad (4)$$

where $f: C \rightarrow R$ is a real-valued convex function. The minimization problem (4) is said to be consistent if it has a solution. In what follows, we shall denote the solution set of problem (4) by S . It is well known that if f is (Fréchet) differentiable, then the gradient-projection method (i.e., GPM) generates a sequence $\{x_n\}$ by using the following recursive formula:

$$x_{n+1} = P_C(x_n - \lambda \nabla f(x_n)), \forall n \geq 1, \quad (5)$$

or in a more general form;

$$x_{n+1} = P_C(x_n - \lambda_n \nabla f(x_n)), \forall n \geq 1, \quad (6)$$

where $x_0 \in C$ is an arbitrary initial guess in both (5) and (6), λ or λ_n are positive real numbers. It is known that if ∇f is α -strongly monotone and L -Lipschitzian with constants $\alpha, L > 0$, then the operator

$$T = P_C(I - \lambda \nabla f) \quad (7)$$

is a contraction; thus the sequence $\{x_n\}$ in (5) converges in norm to the unique minimizer of (4). More generally if the sequence $\{\lambda_n\}$ is such that

$$0 < \liminf \lambda_n \leq \limsup \lambda_n < \frac{2\alpha}{L^2} \quad (8)$$

then the sequence $\{x_n\}$ generated by (6) converges in norm to the unique minimizer of (4). However in a situation where the ∇f fails to be strongly monotone, the operator T defined by (7) would fail to be a contraction, consequently, the sequence $\{x_n\}$ generated by (6) may fail to converge in norm. (see [12], Sect. 4). If ∇f is Lipschitzian, then the schemes (5) and (6) can still converge weakly under certain assumptions.

The GPM for finding the approximate solutions of the constrained convex minimization problem has been studied by several authors; see, for example [9] and the references therein. The convergence of the sequence generated by this method depends largely on the behavior of the gradient of the objective function. If the gradient fails to be strongly monotone, then the sequence generated by the GPM may fail to converge strongly. Recently alternative operator-oriented approach to algorithm (6); namely an average mapping approach; see, for example Xu [12]. Xu [12] also presented two modifications of the gradient-projection algorithms which are shown to be strongly convergent.

Very recently, based on Yamada hybrid steepest descent method, Tian and Huang [10] proposed respectively the following implicit and explicit iterative scheme:

$$x_s = P_C(I - s_n \mu F) T_{\lambda_s}(x_s)$$

and

$$x_{n+1} = P_C(I - s_n \mu F) T_{\lambda_n}(x_n).$$

They proved that the sequence generated by their implicit and explicit schemes converge strongly to a solution of the constrained convex minimization problem, which also solves a certain variational inequality problem.

Also, motivated by the work of Xu [12], Shehu et al [8] proposed the following iterative scheme :

$$\begin{cases} y_n = \alpha_n u + (1 - \alpha_n) x_n \\ x_{n+1} = P_C(y_n - \lambda_n \nabla f(y_n)), \quad n \geq 1, \end{cases} \quad (9)$$

they proved that the sequence generated by their scheme converges strongly to a solution of the constrained convex minimization problem, which also solves a certain variational inequality problem (see [8] for details).

Motivated by the works of Xu [12], Tian and Huang [10] and Shehu et al [8], we propose a new iterative scheme for finding the approximate solution of a constrained convex minimization problem and we proved that the sequence generated by our scheme converges strongly to a solution of the constrained minimization problem.

II PRELIMINARIES

In the sequel, we shall also make use of the following lemmas.

Lemma 2.1 Let H be a real Hilbert space. Then the following inequality holds;

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle$$

for all $x, y \in H$.

Lemma 2.2 Let H be a real Hilbert space. Then the following inequality holds;

$$\begin{aligned} \|\lambda x + (1 - \lambda)y\|^2 &= \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 \\ &\quad - \lambda(1 - \lambda)\|x - y\|^2 \end{aligned}$$

for all $x, y \in H, \lambda \in [0, 1]$.

Lemma 2.3 (Browder [2]) Let H be a real Hilbert space, C a closed convex subset of H and $T: C \rightarrow C$ a nonexpansive mapping with a fixed point. Assume that a sequence $\{x_n\}$ in C is such that $x_n \rightarrow x$ weakly and $x_n - Tx_n \rightarrow y$. Then $x - Tx = y$

Lemma 2.4 (Xu [11]) Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers satisfying the following

$$a_{n+1} \leq (1 + \alpha_n)a_n + \alpha_n\sigma_n + \gamma_n \quad n \geq 0$$

where

- (i) $\{\alpha_n\} \subset [0, 1], \sum_{n=1}^{\infty} \alpha_n = \infty$;
- (ii) $\limsup \sigma_n \leq 0$ (iii) $\gamma_n \geq 0, \sum_{n=1}^{\infty} \gamma_n < \infty$.

Then $a_n \rightarrow 0$ as $n \rightarrow \infty$.

We adopt the following notions :

- $x_n \rightarrow x$ means that $x_n \rightarrow x$ strongly;
- $x_n \rightharpoonup x$ means that $x_n \rightharpoonup x$ weakly;
- $W_w(x_n) = \{x: \exists x_{n_j} \rightharpoonup x\}$ is the weak w -limit set of the sequence $\{x_n\}_{n=1}^{\infty}$

III MAIN RESULTS

In this section we present a modify of the gradient projection method and prove its strong convergence. Our result in this section complements the result of Xu [12].

Furthermore, using the technique in [7, 12], we obtain the following theorem.

Theorem 3.1 Let C be a non empty, closed and convex subset of a real Hilbert space H . Suppose that the minimization problem (4) is consistent and let S denote its solution set. Assume that the gradient ∇f is L -Lipschitzian with constant $L > 0$. For any given $u \in C$, let the sequence $\{x_n\}$ be generated iteratively by, $x_1 \in C$,

$$\begin{aligned} x_{n+1} &= (1 - \beta_n)x_n + \beta_n P_C(x_n - \lambda_n \nabla f(x_n)) \\ &\quad + \alpha_n(u - x_n) \end{aligned} \quad (10)$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{\lambda_n\}$ in $(0, L/2)$ satisfying the following conditions:

$$(C_1) \lim_{n \rightarrow \infty} \alpha_n = 0,$$

$$(C_2) \sum_{n=1}^{\infty} \alpha_n < \infty,$$

$$(C_3) 0 < \liminf \lambda_n \leq \limsup \lambda_n < 2/L$$

Then the sequence $\{x_n\}$ converges strongly to a minimizer \bar{x} of (4), which is closest to u from the solution set S . In other words $\bar{x} = P_S u$.

Proof. Inspired by the method of proof of [8], it is well known that:

(a) $x^* \in C$ solves the minimization problem (4) if and only if x^* solves the fixed point equation

$$x^* = P_C(I - \lambda \nabla f) x^*$$

where $\lambda > 0$ is any fixed positive number. For the sake of simplicity, we assume that (due to condition C_3)

$$0 < a \leq \lambda_n \leq b < \frac{2}{L}, \quad n \geq 1,$$

where a and b are constants.

(b) The gradient ∇f is $\frac{1}{L}$ -ism.

(c) $P_C(I - \lambda \nabla f)$ is $\frac{2+\lambda L}{4}$ -average for $0 < \lambda < \frac{2}{L}$. Hence we have that, for each n , $P_C(I - \lambda \nabla f)$ is $\frac{2+\lambda L}{4}$ -average.

Therefore we can write

$$\begin{aligned} P_C(I - \lambda \nabla f) &= \frac{2 + \lambda_n L}{4} I + \frac{2 - \lambda_n L}{4} T_n \\ &= (1 - \beta_n)I + \beta_n T_n, \end{aligned} \quad (11)$$

where T_n is nonexpansive and $\beta_n = \frac{2+\lambda_n L}{4} \in [a_1, b_1]$ where

$$a_1 = \frac{2+aL}{4} \text{ and } b_1 = \frac{2+bL}{4} < 1.$$

Then we can write (10) as

$$\begin{aligned} x_{n+1} &= (1 - \beta_n)x_n + \beta_n[(1 - \beta_n)x_n + \beta_n T_n x_n] \\ &\quad + \alpha_n(u - x_n) \end{aligned} \quad (12)$$

Let $P \in S$, observing that $T_n P = P$, we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1 - \alpha_n - \beta_n)x_n + (1 - \beta_n)\beta_n + \beta_n^2 T_n x_n \\ &\quad + \alpha_n u - p\| \\ &= \|(1 - \alpha_n)(x_n - p) - \beta_n^2(x_n - p) + \\ &\quad \beta_n^2(T_n x_n - p) + \alpha_n(u - p)\| \\ &\leq (1 - \alpha_n - \beta_n^2)\|x_n - p\| + \beta_n^2\|T_n x_n - p\| \\ &\quad + \alpha_n\|u - p\| \\ &\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\|u - p\| \\ &\leq \max\{\|x_n - p\|, \|u - p\|\}. \end{aligned} \tag{16}$$

Therefore, by induction, we have

$$\|x_n - p\| \leq \max\{\|x_n - p\|, \|u - p\|\}, \quad \forall n \geq 1.$$

Hence the sequence $\{x_n\}$ is bounded and so $\{T_n x_n\}$.

Now we have the following:

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|(1 - \alpha_n - \beta_n) \\ &\quad + \beta_n[(1 - \beta_n)x_n + \beta_n T_n x_n] + \alpha_n u - p\|^2 \\ &= \|(x_n - p) - \beta_n^2(x_n - T_n x_n) \\ &\quad + \alpha_n(u - x_n)\|^2. \end{aligned} \tag{13}$$

We can observe that;

$$\begin{aligned} \|T_n x - p\|^2 &\leq \|x_n - p\|^2 \\ \Rightarrow \langle T_n x - p, T_n x - p \rangle \\ &\leq \langle x - p, x - T_n x \rangle + \langle x - p, T_n x - p \rangle \\ \Rightarrow \langle T_n x - p, T_n x - x \rangle &\leq \langle x - p, x - T_n x \rangle \\ \Rightarrow \langle T_n x - x, T_n x - x \rangle + \langle x - p, T_n x - x \rangle \\ &\leq \langle x - p, x - T_n x \rangle, \end{aligned}$$

which implies that

$$\|T_n x - p\|^2 \leq 2\langle x - p, x - T_n x \rangle. \tag{14}$$

Using Lemma (2.1), (13) and (14) we obtain that

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|(x_n - p) - \beta_n^2(x_n - T_n x_n) \\ &\quad + \alpha_n(u - x_n)\|^2 \\ &\leq \|x_n - p\|^2 + \beta_n^4\|x_n - T_n x_n\|^2 \\ &\quad + \beta_n^2\|T_n x_n - x_n\|^2 - 2\langle x_n - p, \beta_n^2(x_n - T_n x_n) \rangle \\ &\quad + 2\langle \alpha_n(u - x_n), x_{n+1} - p \rangle \\ &\leq \|x_n - p\|^2 + \beta_n^4\|x_n - T_n x_n\|^2 \\ &\quad - \beta_n^2\|T_n x_n - x_n\|^2 - 2\alpha_n\langle x_n - u, x_{n+1} - p \rangle \\ &= \|x_{n+1} - p\|^2 + \beta_n^2(\beta_n^2 - 1)\|x_n - T_n x_n\|^2 \\ &\quad - 2\alpha_n\langle x_n - u, x_{n+1} - p \rangle. \end{aligned}$$

Hence,

$$\beta_n^2(\beta_n^2 - 1)\|x_n - T_n x_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 - 2\alpha_n\langle x_n - u, x_{n+1} - p \rangle. \tag{15}$$

Since $\{x_n\}$ is bounded, then there exists a constant $M \geq 0$ such that $-2\langle x_n - u, x_{n+1} - p \rangle \leq M, \forall n \geq 1$.

Therefore (15) gives

$$\|x_{n+1} - p\|^2 - \|x_n - p\|^2 + \beta_n^2(\beta_n^2 - 1)\|x_n - T_n x_n\|^2$$

$$\leq M\alpha_n. \tag{16}$$

The rest of the proof will be done considering 2 cases.

Case 1: Assume that the sequence $\{\|x_n - x^*\|\}$ is a monotonically decreasing sequence. Then $\{\|x_n - x^*\|\}$ converges.

Clearly, we have that

$$\|x_{n+1} - x^*\|^2 - \|x_n - x^*\|^2 \rightarrow 0.$$

Now by (16) we obtain that

$$\lim_{n \rightarrow \infty} \beta_n^4 \|x_n - T_n x_n\|^2 = 0.$$

Using the condition that $\beta_n \in [a, b] \subset (0, 1)$, we have

$$\lim_{n \rightarrow \infty} \|x_n - T_n x_n\| = 0. \tag{17}$$

Now we show that $W_\omega(x_n) \subset S$.

Let $x^* \in W_\omega(x_n)$ and $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_j} \rightarrow x^*$. We may assume that $\lambda_{n_j} \rightarrow \lambda$; then $0 < \lambda < \frac{2}{L}$. We set $T := P_C(I - \lambda \nabla f)$, hence T is nonexpansive.

From (17), we have

$$\begin{aligned} \|x_{n_j} - T x_{n_j}\| &\leq \|x_{n_j} - T_{n_j} x_{n_j}\| + \|T_{n_j} x_{n_j} - T x_{n_j}\| \\ &= \|x_{n_j} - T_{n_j} x_{n_j}\| + \|P_C(x_{n_j} - \\ &\quad \lambda_{n_j} \nabla f(x_{n_j})) - P_C(x_{n_j} - \lambda \nabla f(x_{n_j}))\| \\ &\leq \|x_{n_j} - T_{n_j} x_{n_j}\| + |\lambda_{n_j} - \lambda| \|\nabla f(x_{n_j})\| \\ &\leq \|x_{n_j} - T_{n_j} x_{n_j}\| + |\lambda_{n_j} - \lambda| M_1 \rightarrow 0, n_j \rightarrow \infty. \end{aligned}$$

Therefore, by Lemma (2.3) we have that

$$W_\omega(x_n) \subset F(T) = S.$$

We next prove that the sequence $\{x_n\}$ converges strongly to $\bar{x} \in S$, where \bar{x} is the solution of (4), which is the closest to u from the solution set S .

Firstly, we show that $\limsup_{n \rightarrow \infty} \langle x_n - \bar{x}, u - \bar{x} \rangle \leq 0$.

Observe that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ satisfying

$$\limsup_{n \rightarrow \infty} \langle x_n - \bar{x}, u - \bar{x} \rangle = \limsup_{j \rightarrow \infty} \langle x_{n_j} - \bar{x}, u - \bar{x} \rangle.$$

Since $\{x_{n_j}\}$ is bounded, there exists a subsequence $\{x_{n_{j_i}}\}$ of $\{x_{n_j}\}$ such that $x_{n_{j_i}} \rightarrow p \in F(T) = S$, without loss of generality, we assume that $x_{n_j} \rightarrow p \in F(T) = S$.

Then we obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle x_n - \bar{x}, u - \bar{x} \rangle &= \limsup_{j \rightarrow \infty} \langle x_{n_j} - \bar{x}, u - \bar{x} \rangle \\ &= \langle p - \bar{x}, u - \bar{x} \rangle \leq 0. \end{aligned} \tag{18}$$

Now, we have

$$\begin{aligned}
 \|x_{n+1} - \bar{x}\|^2 &= \|(1 - \alpha_n - \beta_n)x_n + \beta_n[(1 - \beta_n)x_n \\
 &\quad + \beta_n T_n x_n] + \alpha_n u - \bar{x}\|^2 \\
 &= \|(1 - \alpha_n)(x_n - \bar{x}) + \alpha_n(u - \bar{x}) \\
 &\quad - \beta_n(x_n - T_n x_n)\|^2 \\
 &\leq (1 - \alpha_n)\|x_n - \bar{x}\|^2 + 2\alpha_n\langle u - \bar{x}, x_{n+1} - \bar{x} \rangle \\
 &\quad - 2\beta_n^2\langle x_n - T_n x_n, x_{n+1} - \bar{x} \rangle \\
 &\leq (1 - \alpha_n)\|x_n - \bar{x}\|^2 + 2\alpha_n\langle u - \bar{x}, x_{n+1} - \bar{x} \rangle \\
 &\quad - 2\alpha_n\beta_n^2\langle x_n - T_n x_n, x_{n+1} - \bar{x} \rangle. \quad (19)
 \end{aligned}$$

It is clear that $\lim_{n \rightarrow \infty} \beta_n^2 \langle x_n - T_n x_n, x_{n+1} - \bar{x} \rangle = 0$.

Therefore, by Lemma (2.4) applied to (19),

we obtain that $\lim_{n \rightarrow \infty} \|x_n - \bar{x}\| = 0$.

Case 2: Assume that $\{\|x_n - x^*\|\}$ is not a monotonically decreasing sequence. Set

$\Gamma_n = \{\|x_n - x^*\|\}^2$ and let $\tau : \mathbb{N} \rightarrow \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough)

By $\tau(n) = \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}$.

Clearly, τ is a non-decreasing sequence such that

$$\tau(n) \rightarrow \infty \text{ as } n \rightarrow \infty \text{ and } \Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1} \text{ for } n \geq n_0.$$

From (17) we can see that

$$\beta_{\tau(n)}^2(1 - \beta_{\tau(n)}^2)\|x_{\tau(n)} - T_{\tau(n)}x_{\tau(n)}\| \leq 2\alpha_{\tau(n)}M \rightarrow 0$$

as $n \rightarrow \infty$. Further more, we have that

$$\|x_{\tau(n)} - T_{\tau(n)}x_{\tau(n)}\|^2 \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Using similar argument as in case 1, we can show that $x_{\tau(n)}$ converges weakly to \tilde{x} as $\tau(n) \rightarrow \infty$ and

$\lim_{\tau(n) \rightarrow \infty} \sup \langle u - \tilde{x}, x_{\tau(n)} - \tilde{x} \rangle \leq 0$. We know that for all

$$\begin{aligned}
 n \geq n_0, 0 \leq &\|x_{\tau(n)+1} - \tilde{x}\|^2 - \|x_{\tau(n)} - \tilde{x}\|^2 \\
 &\leq \alpha_n[2\langle u - \tilde{x}, x_{\tau(n)+1} - \tilde{x} \rangle - 2\beta_n^2\langle T_{\tau(n)}x_{\tau(n)} - \\
 &\quad x_{\tau(n)}x_{\tau(n)+1}, x_{\tau(n)+1} - \tilde{x} \rangle - \|x_{\tau(n)} - \tilde{x}\|^2]
 \end{aligned}$$

which implies that

$$\begin{aligned}
 \|x_{\tau(n)} - \tilde{x}\|^2 &\leq 2\langle u - \tilde{x}, x_{\tau(n)+1} - \tilde{x} \rangle \\
 &\quad - 2\beta_n^2\langle T_{\tau(n)}x_{\tau(n)} - x_{\tau(n)}x_{\tau(n)+1}, x_{\tau(n)+1} - \tilde{x} \rangle.
 \end{aligned}$$

Therefore, we conclude that $\lim_{n \rightarrow \infty} \|x_{\tau(n)} - \tilde{x}\| = 0$

Hence, $\lim_{n \rightarrow \infty} \Gamma_{\tau(n)} = \Gamma_{\tau(n)+1} = 0$.

Furthermore, for $n \geq n_0$, one could observe easily that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$, that is $n > \tau(n)$, because $\Gamma_j \leq \Gamma_{j+1}$ for $\tau(n) + 1 \leq j \leq n$.

As a consequence, we obtain for all $n \geq n_0$,

$$0 \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

Thus $\lim_{n \rightarrow \infty} \Gamma_{\tau(n)} = 0$. that is $\{x_n\}$ converges strongly to \tilde{x} . This completes the proof.

Corollary 3.2 Let C be a nonempty, closed and convex subset of a real Hilbert space H . Suppose that the minimization problem (5) is consistent, and let S denote its solution set. Assume that the gradient ∇f is L -Lipschitzian with constant $L > 0$. For any given $u \in C$, let the sequence $\{x_n\}$ be generated iteratively by $x_1 \in C$,

$$\begin{aligned}
 x_{n+1} &= (1 - \beta_n)x_n + \beta_n P_C(x_n - \lambda \nabla f(x_n)) \\
 &\quad + \alpha_n(u - x_n), \quad (20)
 \end{aligned}$$

where $\lambda \in (0, L/2)$ and $\{\alpha_n\} \subset [0, 1]$ satisfies the following conditions:

$$(C_1) \lim_{n \rightarrow \infty} \alpha_n = 0$$

$$(C_2) \sum_{n=1}^{\infty} \alpha_n < \infty.$$

Then the sequence $\{x_n\}$ converges strongly to a minimizer \bar{x} of (4).

IV APPLICATION

We give an application of Theorem 3.1 this section. We apply Theorem 3.1 to the split feasibility problem (denoted as *SPF*), which was introduced by Censor and Elfving [4]. *SFP* has received considerable attention of many authors because of its applications in image reconstruction, signal processing and intensity-modulation therapy (see [3, 8, 10] and the references therein).

SFP can be mathematically formulated as a problem of finding a point x with the property that

$$x \in C \text{ and } Bx \in Q, \quad (21)$$

Where Q and C are nonempty, closed and convex subset of Hilbert spaces H_1 and H_2 respectively and $B: H_1 \rightarrow H_2$ is a bounded linear operator.

Clearly, x^* is a solution of *SFP* (21) if and only if $x^* \in C$ and $Bx^* - P_Q Bx^* = 0$.

The proximity function f is defined by

$$f(x) = \frac{1}{2}\|Bx - P_Q Bx\|^2 \quad (22)$$

and consider the constrained convex minimization problem

$$\min_{x \in C} f(x) = \min_{x \in C} \frac{1}{2}\|Bx^* - P_Q Bx^*\|^2 \quad (23)$$

Then x^* solves the minimization problem (24). In [3] CQ algorithm was introduced to solve *SFP*.

$$x_{n+1} = P_C(I - \lambda B^*(I - P_Q)B)x_n, \quad n \geq n_0, \quad (24)$$

where $0 < \lambda \leq \frac{2}{\|B\|^2}$. It was proved that the sequence generated by (25) converges weakly to a solution of the *SFP*.

Now we introduce the following algorithm to obtain a

strong convergence iterative sequence to solve the *SFP*. Let $u \in C$ be given and the sequence $\{x_n\}$ be generated by $x_1 \in C$,

$$x_{n+1} = (1 - \beta_n)x_n + \beta_n P_C(I - \lambda_n(B^*(I - PQ)B + \gamma I))x_n + \alpha_n(u - x_n), n \geq 1 \quad (25)$$

where $\{\alpha_n\} \subset [0, 1], \gamma > 0$ and $\{\lambda_n\}$ in $(0, 2/L)$ satisfy the following conditions:

$$(C_1) \lim_{n \rightarrow \infty} \alpha_n = 0.$$

$$(C_2) \sum_{n=1}^{\infty} \alpha_n = \infty.$$

$$(C_3^*) 0 < \liminf \lambda_n \leq \limsup \lambda_n < \frac{2}{\|B\|^2 + \gamma}.$$

Applying Theorem 3.1, we obtain the following convergence result for solving the *SFP* (21).

Theorem 4.2 Assume that the split feasibility problem (*SFP*) (21) is consistent. Let the sequence $\{x_n\}$ be generated by (26), where the sequence $\{\alpha_n\} \subset [0, 1]$ and $\{\lambda_n\}$ in $(0, \frac{2}{L})$ satisfy conditions $(C_1 - C_3^*)$. Then the sequence $\{x_n\}$ converges strongly to a solution of the split feasibility problem (21).

Proof From the definition of the proximity function f , we have

$$\nabla f(x) = B^*(I - P_Q)Bx \quad (26)$$

and ∇f is Lipschitz continuous, i.e.,

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\| \quad (27)$$

where $L = \|B\|^2$.

$$\text{Set } f_\gamma(x) = f(x) + \frac{\gamma}{2}\|x\|^2.$$

Consequently,

$$\begin{aligned} \nabla f_\gamma(x) &= \nabla f(x) + \gamma x \\ &= B^*(I - P_Q)Bx + \gamma x \end{aligned}$$

and ∇f_γ is Lipschitzian with Lipschitz constant $\|B\|^2 + \gamma$. Therefore the iterative scheme (25) is equivalent to

$$x_{n+1} = (1 - \beta_n)x_n + \beta_n P_C \left(x_n - \lambda \nabla f_\gamma(x_n) \right) + \alpha_n(u - x_n), \quad (28)$$

where $\{\alpha_n\} \subset [0, 1], \gamma > 0$ and $\{\lambda_n\}$ in $(0, \frac{2}{L^*})$ satisfy the following conditions:

$$(C_1) \lim_{n \rightarrow \infty} \alpha_n = 0$$

$$(C_2) \sum_{n=1}^{\infty} \alpha_n = \infty$$

$$(C_3^*) 0 < \liminf \lambda_n \leq \limsup \lambda_n < \frac{2}{\|B\|^2 + \gamma}$$

Where $L^* = \|B\|^2 + \gamma$

The conclusion follows from Theorem 3.1.

REFERENCES

- [1] D. P. Bertsekas and Gafni, E.M. "Projection methods for variational inequalities with applications to the traffic assignment problem," *Math. Progr. Stud.* 1982;17:139-159.
- [2] F. E. Browder, "Nonexpansive nonlinear mappings in a Banach Space," *Proc. Nat. Acad. Sci. USA.* 1965;54: 1041-1044.
- [3] C. Byrne, "Unified treatment of some algorithms in signal processing and image construction," *Inverse Probl.* 2004;20:103-120.
- [4] Y. Censor and T. A. Elfving, "Multiprojection algorithm using Bregman projection in a product space," *Numer. Algorithms* 1994;8:221-239.
- [5] P. L. Combettes, "Solving monotone inclusion via compositions of nonexpansive averaged operators," *Optimization.* 2004;53:475-504.
- [6] D. Han and H. K. Lo, "Solving non-additive traffic assignment problems: a descent method for co-coercive variational inequalities," *Eur. J. Oper. Res.* 2004;159:529-544.
- [7] P. E. Mainge, "Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization," *Set-Valued Anal.* 2008;16:899-912.
- [8] Y. Shehu, O. S. Iyiola and C. D. Enyi, "Iterative approximation of solutions for constrained convex minimization problem," *Arab J. Math.* 2013;2:393-402.
- [9] M. Su and H. K. Xu, "Remarks on the gradient-projection algorithm," *J. Nonlinear Anal. Optim.* 2010;1(1):35-43.
- [10] M. Tian and L. Huang, "Iterative methods for constrained convex minimization problem in Hilbert spaces," *Fixed Point Theory Appl.* 2013; doi:10.1186/1687-1812-2013-105.
- [11] H. K. Xu, "Iterative algorithms for nonlinear operators," *J. London Math. Soc.* 2002;66(2):240-256.
- [12] H. K. Xu, "Averaged mappings and the gradient-projection algorithm," *J. of Opt. Theory Appl.* 2011;150(2):360-378.