Valuing Multi-asset Spread Options by
the Lie-Trotter Operator Splitting Method

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Abstract — In this paper, by means of the Lie-Trotter operator splitting method, we have presented a new unified approach not only to rigorously derive Kirk’s approximation but also to obtain a generalisation for multi-asset spread options straightforwardly. The derived price formula for the multi-asset spread option bears a great resemblance to Kirk’s approximation in the two-asset case. More importantly, our approach is able to provide a new perspective on Kirk’s approximation and the generalisation; that is, they are simply equivalent to the Lie-Trotter operator splitting approximation to the Black-Scholes equation.

Keywords: Spread options; Kirk’s approximation; Lie-Trotter operator splitting method

1. Introduction

Despite the rapid development of spread options whose payoff is contingent upon the price difference of two underlying assets, pricing spread options is a very challenging task and receives much attention in the literature.[1] The major difficulty lies in the lack of knowledge about the distribution of the spread of two correlated lognormal random variables. The simplest approach is to evaluate the expectation of the final payoff over the joint probability distribution of the two correlated lognormal underlyings by means of numerical integration. However, practitioners often prefer to use analytical approximations rather than numerical methods because of their computational ease. Among various analytical approximations, e.g. Carmona and Durrleman (2003), Deng et al. (2008), Bjerk sund and Stensland (2011), Venkatramana and Alexander (2011),[1-4] Kirk’s approximation seems to be the most widely used and is the current market standard.[5] It is well known that Kirk’s approximation extends from Margrabe’s exchange option formula with no rigorous derivation.[6] Recently, Lo (2013) applied the idea of WKB method to provide a derivation of Kirk’s approximation and discuss its validity.[7] Nevertheless, it is not straightforward to provide a generalisation of Kirk’s approximation for the case of multi-asset spread option via this approach.

Accordingly, it is the aim of this paper to present a simple unified approach, namely the Lie-Trotter operator splitting method,[8] not only to rigorously derive Kirk’s approximation but also to obtain a generalisation for the case of multi-asset spread option in a straightforward manner. The derived price formula for the multi-asset spread option bears a great resemblance to Kirk’s approximation in the two-asset case. More importantly, the proposed approach is able to provide a new perspective on Kirk’s approximation and the generalisation; that is, they are simply equivalent to the Lie-Trotter operator splitting approximation to the Black-Scholes equation. Illustrative numerical examples for the three-asset spread options are also shown to demonstrate both the accuracy and efficiency of our generalised Kirk approximation.

2. Multi-asset spread options

To price a European $N$-asset call spread option, we need to solve the $N$-dimensional Black-Scholes equation

$$\left\{ \hat{L}_{BS} - \frac{\partial}{\partial \tau} \right\} P (\{F_i\}, \tau) = 0$$

(1)

with

$$\hat{L}_{BS} = \sum_{i,j=1}^{N} \frac{1}{2} \rho_{ij} \sigma_i \sigma_j F_i F_j \frac{\partial^2}{\partial F_i \partial F_j} - r ,$$

(2)
subject to the final payoff condition

\[ P \left( \{ F_i \}, 0 \right) = \max \left( F_N - \sum_{j=1}^{N-1} F_j - K, 0 \right) , \] (3)

where \( F_i \) is the future price of the lognormal underlying asset \( i \) with the volatility \( \sigma_i \), \( \rho_{ij} \) is the correlation between the assets \( i \) and \( j \), \( K \) is the strike price, \( r \) is the risk-free interest rate, and \( \tau \) is the time-to-maturity. Unfortunately, this is a very formidable task. In the following we apply the Lie-Trotter operator splitting method to derive a closed-form approximate price formula for the multi-asset spread option, which bears a great resemblance to the price formula of Kirk’s approximation in the two-asset case.

**Proposition:**

The price of the \( N \)-asset spread option can be approximated by

\[ P \left( \{ F_i \}, \tau \right) \approx \{ F_N N (\theta_1) - (F_+ + K) N (\theta_2) \} e^{-r\tau} \] (4)

where

\[ F_+ = \sum_{j=1}^{N-1} F_j \] (5)

\[ \theta_1 = \frac{1}{\sigma_- \sqrt{\tau}} \left\{ \ln \left( \frac{F_N}{F_+ + K} \right) + \frac{1}{2} \sigma_+^2 \tau \right\} \] (6)

\[ \theta_2 = \theta_1 - \alpha \sqrt{\tau} \] (7)

\[ \tilde{\sigma}_- = \sqrt{\sigma_N^2 - 2 \rho \sigma_N \sigma_+ + \sigma_+^2} \] (8)

\[ \tilde{\sigma}_+ = \tilde{\sigma}_+ \left( \frac{F_+}{F_+ + K} \right) \] (9)

\[ \tilde{\sigma}_+ = \sqrt{\sum_{j,k=1}^{N-1} \rho_{jk} \sigma_j \sigma_k F_j F_k} \] (10)

\[ \tilde{\rho} = \frac{1}{\tilde{\sigma}_+} \left( \sum_{j=1}^{N-1} \rho_{jj} \sigma_j F_j \right) \] (11)

**Proof:**

Introducing the new variables: \( f_i \equiv F_i - F_+ \) for \( i = 1, 2, \ldots, N - 2 \) and \( F_+ \equiv \sum_{j=1}^{N-1} F_j \), Eq.(1) can be cast in the form

\[ \left\{ \hat{L}_N + \hat{L}_+ + \hat{L}_{N+} + \hat{L}_R - r \right\} P \left( \{ F_i \}, \tau \right) = \frac{\partial P \left( \{ F_i \}, \tau \right)}{\partial \tau} , \] (12)

where

\[ \hat{L}_N = \frac{1}{2} \sigma_N^2 F_N^2 \frac{\partial^2}{\partial F_N^4} \] (13)

\[ \hat{L}_+ = \frac{1}{2} \sigma_+^2 F_+^2 \frac{\partial^2}{\partial F_+^4} \] (14)

\[ \hat{L}_{N+} = \tilde{\rho} \sigma_N \tilde{\sigma}_+ F_+ F_N \frac{\partial^2}{\partial F_+ \partial F_N} \] (15)

and

\[ \hat{L}_R = \frac{1}{2} \sum_{k=1}^{N-2} \sum_{n=1}^{N-2} \left\{ \left( \rho_{kn} \sigma_k F_k - 2 \sum_{m=1}^{N-1} \rho_{mn} \sigma_m F_m \right) \times \sigma_n F_n + \sum_{p=1}^{N-1} \sum_{q=1}^{N-1} \rho_{pq} \sigma_p \sigma_q F_p F_q \right\} \frac{\partial^2}{\partial f_n \partial f_k} + \frac{1}{2} \sum_{k=1}^{N-2} \sum_{n=1}^{N-2} \left( \rho_{kn} \sigma_k F_k - 2 \sum_{m=1}^{N-1} \rho_{mn} \sigma_m F_m \right) \times \sigma_n F_n \frac{\partial^2}{\partial f_+ \partial f_k} . \] (16)

The formal solution of Eq.(12) is given by

\[ P \left( \{ F_i \}, \tau \right) = \exp \left\{ \tau \left( \hat{L}_N + \hat{L}_+ + \hat{L}_{N+} + \hat{L}_R - r \right) \right\} \times \max (F_N - F_+ - K, 0) . \] (17)

Then we apply the Lie-Trotter operator splitting method to obtain an approximation to the formal solution \( P \left( \{ F_i \}, \tau \right) \) namely (see the Appendix)

\[ P^{LT} \left( \{ F_i \}, \tau \right) = \exp \left\{ \frac{\tau}{N-2} \left( \hat{L}_N + \hat{L}_+ + \hat{L}_{N+} + \hat{L}_R - r \right) \right\} \times \max (F_N - F_+ - K, 0) \] (18)

where the relation

\[ \hat{L}_R \max (F_N - F_+ - K, 0) = 0 . \] (19)

is utilized.

Next, in terms of the two new variables

\[ R_1 = \frac{F_N}{F_+ + K} \quad \text{and} \quad R_2 = F_+ + K , \] (20)

we rewrite \( P^{LT} \left( \{ F_i \}, \tau \right) \) as follows:

\[ P^{LT} \left( \{ F_i \}, \tau \right) = e^{-r\tau} \exp \left\{ \tau \left( \hat{L}_A + \hat{L}_B \right) \right\} \times R_2 \max (R_1 - 1, 0) , \] (21)
where

\begin{align*}
\hat{L}_A &= \frac{1}{2} \hat{\sigma}^2 R_1^2 \frac{\partial^2}{\partial R_1^2} - \frac{\partial}{\partial R_1} \sigma_+ R_1 \hat{\sigma}, \\
\hat{L}_B &= \frac{1}{2} \hat{\sigma}^2 R_2^2 \frac{\partial^2}{\partial R_2^2} - \left( \hat{\rho} \sigma_N - \sigma_+ \right) \sigma_+ R_2 \frac{\partial}{\partial R_2} \\
&\quad + \left( \hat{\rho} \sigma_N - \sigma_+ \right) \sigma_+ R_2 \frac{\partial^2}{\partial R_1 \partial R_2} \\
\hat{\sigma}_- &= \sqrt{\sigma_N^2 - 2 \hat{\rho} \sigma_N \sigma_+ + \sigma_+^2}, \\
\hat{\sigma}_+ &= \sigma_+ \left( \frac{F_+}{F_+ + K} \right).
\end{align*}

It is clear that the exponential operator \(\exp \{\tau \hat{L}_A \} \) is difficult to evaluate, so we need to apply the Lie-Trotter operator splitting method again to approximate the operator by \(\exp \{\tau \hat{L}_A \} \exp \{\tau \hat{L}_B \} \). As a result, we obtain

\begin{align*}
P^{LT} \{ \{ F_i \}, \tau \} &\approx e^{-\tau \hat{L}_A} \exp \{\tau \hat{L}_B \} R_2 \max (R_1 - 1, 0) \\
&\approx e^{-\tau R_2} \max (R_1 - 1, 0)
\end{align*}

for

\[ \hat{L}_B R_2 \max (R_1 - 1, 0) = 0. \] (27)

Since \(\hat{\sigma}_-\) is independent of \(R_1\), the standard workhorse of the Black-Scholes model can be used to evaluate \(\exp \{\tau \hat{L}_A \} \max (R_1 - 1, 0)\) such that

\[ P^{LT} \{ \{ F_i \}, \tau \} \approx e^{-\tau \hat{L}_A} R_2 \{ R_1, N (\theta_1) - N (\theta_2) \}. \] (28)

where

\begin{align*}
\theta_1 &= \frac{1}{\hat{\sigma}_- \sqrt{T}} \left( \ln \left( \frac{F_+}{F_+ + K} \right) + \frac{1}{2} \hat{\sigma}^2 \tau \right), \\
\theta_2 &= \theta_1 - \hat{\sigma}_- \sqrt{T}.
\end{align*}

It is obvious that this approximate solution is identical to the approximate price formula given in Eq.(4). (Q.E.D.)

In terms of the spot asset prices, namely \(S_1 = F_i \exp (-\tau r)\), the price formula given in Eq.(4) becomes

\[ P_{Kirk} \{ \{ S_i \}, \tau \} = S_N (d_1) - \left( \sum_{j=1}^{N-1} S_j + Ke^{-\tau} \right) N (d_2) \] (31)

where

\begin{align*}
d_1 &= \ln S_N - \ln \left( \sum_{j=1}^{N-1} S_j + K e^{-\tau} \right) \\
&\quad + \frac{1}{2} \hat{\sigma}^2 \tau \\
d_2 &= d_1 - \hat{\sigma}_- \sqrt{T} \\
\hat{\sigma}_- &= \sqrt{\sigma_N^2 - 2 \hat{\rho} \sigma_N \sigma_+ + \sigma_+^2} \\
\hat{\sigma}_+ &= \sigma_+ \left( \frac{F_+}{F_+ + K} \right).
\end{align*}

Obviously, this approximate price formula resembles the price formula of Kirk’s approximation in the two-asset case very closely. In fact, by setting \(N = 2\) we can recover Kirk’s approximation readily. Moreover, for the Lie-Trotter splitting approximation to be valid, we need to require \(\hat{\sigma}_+^2 \tau \) to be sufficiently small, namely \(\hat{\sigma}_+^2 \tau \ll 1\).

3. Illustrative numerical examples

In this section illustrative numerical examples are presented to demonstrate the accuracy of our generalised Kirk approximation for the three-asset spread options. We examine a simple three-asset spread option with the final payoff \(\max (S_3 - S_1 - S_2 - K, 0)\). Table I tabulates the approximate option prices estimated by our generalised Kirk approximation for different values of the strike price \(K\) and time-to-maturity \(T\). Other input model parameters are set as follows: \(r = 0.05\), \(\sigma_1 = \sigma_2 = \sigma_3 = 0.3\), \(\rho_{12} = 0.4\), \(\rho_{23} = 0.2\), \(\rho_{13} = 0.8\), \(S_1 = 50\), \(S_2 = 60\) and \(S_3 = 150\). Monte Carlo estimates and the corresponding standard deviations are also presented for comparison. It is observed that the computed errors of the approximate option prices are capped at 1% (in magnitude). In fact, most of them are less than 0.5%. The effect of increasing the three volatilities (from 0.3 to 0.6) upon the approximate estimation of the option prices is also investigated. Only a slight increase occurs in the computed errors, and these errors are still less than 1% (in magnitude). As a result, it can be concluded that our generalised Kirk approximation for the three-asset spread option is found to be very accurate and efficient.

4. Conclusion

By means of the Lie-Trotter operator splitting method, we have presented a new unified approach...
not only to rigorously derive Kirk’s approximation but also to obtain a generalisation for multi-asset spread options in a straightforward manner. The derived price formula for the multi-asset spread option bears a great resemblance to Kirk’s approximation in the two-asset case. In fact, by setting the total number of assets to be two, we can recover Kirk’s approximation readily. Thus, our generalisation possesses the same nice features as Kirk’s approximation. Moreover, our approach is able to provide a new perspective on Kirk’s approximation and the generalisation; that is, they are simply equivalent to the Lie-Trotter operator splitting approximation to the Black-Scholes equation.

Table I: Prices of a European three-asset call spread option. Other input parameters are: \( r = 0.05 \), \( \sigma_1 = \sigma_2 = \sigma_3 = 0.3 \), \( \rho_{12} = 0.4 \), \( \rho_{23} = 0.2 \), \( \rho_{13} = 0.8 \), \( S_1 = 50 \), \( S_2 = 60 \) and \( S_3 = 150 \). Here “GK” refers to our generalised Kirk approximation while “MC” denotes the Monte Carlo estimates with 900,000 replications. The relative errors of the “GK” option prices with respect to the “MC” estimates are also presented.

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Appendix:

Suppose that one needs to exponentiate an operator \( \hat{C} \) which can be split into two different parts, namely \( \hat{A} \) and \( \hat{B} \). For simplicity, let us assume that \( \hat{C} = \hat{A} + \hat{B} \), where the exponential operator \( \exp(\hat{C}) \) is difficult to evaluate but \( \exp(\hat{A}) \) and \( \exp(\hat{B}) \) are either solvable or easy to deal with. Under such circumstances the exponential operator \( \exp(\varepsilon \hat{C}) \), with \( \varepsilon \) being a small parameter, can be approximated by the Lie-Trotter operator splitting formula:

\[
\exp(\varepsilon \hat{C}) = \exp(\varepsilon \hat{A}) \exp(\varepsilon \hat{B}) + \mathcal{O}(\varepsilon^2). \tag{A.1}
\]

This can be seen as the approximation to the solution at \( t = \varepsilon \) of the equation \( d\hat{Y}/dt = (\hat{A} + \hat{B})\hat{Y} \) by a composition of the exact solutions of the equations \( d\hat{Y}/dt = \hat{A}\hat{Y} \) and \( d\hat{Y}/dt = \hat{B}\hat{Y} \) at time \( t = \varepsilon \). Details of the Lie-Trotter splitting approximation can be found in the references [8–13]. The Lie-Trotter splitting approximation is particularly useful for studying the short-time behaviour of the solutions of evolutionary partial differential equations of parabolic type because for this class of problems it is sensible to split the spatial differential operator into several parts each of which corresponds to a different physical contribution (e.g., reaction and diffusion).

References:


