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Existence Results for a Second-order Difference Equation with Summation Boundary Conditions at Resonance

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***Abstract*—In this paper, we study the existence of solutions of a second-order difference equation with summation boundary value problem at resonance by using intermediate value theorems and Schaefer's fixed point theorem, we obtain a sufficient condition for the existence of the solution for the problem.**

***Index Terms*—positive solution, boundary value problem, fixed point theorem, cone.**

I. Introduction

THE study of the existence of solutions of multipoint boundary value problems for linear second-order ordinary differential and difference equations was initiated by Ilin [1]. Then Gupta [2] studied three-point boundary value problems for nonlinear second-order ordinary differential equations. Since then, nonlinear second-order three-point boundary value problems have also been studied by many authors, one may see the text books [3-4]. We refer the readers to [6-13] and references therein. Also, there are a lot of papers dealing with the resonant case for multi-point boundary value problems, see [14-19].

In this paper, we study the existence of solutions of a second-order difference equation with summation boundary value problem at resonance

$$\Delta^2 u(t-1) + f(t,u(t)) = 0, \qquad t \in \{1,2,...,T\}, \tag{1}$$

with summation boundary condition

$$u(0) = 0, \qquad u(T+1) = \alpha \sum_{s=1}^{\eta} u(s), \tag{2}$$

where f is continuous, $T \geq 3$ is a fixed positive integer, $\eta \in \{1,2,...,T-1\}$, $\frac{2(T+1)}{\alpha\eta(\eta+1)} = 1$. we are interested in the existence of the solution for problem (1)-(2) under the condition $\frac{2(T+1)}{\alpha\eta(\eta+1)} = 1$, which is a resonant case. Using some properties of the Green function $G(t,s)$, intermediate value theorems and Schaefer's fixed point theorem, we establish a sufficient condition for the existence of positive solutions of problem $\frac{2(T+1)}{\alpha\eta(\eta+1)} = 1$.

Let $\mathbb{N}$ be the nonnegative integer, we let $\mathbb{N}_{i,j} = \{k \in \mathbb{N}| \ i \leq k \leq j\}$ and $\mathbb{N}_p = \mathbb{N}_{0,p}$.

Throughout this paper, we suppose the following conditions hold:
(H) $f(t,u) \in C(\mathbb{N}_{T+1} \times R, R)$ and there exist two positive continuous functions $p(t), q(t) \in C(\mathbb{N}_{T+1}, R^+)$ such that

$$|f(t,tu)| \leq p(t) + q(t)|u|^m, t \in \mathbb{N}_{T+1}, \tag{3}$$

where $0 \leq m \leq 1$. Furthermore,

$$\lim_{u\to\pm\infty} f(t,tu) = \infty. \tag{4}$$

for any $t \in \mathbb{N}_{1,T}$.

To accomplish this, we denote $C(\mathbb{N}_{T+1}, R)$,the Banach space of all function u with the norm defined by $\|u\| = \max\{u(t) \mid t \in \mathbb{N}_{T+1}\}$

The proof of the main result is based upon an application of the following theorem.

Theorem 1. ([5]). *Let X be a Banach space with $C \subset X$ closed and convex. Assume that U is a relatively open subset of C with $0 \in U$ and $T : \overline{U} \to C$ is completely continuous. Then either*
(i) T has a fixed point in $\overline{U}$, or
(ii) there exist $u \in \partial U$ and $\mu \in (0,1)$ with $u = \mu Tu$.

The plan of the paper is follows. In Section 2, we recall some lemmas. In Section 3, we prove our main result. Some illustrate example are presented in Section 4.

II. Preliminaries

We now state and prove several lemmas before stating our main results.

Lemma 1. *The problem* (1)-(2) *is equivalent to the following*

$$u(t) = \sum_{s=1}^{T} G(t,s)f(s,u(s)) + \frac{u(T+1)}{T+1}t, \tag{5}$$

where

$$G(t,s) = \frac{1}{(T+1)(\alpha-1)} \times \tag{6}$$

$$\begin{cases} \alpha t(T+1-s) - \frac{1}{2}\alpha t(\eta-s)(\eta-s+1) & \\ \qquad - (T+1)(\alpha-1)(t-s), & s \in \mathbb{N}_{1,t-1} \cap \mathbb{N}_{1,\eta-1} \\ \alpha t(T+1-s) - \frac{1}{2}\alpha t(\eta-s)(\eta-s+1), & s \in \mathbb{N}_{t,\eta-1} \\ \alpha t(T+1-s) - (T+1)(\alpha-1)(t-s), & s \in \mathbb{N}_{\eta,t-1} \\ \alpha t(T+1-s), & s \in \mathbb{N}_{t,T} \cap \mathbb{N}_{\eta,T} \end{cases}$$

Proof. Assume that u(t) is a solution of problem (1)-(2) , then it satisfies the following equation:

$$u(t) = C_1 + C_2 t - \sum_{s=1}^{t-1}(t-s)f(s,u(s))$$

where C_1, C_2 are constants. By the boundary value condition (2), we obtain $C_1 = 0$. So,

$$u(t) = C_2 t - \sum_{s=1}^{t-1}(t-s)f(s,u(s)) \tag{7}$$

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From (2.3),

$$\sum_{s=1}^{\eta} u(s) = \frac{\eta(\eta+1)}{2}C_2 - \frac{1}{2}\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)y(s)$$

From the second boundary condition, we have

$$(2T+2-\alpha\eta(\eta+1))C_2 = 2\sum_{s=1}^{T}(T+1-s)f(s,u(s)) + \alpha\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)f(s,u(s)). \quad (8)$$

Since $\frac{2(T+1)}{\alpha\eta(\eta+1)} = 1$, then (10) is solvable if and only if

$$\sum_{s=1}^{T}(T+1-s)f(s,u(s)) = \frac{\alpha}{2}\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)f(s,u(s)).$$

Note that

$$\begin{aligned}u(T+1) - \sum_{s=1}^{\eta}u(s) &= (T+1)C_2 \\ &- \sum_{s=1}^{T}(T+1-s)f(s,u(s)) - \frac{\eta(\eta+1)}{2}C_2 \\ &+ \frac{1}{2}\sum_{s=1}^{\eta-1}(\eta-s)(\eta+1-s)f(s,u(s)),\end{aligned}$$

and then

$$\begin{aligned}C_2 =& \frac{\alpha}{(T+1)(\alpha-1)}\Bigg[u(T+1) - \sum_{s=1}^{\eta}u(s) \\ &+ \sum_{s=1}^{T}(T+1-s)f(s,u(s)) \\ &- \frac{1}{2}\sum_{s=1}^{\eta-1}(\eta-s)(\eta+1-s)f(s,u(s))\Bigg].\end{aligned}$$

We now use that $u(T+1) = \frac{2(T+1)}{\eta(\eta+1)}\sum_{s=1}^{\eta}u(s)$ to get

$$\frac{\alpha}{(T+1)(\alpha-1)}\left[u(T+1) - \sum_{s=1}^{\eta}u(s)\right] = \frac{u(T+1)}{T+1},$$

and

$$\begin{aligned}C_2 =& \frac{\alpha}{(T+1)(\alpha-1)}\Bigg[\sum_{s=1}^{T}(T+1-s)f(s,u(s)) \\ &- \frac{1}{2}\sum_{s=1}^{\eta-1}(\eta-s)(\eta+1-s)f(s,u(s))\Bigg] \\ &+ \frac{u(T+1)}{T+1}.\end{aligned}$$

Hence the solution of (1)-(2) is given, implicity as

$$\begin{aligned}u(t) =& \frac{\alpha t}{(T+1)(\alpha-1)}\Bigg[\sum_{s=1}^{T}(T+1-s)f(s,u(s)) \\ &- \frac{1}{2}\sum_{s=1}^{\eta-1}(\eta-s)(\eta+1-s)f(s,u(s))\Bigg] \\ &- \sum_{s=1}^{t-1}(t-s)f(s,u(s)) + \frac{u(T+1)}{T+1}t. \quad (9)\end{aligned}$$

According to (211) it is easy to show that (7) holds. Therefore, problem (1)-(2) is equivalent to the equation (7) with the function $G(t,s)$ defined in (8). The proof is completed. □

Lemma 2. *For any* $(t,s) \in \mathbb{N}_{T+1} \times \mathbb{N}_{T+1}, G(t,s)$ *is continuous, and* $G(t,s) > 0$ *for any* $(t,s) \in \mathbb{N}_{1,T} \times \mathbb{N}_{1,T}$.

Proof. The continuity of $G(t,s)$ for any $(t,s) \in \mathbb{N}_{T+1} \times \mathbb{N}_{T+1}$, is obvious. Let

$$g_1(t,s) = \alpha t(T+1-s) - \frac{1}{2}\alpha t(\eta-s)(\eta-s+1) - (T+1)(\alpha-1)(t-s),$$

where $s \in \mathbb{N}_{1,t-1} \cap \mathbb{N}_{1,\eta-1}$

Here we only need to prove that $g_1(t,s) > 0$ for $s \in \mathbb{N}_{1,t-1} \cap \mathbb{N}_{1,\eta-1}$, the rest of the proof is similar. So, from the definition of $g_1(t,s), \eta \in \mathbb{N}_{1,T-1}$ and the resonant condition $\frac{2(T+1)}{\alpha\eta(\eta+1)} = 1$, we have

$$\begin{aligned}g_1(t,s) &= \alpha t(T+1-s) - \frac{1}{2}\alpha t(\eta-s)(\eta-s+1) \\ &\quad - (T+1)(\alpha-1)(t-s) \\ &> (T+1)(t-s) - \frac{\alpha}{2} \\ &> (T+1)(t-s) - \frac{T+1}{\eta(\eta+1)} \\ &> (T+1)(t-s) - (T+1) \\ &= (T+1)(t-s-1) \\ &\geq 0,\end{aligned}$$

for $s \in \mathbb{N}_{1,t-1} \cap \mathbb{N}_{1,\eta-1}$.
Since $t > s$ and $\eta(\eta+1) \geq 2(T+1-t)$ where $T \geq 3$. The proof is completed. □

Let

$$G^*(t,s) = \frac{1}{t}G(t,s). \quad (10)$$

Then

$$G^*(t,s) = \frac{1}{(T+1)(\alpha-1)} \times \begin{cases}\alpha(T+1-s) - \frac{1}{2}\alpha(\eta-s)(\eta-s+1) \\ \quad - \frac{1}{t}(T+1)(\alpha-1)(t-s), s \in \mathbb{N}_{1,t-1} \cap \mathbb{N}_{1,\eta-1} \\ \alpha(T+1-s) - \frac{1}{2}\alpha(\eta-s)(\eta-s+1), s \in \mathbb{N}_{t,\eta-1} \\ \alpha(T+1-s) - \frac{1}{t}(T+1)(\alpha-1)(t-s), s \in \mathbb{N}_{\eta,t-1} \\ \alpha(T+1-s), s \in \mathbb{N}_{t,T} \cap \mathbb{N}_{\eta,T}\end{cases} \quad (11)$$

Thus, problem (1)-(2) is equivalent to the following equation:

$$u(t) = \sum_{s=1}^{T} tG^*(t,s)f(s,u(s)) + \frac{u(T+1)}{T+1}t, \quad (12)$$

By a simple computation, the new Green function $G^*(t,s)$ has the following properties.

Lemma 3. *For any* $(t,s) \in \mathbb{N}_{T+1} \times \mathbb{N}_{T+1}, G^*(t,s)$ *is continuous, and* $G^*(t,s) > 0$ *for any* $(t,s) \in \mathbb{N}_{1,T} \times \mathbb{N}_{1,T}$. *Furthermore,*

$$\begin{aligned}&\lim_{t\to 0} G^*(t,s) := G^*(0,s) \\ &= \frac{1}{(T+1)(\alpha-1)}\begin{cases}\alpha(T+1-s) - \frac{1}{2}\alpha(\eta-s)(\eta-s+1), \\ \quad s \in \mathbb{N}_{1,\eta-1} \\ \alpha(T+1-s), s \in \mathbb{N}_{\eta,T}\end{cases}\end{aligned} \quad (13)$$

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Lemma 4. *For any $s \in \mathbb{N}_{1,T}, G^*(t,s)$ is nonincreasing with respect to $t \in \mathbb{N}_{T+1}$, and for any $s \in \mathbb{N}_{T+1}$, $\frac{\triangle_t G^*(t,s)}{\triangle t} < 0$, and $\frac{\triangle_t G^*(t,s)}{\triangle t} = 0$ for $t \in \mathbb{N}_s$. That is, $G^*(T+1,s) \le G^*(t,s) \le G^*(s,s)$ where*

$$\begin{aligned} G^*(t,s) \le\ & G^*(s,s) \\ = \frac{1}{(T+1)(\alpha-1)} & \begin{cases} \alpha(T+1-s) - \frac{1}{2}\alpha(\eta-s)(\eta-s+1), \\ \quad s \in \mathbb{N}_{1,\eta-1} \\ \alpha(T+1-s), s \in \mathbb{N}_{\eta,T} \end{cases} \end{aligned} \tag{14}$$

$$\begin{aligned} G^*(t,s) \ge\ & G^*(T+1,s) \\ = \frac{1}{(T+1)(\alpha-1)} & \begin{cases} (T+1)(T+1-s) - \frac{1}{2}\alpha(\eta-s)\times \\ \quad (\eta-s+1), s \in \mathbb{N}_{1,\eta-1} \\ (T+1)(T+1-s), s \in \mathbb{N}_{\eta,T} \end{cases} \end{aligned} \tag{15}$$

Let

$$u(t) = tw(t). \tag{16}$$

Then $u(T+1) = (T+1)w(T+1)$, and equation (14) gives

$$w(t) = \sum_{s=1}^{T} G^*(t,s) f(s, sw(s)) + w(T+1), \tag{17}$$

Now we have

$$y(t) = w(t) - w(T+1), \tag{18}$$

Then $y(T+1) = w(T+1) - w(T+1) = 0$, and equation (19) gives

$$y(t) = \frac{1}{T+1} \sum_{s=1}^{T} G^*(t,s) f(s, s(y(s) + w(T+1))), \tag{19}$$

We replace $w(T+1)$ by any real number λ, then (21) can be rewritten as

$$y(t) = \frac{1}{T+1} \sum_{s=1}^{T} G^*(t,s) f(s, s(y(s)+\lambda)), \tag{20}$$

The following result is based on Schaefer's fixed point theorem. We define an operator T on the set $\Omega = C(\mathbb{N}_{T+1})$ as follows:

$$Ty(t) = \frac{1}{T+1} \sum_{s=1}^{T} G^*(t,s) f(s, s(y(s)+\lambda)), \tag{21}$$

Lemma 5. *Assume that $f \in C(\mathbb{N}_{T+1} \times R, R)$, $\sum_{s=1}^{T} G^*(t,s)q(s) < T+1$ and (1.5) holds. Then the equation* (20) *has at least one solution for any real number λ.*

Proof. We divide the proof into four steps.
Step I. *Continuity of T.* Let y_n be a sequence such that $y_n \to y$ in Ω. Then, for each $t \in \mathbb{N}_{T+1}$, we get

$$\begin{aligned} & \|(Ty_n)(t) - (Ty)(t)\| \\ = & \ \|\frac{1}{T+1} \sum_{s=1}^{T} G^*(t,s)[f(s, s(y_n(s) \\ & +\lambda)) - f(s, s(y(s)+\lambda))]\| \\ \le & \ \frac{1}{T+1} \| \sum_{s=1}^{T} G^*(t,s)\| \|f(s, s(y_n(s) \\ & +\lambda)) - f(s, s(y(s)+\lambda))\| \end{aligned}$$

Since $f(t, ty)$ is continuous function and from Lemma 4, it is continuous with respect to $(t,s) \in \mathbb{N}_{T+1} \times \mathbb{N}_{T+1}$, we have $\|(Ty_n)(t) - (Ty)(t)\| \to 0$ as $n \to \infty$. This means that T is continuous in Ω.
Step II. *T maps bounded sets into bounded sets in Ω.* Let us prove that for any $R > 0$, there exists a positive constant L such that for each $y \in B_R = \{y \in C(\mathbb{N}_{T+1} \times R) : \|y\| \le R\}$, we have $\|(Ty)(t)\| \le L$. Indeed, for any $y \in B_R$,we obtain

$$\begin{aligned} & \|(Ty)(t)\| \\ = & \ \|\frac{1}{T+1} \sum_{s=1}^{T} G^*(t,s) f(s, s(y(s)+\lambda))\| \\ \le & \ \frac{1}{T+1} \sum_{s=1}^{T} G^*(t,s)p(s) + \frac{1}{T+1} \sum_{s=1}^{T} G^*(t,s)q(s) \times \\ & (\|y(s)\| + \|\lambda\|)^m \\ \le & \ \frac{1}{T+1} \sum_{s=1}^{T} G^*(s,s)p(s) + \frac{(R+\|\lambda\|)^m}{T+1} \times \\ & \sum_{s=1}^{T} G^*(s,s)q(s) \\ := & \ L. \end{aligned} \tag{22}$$

Step III. *$T(B_R)$ is equicontinuous with B_R defined as in Step II.* Since B_R is bounded, then there exists $M > 0$ such that $|f| \le M$. For any $\varepsilon > 0$, there exist $t_1, t_2 \in \mathbb{N}_{T+1}, t_1 \le t_2$ such that

$$\frac{M}{T+1}\left[\frac{(t_1-1)(t_2-t_1)}{2} + \frac{t_2+1}{2} + \frac{(t_1+1)(t_1-2t_2)}{2t_2}\right] < \epsilon.$$

Then we have

$$\begin{aligned} & \|(Ty)(t_2) - (Ty)(t_1)\| \\ \le & \ \|\frac{1}{T+1} \sum_{s=1}^{T} |G^*(t_2,s) - G^*(t_1,s)||f(s, s(y(s)+\lambda))| \\ \le & \ \frac{M}{T+1}\left[\sum_{s=1}^{t_1-1} \frac{s(t_2-t_1)}{t_1 t_2} + \sum_{s=t_1}^{t_2-1}\left(1 - \frac{s}{t_2}\right)\right] \\ = & \ \frac{M}{T+1}\left[\frac{(t_1-1)(t_2-t_1)}{2} + \frac{t_2+1}{2}\right. \\ & \left. + \frac{(t_1+1)(t_1-2t_2)}{2t_2}\right] \\ \le & \ \epsilon. \end{aligned}$$

This means that the set $T(B_R)$ is an equicontinuous set. As a consequence of Steps I to III together with the Arzela'-Ascoli theorem, we get that T is completely continuous in Ω.
Step IV. A priori bounds. We show that the set

$$E = \{y \in C(\mathbb{N}_{T+1}, \mathbb{R}) \ / \ y = \mu Ty \ \text{ for some } \mu \in (0,1)\}$$

is bounded.

By Lemma 1, assume that there exist $y \in \partial B_R$ with $\|y(t)\| = R$ and $\mu \in (0,1)$ such that $y = \mu Ty$. It follows that

$$y(t) = \mu|(Ty)(t)| = \frac{\mu}{T+1}\left|\sum_{s=1}^{T} G^*(t,s) f(s, s(y(s)+\lambda))\right|$$

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and

$$\begin{aligned}
\|y(t)\| &= \left\| \frac{\mu}{T+1} \sum_{s=1}^{T} G^*(t,s) f(s, s(y(s)+\lambda)) \right\| \\
&< \frac{1}{T+1} \Bigg[\sum_{s=1}^{T} G^*(s,s) p(s) + \sum_{s=1}^{T} G^*(s,s) q(s) \times \\
&\quad (\|y(s)\| + \|\lambda\|)^m \Bigg] \\
&\leq \frac{1}{T+1} \sum_{s=1}^{T} G^*(s,s) p(s) + \frac{(R+\|\lambda\|)^m}{T+1} \times \\
&\quad \sum_{s=1}^{T} G^*(s,s) q(s) \\
&:= L.
\end{aligned} \tag{23}$$

This shows that the set E is bounded. As a consequence of Schaefer's fixed point theorem, we conclude that T has a fixed point which is a solution of problem (3)-(4). □

III. Main Results

In this section, we prove our result by using Lemmas 2.5-2.7 and the intermediate value theorem.

Theorem 2. *Assume that $(H1)$ holds. If $\sum_{s=1}^{T} G^*(s,s)q(s) < 1$, then the problem (1)-(2) has at least one solution, where*

$$G^*(s,s) = \frac{1}{(T+1)(\alpha-1)} \begin{cases} \alpha(T+1-s) - \frac{1}{2}\alpha(\eta-s)(\eta-s+1), \\ \quad s \in \mathbb{N}_{1,\eta-1} \\ \alpha(T+1-s), s \in \mathbb{N}_{\eta,T} \end{cases}$$

Proof. Since (25) is continuously dependent on the parameter λ. So, we should only investigate λ such that $y(T+1) = 0$ in order that $u(T+1) = \lambda$.

Equation (22) is rewrite as

$$y_\lambda(t) = \frac{1}{T+1} \sum_{s=1}^{T} G^*(t,s) f(s, s(y_\lambda(s)+\lambda)), \quad t \in \mathbb{N}_{T+1}. \tag{24}$$

where λ is any given real number.
Equation (26) show that there exists λ such that

$$L(\lambda) := y_\lambda(T+1) = \frac{1}{T+1} \sum_{s=1}^{T} G^*(T+1,s) f(s, s(y_\lambda(s)+\lambda)) \tag{25}$$

and we can observe that, $y_\lambda(T+1)$ is continuously dependent on the parameter λ.

To prove that there exists λ^* such that $y_{\lambda^*}(T+1) = 0$, we must to show that $\lim_{\lambda\to\infty} L(\lambda) = \infty$ and $\lim_{\lambda\to-\infty} L(\lambda) = -\infty$.

Firstly, we prove that $\lim_{\lambda\to\infty} L(\lambda) = \infty$ by supposing that $\lim_{\lambda\to\infty} L(\lambda) < \infty$ as acontradiction. Therefore there exists a sequence $\{\lambda_n\}$ with $\lim_{n\to\infty} L(\lambda) = \infty$ such that $\lim_{\lambda_n\to\infty} L(\lambda_n) < \infty$. This implies that the sequence $\{L(\lambda_n)\}$ is bounded. Since the function $f(t,ty)$ is continuous with respect to $t \in \mathbb{N}_{T+1}$ and $y \in R$, we have

$$f(t, t(y_{\lambda_n}(t)+\lambda_n)) \geq 0\ , \quad t \in \mathbb{N}_{T+1} \tag{26}$$

where λ_n is large enough, Assuming that (28) is true and using (26), we have

$$y_\lambda \geq 0\ , \quad t \in \mathbb{N}_{T+1}. \tag{27}$$

Therefore,

$$\lim_{\lambda_n\to\infty} f(t, t(y_{\lambda_n}(t)+\lambda_n)) = \infty\ , \quad t \in \mathbb{N}_{T+1}. \tag{28}$$

From (H), we get

$$\lim_{\lambda\to\infty} f(t, tu) = \infty\ , \quad t \in \mathbb{N}_{T+1}. \tag{29}$$

From (27),(30) and (31), we have

$$\begin{aligned}
&\lim_{\lambda_n\to\infty} y_{\lambda_n}(T+1) \\
&= \lim_{\lambda_n\to\infty} \sum_{s=1}^{T} G^*(T+1,s) f(s, s(y_{\lambda_n}(s)+\lambda_n)) \\
&\geq \lim_{\lambda_n\to\infty} \sum_{s=\frac{1}{4}(T-1)}^{\frac{3}{4}(T-1)} G^*(T+1,s) f(s, s(y_{\lambda_n}(s)+\lambda_n)) \\
&= \infty,
\end{aligned} \tag{30}$$

we find that this result contradicts our assumption.

We define

$$S_n = \{t \in \mathbb{N}_{T+1} \mid f(t, t(y_{\lambda_n}(t)+\lambda_n)) < 0\}.$$

where λ_n is large. Note that S_n is not empty.

Secondly, we divide the set S_n into set $\widetilde{S}_n$ and set $\widehat{S}_n$ as follows:

$$\begin{aligned}
\widetilde{S}_n &= \{t \in S_n \mid y_{\lambda_n} + \lambda_n > 0\}, \\
\widehat{S}_n &= \{t \in S_n \mid y_{\lambda_n} + \lambda_n \leq 0\}
\end{aligned}$$

where $\widetilde{S}_n \cap \widehat{S}_n = \emptyset$, $\widetilde{S}_n \cup \widehat{S}_n = S_n$. So, we have from (H) that $\widehat{S}_n$ is not empty.

In addition, we find from (H) that the function $f(t,tu)$ is bounded below by a constant for $t \in \mathbb{N}_{T+1}$ and $\lambda \in [0,\infty)$. Thus, there exists a constant $M(<0)$ which is independent of t and λ_n, such that

$$f(t, t(y_{\lambda_n}(t)+\lambda_n)) \geq M\ , \quad t \in \widetilde{S}_n, \tag{31}$$

Let

$$h(\lambda_n) = \min_{t\in S_n} y_{\lambda_n}(t)$$

and using the definitions of $\widetilde{S}_n$ and set $\widehat{S}_n$, we have

$$h(\lambda_n) = \min_{t\in\widehat{S}_n} y_{\lambda_n}(t) = -\|y_{\lambda_n}(t)\|_{\widehat{S}_n}.$$

It follows that $h(\lambda_n) \to -\infty$ as $\lambda_n \to \infty$ since if $h(\lambda_n)$ is bounded below by a constant as $\lambda_n \to \infty$, then (32) holds. Therefore, we can choose largeλ_{n_1} such that

$$\begin{aligned}
h(\lambda_n) &< \tfrac{1}{T+1} \times \\
&\quad \max\left\{-1, \tfrac{M\sum_{s=1}^{T} G^*(s,s) - \sum_{s=1}^{T} G^*(s,s)p(s)}{1-\sum_{s=1}^{T} G^*(s,s)q(s)}\right\}
\end{aligned} \tag{32}$$

for $n > n_1$. Employing (H), (26), (33), (34), the definitions of $\widetilde{S}_n$, and set $\widehat{S}_n$, for any $\lambda_n > \lambda_{n_1}$, we have

$$\begin{aligned} y_{\lambda_n}(t) &= \frac{1}{T+1}\sum_{s=1}^{T} G^*(t,s)f(s, s(y_{\lambda_n}(s)+\lambda_n)) \\ &\geq \frac{1}{T+1}\Bigg[M\sum_{s\in\widetilde{S}_n} G^*(s,s) - \sum_{s\in\widehat{S}_n} G^*(s,s)p(s) \\ &\quad - \sum_{s\in\widehat{S}_n} G^*(s,s)q(s)\|y_{\lambda_n}(s)+\lambda_n\|^m\Bigg]. \end{aligned}$$

It follows that

$$\begin{aligned} y_{\lambda_n}(t) &\geq \frac{1}{T+1}\Bigg[M\sum_{s=1}^{T} G^*(s,s) - \sum_{s=1}^{T} G^*(s,s)p(s) \\ &\quad - \sum_{s=1}^{T} G^*(s,s)q(s)\|y_{\lambda_n}(s)+\lambda_n\|^m_{S_n}\Bigg], \\ &\geq \frac{1}{T+1}\Bigg[M\sum_{s=1}^{T} G^*(s,s) - \sum_{s=1}^{T} G^*(s,s)p(s) \\ &\quad - \sum_{s=1}^{T} G^*(s,s)q(s)h(\lambda_n)\Bigg], \quad t\in S_n \end{aligned}$$

which implies that

$$h(\lambda_n) \geq \frac{1}{T+1}\left[\frac{M\sum_{s=1}^{T} G^*(s,s) - \sum_{s=1}^{T} G^*(s,s)p(s)}{1-\sum_{s=1}^{T} G^*(s,s)q(s)}\right].$$

This result contradicts (34). Thus, the proof that $\lim\limits_{\lambda\to\infty} L(\lambda) = \infty$ is done. using a similar method, we can prove that $\lim\limits_{\lambda\to-\infty} L(\lambda) = -\infty$.

Notice that $L(\lambda)$ is continuous with respect to $\lambda \in (-\infty,\infty)$. From the intermediate value theorem, there exists $\lambda^* \in (-\infty,\infty)$ such that $L(\lambda^*) = 0$, that is, $y(T+1) = y_{\lambda^*}(T+1) = 0$, which satisfies the second boundary value condition of (4). The proof is completed. □

IV. Some Examples

In this section, we give an example to illustrate our result.

Example Consider the BVP

$$\Delta^2 u(t-1) + t^2 + \frac{1}{2}u(t) = 0, \qquad t\in N_{1,4}, \tag{33}$$

$$u(0) = 0, \qquad u(5) = \frac{5}{6}\sum_{s=1}^{2} u(s). \tag{34}$$

Set $\alpha = \frac{5}{6},\ \eta = 2,\ T = 4,\ f(t,u) = t^2 + \frac{1}{2}u(t)$.
So we have

$$\frac{\alpha\eta(\eta+1)}{2(T+1)} = 1$$

and

$$f(t,tu) = t^2 + \frac{t}{2}u(t).$$

Now we take $q(t) = \frac{t}{5}$
It is easy to check that

$$\lim_{u\to\pm\infty} f(t,tu) = \pm\infty$$

and

$$\sum_{s=1}^{4} G^*(s,s)q(s) \leq \frac{1}{25}\sum_{s=1}^{4}(5-s)s = \frac{4}{5} < 1.$$

Thus the conditions of Theorem 2 are satisfied. Therefore problem (35)-(36) has at least a nontrivial solution.

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