Proceedings of the World Congress on Engineering 2015 Vol I
WCE 2015, July 1 - 3, 2015, London, U.K.

Interpolation Theorems for Weighted Sobolev Spaces

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***Abstract*—The description of Peetre interpolation space for pair of weighted Sobolev spaces with different summability dimensions is obtained**

***Index Terms*—weighted spaces, weighted Sobolev spaces, embedding, interpolation theorem, Peetre interpolation space**

I. Introduction

In this work we will give the description of Peetre interpolation spaces of weighted Sobolev spaces. At first we will say a few words about the history and development of interpolation theory of function spaces.

Interpolation theory of function spaces is a new chapter of functional analysis, which finds application in wide areas of theory of functions, as well as in other areas of mathematics. The most important fields of application of interpolation theory are: the theory of function spaces and differential and integral operators, the theory of partial differential equations, the theory of Fourier series, approximation theory in Banach spaces, integral inequalities, singular integrals, theory of multipliers.

Now we will describe the purpose of interpolation theory in Banach spaces. Let $\mathcal{A}$ be a Hausdorff linear topological space. We say that a pair $\{A_0, A_1\}$ of Banach spaces is an interpolation pair in $\mathcal{A}$ provided that A_0 and A_1 are Banach spaces continuously embedded into $\mathcal{A}$. Let $\mathcal{B}$ be an other Hausdorff linear topological space. Let $\{B_0, B_1\}$ be an interpolation pair in $\mathcal{B}$. An interpolation theory aim at associating to each interpolation pair $\{A_0, A_1\}$ a Banach space A such that $A_0 \cap A_1 \subseteq A \subseteq A_0 + A_1$ and to each pair $\{B_0, B_1\}$ a Banach space B such that $B_0 \cap B_1 \subseteq B \subseteq B_0 + B_1$ in such a way that if T is a linear operator from $\mathcal{A}$ to $\mathcal{B}$ such that the restriction of T to A_i induces a linear and continuous map from A_i into B_i, $(i = 0, 1)$ that T induces necessarily a linear and continuous operator from A into B.

Let A_i $(i = 0, 1)$ be Banach spaces. Let A_0 be continuously embedded into A_1. Then a Banach space A is said to be an interpolation space between A_0 and A_1 provided that A_0 is continuously embedded into A and that A is continuously embedded into A_1, and provided that a linear operator T is continuous from A to A whenever T is a linear and continuous operator from A_1 to A_1 which restricts a linear and continuous operator from A_0 to A_0.

For example, one can show that the space $L_p(\mathbb{R}^n)$, $p \in [1, +\infty)$, continuously embedded to $L_1(\mathbb{R}^n) + L_\infty(\mathbb{R}^n)$, that is $(L_1(\mathbb{R}^n), L_2(\mathbb{R}^n))$ and $(L_2(\mathbb{R}^n), L_\infty(\mathbb{R}^n))$ are interpolation pairs. We know that $L_2(\mathbb{R}^n)$ continuously embedded to $L_2(\mathbb{R}^n)$ and that $L_1(\mathbb{R}^n)$ continuously embedded to $L_\infty(\mathbb{R}^n)$. That is by above definition $L_p(\mathbb{R}^n)$, $1 \le p \le 2$, is an interpolation space between $L_1(\mathbb{R}^n)$ and $L_2(\mathbb{R}^n)$. And by the definition $L_p(\mathbb{R}^n)$ continuously embedded to the space $L_q(\mathbb{R}^n)$, which is an interpolation space between $L_2(\mathbb{R}^n)$ and $L_\infty(\mathbb{R}^n)$.

The classical Riesz-Thorin theorem ([1]) is well-known and widely used. It is one of the results at the basis of further developments of interpolation theory of function spaces. This theorem was formulated in 1926 by M. Riesz. An extended version of this theorem was given by G. O. Thorin in 1939 and 1948 and nowadays is known as the Convexity theorem of M. Riesz and G. O. Thorin.

The interpolation theory of Banach spaces has developed as an independent area of research in linear operator theory in Banach spaces in 60's of the last century (1958-61). In particular, we mension the works of J.-L. Lions, J. Peetre, A. P. Calderon, S. G. Krejn, E. Gagliardo. Then it was further developed in the works of E. Magenes, P. I. Lizorkin, H. Triebel, O. V. Besov and many others. Interpolation theorems of nonweighted Besov and Sobolev spaces were obtained in the works of J.-L. Lions, J. Peetre, E. Magenes, Calderon, Littlewood, Lizorkin, Nikolsky, Mityagin, Triebel, Besov and many others.

In order to analyze singular boundary value problems of mathematical physics one naturally resorts to results of embedding theory of weighted spaces of differentiable functions. Development of the embedding theory of spaces of the differentiable functions as a new direction in mathematics begun with works of S. L. Sobolev. A systematic study in general weighted spaces has begun with works of L. D. Kudryavtsev [2], [3]. Many of these works have applications in the theory of degenerate elliptic differential operators. For this reason a considerable part of these works belongs to the area of research of weighted Sobolev spaces with weights expressed as powers of the distance function to some manifolds (V. S. Guliyev, A. D. Dzhabrailov, V. P. Il'in, N. I. Imranov, I. A. Kiprianov, S. M. Nikolsky, V. G. Perepelkin, S. L. Sobolev, S. V. Uspensky, P. Grisvard, A. Kufner, B. Opic, H. Toshio and others).

Works, devoted to the interpolation theory of weighted spaces of differentiable functions, appeared at the end of 50th - 60th. The works of the mathematicians Grisvard P., Hanouzet B., Favini A., Goudjo C. were devoted to the study of weighted spaces. The first results of interpolation of weighted Sobolev spaces have been obtained in the works of the mathematicians C. Goudjo, A. Favini [4]. In those works Sobolev spaces on an interval $I = \mathbb{R}$ $(I = (0, \infty))$ with power weights (that is with weights of type $\rho(x) = (1+|x|)^\nu$, $-\infty < \nu < \infty$)) have been considered. Furthermore H.

Manuscript received March 23, 2015.

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Triebel [5, 3.2], Guliyev [6], were engaged in the theory of interpolation of Sobolev spaces with such weights, and also with weights of more general type, satisfying certain regularity conditions. Also Besov [7], Imranov [8], Lizorkin [9], Nikolsky [10], Ovchinnikov [11]-[19], Toshio [20]. The most complete and systematic description of the interpolation theory of weighted spaces of the differentiable functions with the weights of regular (special) type was given by Triebel [5]. He has brought a considerable contribution to the interpolation theory and has investigated the weighted spaces of both integer and fractional exponent in domains G of $\mathbb{R}^n$. He has considered positive weight functions of class $C^\infty(G)$, satisfying the certain growth conditions. So for example he has considered weights ρ which grow uniformly to infinity as x in G approaches the boundary G or as the modulus of x approaches infinity, and satisfies a condition as $|\nabla\rho| \leq c\rho^2$. Triebel obtained embedding and interpolation theorems for spaces of Sobolev and Besov. In particular, Triebel has considered power weights ρ^μ.

The problem of interpolation of Sobolev spaces with the weights of nonpowered character was for the first time considered by L. Kussainova. Many of these results are contained in her proceeding papers [21]-[26].

The theory of interpolation of weighted spaces of the differentiable functions with the weights of general form still remains comparatively a new section in the general theory of interpolation. A large contribution in the last decade in the development of theory of interpolation of function spaces has been done by the mathematicians E. D. Nursultanov, V. I. Burenkov, K. A. Bekmaganbetov and others [27]-[29]. They have obtained considerable results in interpolation theory.

II. Interpolation theorems

Let's state our results. Let (A_0, A_1) be an interpolation pair. The real interpolation space (Peetre space) is defined as follows. Let $0 < \theta < 1$ and $1 \leq q < \infty$. We set by definition

$$(A_0, A_1)_{\theta,q} = \Bigg\{ a : a \in A_0 + A_1,$$

$$\|u\|_{(A_0,A_1)_{\theta,q}} = \left(\int_0^\infty (t^{-\theta} K(t,a))^q \frac{dt}{t} \right)^{1/q} < \infty \Bigg\},$$

where

$$A_0 + A_1 = \{a : \ a = a_0 + a_1, \ a_0 \in A_0, \ a_1 \in A_1\},$$

$$K(t,a) = \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + t\,\|a_1\|_{A_1}), \qquad a \in A_0 + A_1 \quad (0 < t < \infty).$$

Example. Let $0 < p_0 < p_1 \leq \infty$. Let $\theta \in (0,1)$ such that $1/p = (1-\theta)/p_0 + \theta/p_1$, and $q = p$. Then one can prove that $(L_{p_0}, L_{p_1})_{\theta,p} = L_p$.

Now we introduce some notation. Let $L_p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, be the usual complex space of $p-$intergrable functions in the n-dimensional real Euclidean space $\mathbb{R}^n$ with the usual identification of functions which are equal almost everywhere. Define the two-weighted Sobolev space. Let $\rho(x)$, $\upsilon(x)$ be the weight functions given on the entire exis $\mathbb{R} = (-\infty, \ \infty)$. Let $m \geq 1$ be an integer. Let $1 < p < \infty$. We define the Sobolev space $\mathcal{W}_p^m(\rho;\upsilon)$ as follows:

$$\mathcal{W}_p^m(\rho;\upsilon) = \Bigg\{ u \in L^1_{loc}(\mathbb{R}) : \ \left\|u; \mathcal{W}_p^m(\rho,\upsilon)\right\| = \left\|u^{(m)}; L_p(\rho)\right\| + \|u; L_p(\upsilon)\| = \left(\int_{-\infty}^\infty |u^{(m)}(x)|^p \rho(x) dx \right)^{1/p} + \left(\int_{-\infty}^\infty |u(x)|^p \upsilon(x) dx \right)^{1/p} < \infty \Bigg\}. \quad (1)$$

Let $W_p^m(\rho;\upsilon)$ be a two-weighted Sobolev space defined as completion of the space $C^\infty\mathcal{W}_p^m(\rho;\upsilon) = C^\infty \cap \mathcal{W}_p^m(\rho;\upsilon)$ with respect to the norm $\left\|\cdot;\mathcal{W}_p^m(\rho,\upsilon)\right\|$ of (1). $W_p^m(\upsilon)$ denotes $W_p^m(1;\upsilon)$.

So, we will give the description of Peetre interpolation spaces when A_0 and A_1 are weighted Sobolev spaces defined on $\mathbb{R}$.

Let $Q_d(x)$ be an open interval of length d centered at x

$$Q(x) = Q_d(x) \equiv \left(x - \frac{d}{2}, \ x + \frac{d}{2} \right).$$

Definition. *We say that d is function of length of edge for a cube if d is a positive bounded function from $\mathbb{R}$ to $(0,+\infty)$ satisfying condition: there exists $\eta \in (0,1)$ such that*

$$\eta < \frac{d(y)}{d(x)} < \eta^{-1} \text{ if } x \in \mathbb{R} \text{ and } y \in Q(x). \quad (2)$$

Now introduce some examples of weights satisfying the condition.

Examples. *The following functions satisfy the condition (2):*
1) $d(x) \equiv 1$.
2) $d(x) = \min\{x-a, \ b-x\}$ on $G = (a,b)$.
3) $d(x) = (1+|x|^\mu)^{-1}$.

Next we turn to define the weighted Besov space which is necessary to store our results on the weighted Sobolev spaces.

Let $0 < s < \infty$, $1 \leq p < \infty$. We define a two-weighted space of Besov as follows:

$$B_p^s(\rho,\upsilon) \stackrel{def}{=} \left\{ u \in L^1_{loc}(\mathbb{R}) : \left\|u; B_p^s(\rho,\upsilon)\right\| < \infty \right\},$$

where

$$\left\|u; B_p^s(\rho,\upsilon)\right\| \equiv \left\{ \int_0^\infty h^{-(s-k)p-1} \int_{\mathbb{R}} \left| \triangle^m(h) u^{(k)}(x) \right|^p \rho(x) dx dh \right\} + \|u; L_p(\upsilon)\|.$$

Here $m, \ k$ – two integer non-negative numbers, satisfying the condition $m + k > s > k$. Let $\Delta^m(h)$ be the difference operator of order m $(m = 1, 2...)$ with step h defined as

$$\Delta^m(h) f(x) = \sum_{r=0}^m (-1)^{m-r} \frac{m!}{r!(m-r)!} f(x+rh).$$

One can prove that the above definition is independent of our specific choice of m and k as long as the above conditions are satisfied. We denote by $B_p^s(\upsilon)$ the space $B_p^s(1,\upsilon)$.

ISBN: 978-988-19253-4-3
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Class $(\Pi_{m,p})$. *We say that a weight* υ *on* $\mathbb{R}$ *satisfies condition* $(\Pi_{m,p})$ *with respect to the function* d, *and write* $\upsilon \in (\Pi_{m,p})$, *if there are numbers* $0 < \delta,\ \gamma < 1$ *such that*

$$d(x)^{mp-1}\inf_{e}\int_{Q\setminus e}\upsilon(y)dy \geq \gamma$$
$$\forall Q = Q(x) \ \ \text{for almost all } x \in \mathbb{R} \quad (3)$$

In (3) the infimum is taken over all measurable subset e *of* Q *such that* $|e| \leq \delta\,|Q|$.

Example 1. *The weight* $\upsilon = 1$ *satisfies condition* $(\Pi_{m,p})$ *with respect to* $d(x) = 1$. *Indeed,*

$$d(x)^{mp-1}\inf_{e}\upsilon\left(Q(x)\setminus e\right) \geq 1-\delta = \gamma > 0, \ \text{for all } \delta \in (0,1).$$

Example 2. *Let* $\varepsilon \in (0,1]$. *Let* $\upsilon(x) = |x - x_0|^{\mu}$, $0 < \mu < 1$, *if* $|x - x_0| < 1$ *and* $\upsilon(x) = \varepsilon$ *if* $|x - x_0| \geq 1$. *Then weight* υ *satisfies condition* $(\Pi_{m,p})$ *with respect to the function*

$$d(x) = \begin{cases} (x - x_0)^{\gamma}, & |x - x_0| < 1, \\ 1, & |x - x_0| \geq 1, \end{cases}$$

where $\gamma = \mu/mp$.

Example 3. *Let* G *be a domain with nonempty boundary,*

$$d(x) = \left\{ \begin{array}{c} 1/(2\sqrt{n}), \ \text{if } \sigma(x) \geq 1, \\ \sigma(x)/(2\sqrt{n}), \ \text{if } \sigma(x) < 1. \end{array} \right.$$

Function $\upsilon(x) \equiv e^{1/\sigma(x)}$ $(\sigma(x) = dist(x, \partial G))$ *satisfies condition* $(\Pi_{m,p})$ *with respect to the function* $d(x)$.

Theorem 1. Let $0 < m_1 < m_0$ be integers, $1 < p_0 \leq p_1 < \infty$, $0 < \theta < 1$, $s = (1-\theta)m_0 + \theta m_1$, $1/p = (1-\theta)/p_0 + \theta/p_1$. Let $\upsilon_i(x)$ $(i = 0, 1)$ be weights on $\mathbb{R}$, satisfying the following conditions:

1. $\upsilon_i \in \Pi_{m_i,p_i}$ with respect to function $d(x)$.
2. $K_i = \sup\limits_{x\in\mathbb{R}} d(x)^{m_i - 1/p_i}\left(\upsilon_i\left(Q(x)\right)\right)^{1/p_i} < \infty$.

Then

$$\left(W^{m_0}_{p_0}(\upsilon_0),\ W^{m_1}_{p_1}(\upsilon_1)\right)_{\theta,p} = B^s_p(d(\cdot)^{-sp}).$$

Theorem 2. Let $0 < m_1 < m_0$ be integers, $1 < p_0 \leq p_1 < \infty$, $0 < \theta < 1$. Let $\upsilon_i = \rho_i d(\cdot)^{-m_i p_i}$ $(i = 0, 1)$, ρ_i be weights on $\mathbb{R}$ satisfying the following conditions. There exist constants $K > 0$, $b_i > 0$ $(i = 0, 1)$ such that:

1) $b_i^{-1} \leq \rho_i(y)/\rho_i(x) \leq b_i$ if $y \in Q(x)$ for all $x \in\mathbb{R}$ (bounded oscillation condition);

2) $d(x)^{m_0 - m_1}\rho_1(Q(x))^{1/p_1} \leq K\rho_0(Q(x))^{1/p_0}$ for all $x \in\mathbb{R}$.

Then the equality

$\left(W^{m_0}_{p_0}(\rho_0, \upsilon_0),\ W^{m_1}_{p_1}(\rho_1, \upsilon_1)\right)_{\theta,p} = B^s_p(\rho_\theta, \upsilon_\theta)$ holds,where

$\rho_\theta^{1/p} = \rho_0^{(1-\theta)/p_0}\rho_1^{\theta/p_1}$, $\upsilon_\theta = \rho_\theta d(\cdot)^{-sp}$, $s = (1-\theta)m_0 + \theta m_1$, $1/p = (1-\theta)/p_0 + \theta/p_1$.

In this Theorem the basic condition imposed on the weights is a bounded oscillation condition. This condition can be a consequence of any properties of behaviour of the gradient if the weight ρ is differentiable. But it is not such a rigid condition. Considering for simplicity one-dimensional case, we can obtain such a bounded oscillation condition from the condition $\max\limits_{Q(x)}|\rho'| \leq K\frac{\rho(x)}{d(x)}$, a condition which comes up in the analysis of weighted spaces.

Example 1. From the Theorems in case $\upsilon_0(x) = 1 = \upsilon_1(x)$ and $\rho_0(x) = 1 = \rho_1(x)$ we obtain the description of the spaces $\left(W^{m_0}_{p_0}(\mathbb{R}), W^{m_1}_{p_1}(\mathbb{R})\right)_{\theta,p}$.

Example 2. If in the Theorem 2 we assume that weights $\rho_i(x) = (1 + |x|)^{\mu_i}$, $i = 0, 1$, and that $d(x) \equiv 1$, then we obtain a one-dimensional result of H.Triebel.

Example 3. If in the example 2 $\mu_0 = \mu_1 = 0$ then we obtain the known interpolation equality: $\left(W^{m_0}_{p_0}(\mathbb{R}),\ W^{m_1}_{p_1}(\mathbb{R})\right)_{\theta,p} = \left(W^{m_0}_{p_0},\ W^{m_1}_{p_1}\right)_{\theta,p} = B^s_p(\mathbb{R})$.

Remark. It's easy to obtain the description of Petree unterpolation spaces for the following pairs of spaces on the real axis $\mathbb{R}$:

$$(W^m_{p_0}(\upsilon), L_{p_1}), (W^m_{p_0}(\upsilon_0), L_{p_1}(\upsilon_1)).$$

Also we can obtain embedding theorems and the description of the Peetre interpolation spaces for two-weighted Sobolev spaces in a arbitrary domain in $\mathbb{R}^n$, was considered in our work too.

The results we have presented have been obtained with the use of some theorems of Kusainova (the covering theorem and the partition of unity theorem). L. Kussainova has developed in detail a new approach to discretization of spaces, which is a main problem in the description of interpolation spaces ([21]). She gave in fact a new method of discretization which allowed to extend her class of weights. Consequently, she could obtain new interpolation theorems for weighted Sobolev spaces for her new class of weights.

What we present here is devoted to the problem of interpolation for Sobolev spaces with a new class of weights. Our results generalize those of L. Kussainova. We prove our interpolation theorems with the use of the discretization method of Kussainova . The classical non-weighted interpolation theorems follow from such theorems as a consequence. Moreover, many known weighted interpolation theorems with weights described above, and, in particular, with weights of the form $(1 + |x|^{\eta})^{\nu}$ follow from our theorems (see [4], [5], [33]). Note that part of presented results was obtained in ([34]) but here we give they in more general sence.

III. Conclusion

The main aim of interpolation theory is to obtain a construction of interpolation spaces in explicit form. There are two interpolation methods of construction of interpolation spaces: the so-called real method and the complex method. Interpolation spaces built by the real method are called the Peetre spaces. We consider this method because it allows to obtain describe the properties of many function spaces. The method has many applications in function theory. For this reason the Peetre spaces are interesting and are worthy.

Weighted spaces quite naturally arise in theory of non-weighted function spaces. Also in applications of function theory, for example, in the theory of boundary value problems for partial differential equations. That is why the interpolation of weighted Sobolev spaces is important and actual.

There has been a recent interest in considering generalizations of spaces of differentiable functions, and in particular, in considering the interpolation of Sobolev space with several weights (H. Triebel, O. V. Besov, V. I. Ovchinnikov, V. I. Burenkov, S. K. Vodop'yanov, J. G. Resetnyak, Genebashvili and others). Also, the localization methods connected with the use of singular integrals and differential operators has attracted a special attention in the last decades (E. Stein, Kokilashvili, M. L. Goldman, O. V. Besov, M. Otelbaev, R. O. Oynarov, L. K. Kussainova, A. Kufner and others). The interpolation of multi-weighted spaces of smooth functions with the weights of general form (in particular, Sobolev spaces) still remains in the future tasks and undoubtedly can have further development.

The local inequalities used in the work and our difference embedding theorems ([37]) may give an origin to new researches in the difference interpolation theory. Also this results can be applied to other problems in difference theory, as in the works ([35], [36]).

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ISBN: 978-988-19253-4-3
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)