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A General Attribute Diffusion Mechanism on Social Networks

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***Abstract*—Diffusion is a common phenomenon in physical, biological or social sciences, but it is not trivial to be analytically described when commodity or information is spread through a social network in our daily life. This work proposed a mathematical model to trace the change of a target attribute under a diffusion mechanism. Such model is applied to a commodity sharing game where participants could donate their tokens to each other in a defined network relationship. The diffusion and decay rates of the model govern respectively the flow directions and magnitude decreases towards the steady state.**

***Index Terms*—social network, diffusion, Markov chain.**

I. Introduction

DONATIONS is an important part of world economy. Many universities, health services and non-profit organizations relied on donations to remain viable. There were some social, economical and psychological factors that were known to have influence on people's decision-making process of charitable donations [1], [12], [18]. For examples, charitable donation can strongly be influenced by comments from people's social groups, like employer recommendations and colleague suggestions. Furthermore, organizations provide services like medical care, babysitting and teaching to attract individuals to increase investments. Psychologically, [13] found that former patients were likely to make donations to hospitals as an expression of gratitude. In addition, there are other factors, including personal attributes like age, income, marital status and education that have great impacts on charitable giving behavior. In terms of the recognition of a charity, people are likely to make their charitable decision on the reputation of a charity. [11] found the gender difference matters that women are more likely to donate than men. People aged 45 to 54 were, on average, more likely to donate. [3] found that people who received higher education than a high school degree were more likely to make donations than those with lower education. Donors with large income are likely to make the largest average donations. See [16] for a more complete description.

Commodity and/or information diffusion exists among people in the modern society through their relationships, like classmates, colleagues or friends in social media. Such mechanism is important in understanding the social networks via a general frameworks for descriptive and analytical purposes. There have been theories on diffusion processes in physical sciences, and one of these famous theories is the reaction-diffusion (RD) process. It is used to model phenomena as diverse as chemical reactions, population evolution, epidemic spreading and other spatially distributed systems. The RD process is a dynamical system defined by many kinds of statess, which diffuse stochastically and interact among them according to a fixed set of diffusion mechanisms. A detailed introduction on the RD process is referred to [5], [4], [6], [7], [8], [14], [15], [17].

Inspired by a token donation game developed in [2], we develop a mathematical model that describes the diffusion mechanism of the commodity among players in a defined network relationship in this paper. The diffusion mechanism is similar to the RD process, in which each player holds different amount of tokens, and the donation (mediated by diffusion) takes place among players who are connected by edges (thus representing tie matrix) in a predefined network. Such mechanism includes several calibration parameters that decide how the tokens are distributed towards the steady states. [2] provided a detailed explanation on the general setup of the physical experiment and thus we omit here. Readers who are interested in its detailed setup and observational studies can refer to [2].

The paper is organized as follows. Sections 2 and 3 provides definitions and theorems that are related to the social networks and our diffusion algorithm. A mathematical model is built, its implicit idea is described and its complexity is calculated afterwards. Section 4 studies how the calibration parameters affect the eventual outcomes of the algorithm. We draw our concluding remarks in the last section.

II. Initialization of the Experiment

We project the token donation game into a network diffusion model as follows. (1) Each player in the game is considered as a node in a network and we denote n as the number of players in the game or the number of nodes of a network. (2) A connective relationship between two players is described as a link between two nodes in the network. The whole connective relationship is thus described as a tie matrix T (dimension $n \times n$) of a network. If there is a connective relationship between two players, the corresponding entry of T is 1, or otherwise 0. (3) The randomly assigned amount of tokens of all players in the initial round is recorded in a token vector $M(0)$ (dimension $n \times 1$). (4) The round number is denoted as t and thus the token vector at round t is $M(t)$. In addition, due to limited amount of playing time, it usually sets a maximum number of rounds for a game and denoted as t_{max}.

It is clear that every player's generosity to donation is different. In our model, we define a generosity constant g to

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each player as a representation of its personal generosity. It is scaled from 0 to 1. A player with $g = 0$ is ultimately selfish and never shares any tokens to any players, while a player with $g = 1$ is ultimately generosity and shares all tokens to his or her connected players in a round. It is also reasonable that every player's fatigue to the continuation of the game is different. Some players may feel excited about the game after many rounds, but others may feel bored and do not want to play anymore. In our model, we define a tolerance constant r to each player as a representation of its personal tolerance to the game. It is a nonnegative number from 0 to infinity. A player with $r = 0$ feels the game is ultimately exciting and never stops playing the game. A player with $r = \infty$ feels the game is ultimately boring and gives up the game even before the game started. It is unrealistic to have players with $r = \infty$ in reality, so we only expect players with low tolerance to have large r. According to the game rule described in the initial part of the section, the game stops when there is no player to share tokens with one another anymore. It is equivalent to setting a threshold, denoted as ϵ, in a simulation of a mathematical model.

III. Some Mathematical Details in the Mechanism

A. Different Player's Viewpoint on Tokens

Due to different generosity for different players, they may feel differently on the tokens owned by theirs and their neighbors'. Assume that a player X with generosity g_X is connected to his neighbor Y with generosity g_Y in T. At round t, X and Y have $M_X(t)$ and $M_Y(t)$ respectively. We express the adjusted token amount of X from Y's viewpoint as

$$M_{XY} = e^{(\ln(M_X(t)) + \frac{\ln(M_X(t)) - \ln(M_Y(t))}{\ln(M_Y(t))})} \times (1 - g_Y). \quad (1)$$

The token amount has to be non-negative, which is guaranteed by the exponential function. Moreover, if $M_X(t) = M_Y(t)$, (1) is reduced to a simpler formula $M_{XY} = M_X(t) \times (1 - g_Y)$.

B. Transition Matrix and Columnwise Normalization

In the traditional definition of a transition matrix from [9]. One of its properties is that its overall entry sum equals to 1, which does not fit in our diffusion process. Instead, we need a matrix that the sum of entries in each column is 1. Given a matrix $H = [h_{ij}] \in R^{n \times n}$, we define a columnwise normalization on H that leads to a new matrix P such that $P = \frac{(h)_j}{H \bullet j}$. where $(h)_j$ is the j-th column vector of H and $H \bullet j \equiv \Sigma_j h_{1j} + h_{2j} + \cdots + h_{nj}$, for $j = 1, 2, \ldots, n$. Then the transition matrix we refer to is not the one from the traditional definition, but the one from the definition of P with columnwise normalization. Then we can express the update of the token vector in each round via Theorem 2.4 of [9] rephrased below:

Theorem. *If P is the transition matrix of a Markov chain, then the state vector $M(k+1)$, at the $(k+1)$th period, can be devised from the state vector $M(k)$, at the kth period, as $M(k+1) = PM(k)$. Note that the transition matrix P changes according to the the amount of each person's tokens at a certain period.*

C. A Theorem on the Observed and Adjusted Token Difference

The following theorem describes the relationship between the observed token amount of players and their adjusted token amount in each player's mind.

Theorem. *Given two nodes X and Y, the amount tokens of $M_X(t)$ and $M_Y(t)$, generosity coefficients g_X and g_Y. If $M_X(t) > M_Y(t)$, then it implies $M_{XX} - M_{YX} > 0$ and $M_{YY} - M_{XY} < 0$, where M_{XX}, M_{YY}, M_{XY} and M_{YX} are defined in the subsection A.*

D. Token Difference Matrix with Negative Value Adjustment

We first rewrite the token adjustment equation into matrix condensed form. $R = \{\exp(B + (B - C) \circ \{[(C \circ I_n)^{-1} E_n] E_n^\top\})\} \circ (E_n E_n^\top - g E_n^\top)$, where $B \equiv E_n (\ln M(t))^\top$, $C \equiv (\ln M(t)) E_n^\top$, $E_n \in R^{n \times 1}$ is a column of 1, I_n is an $n \times n$ diagonal matrix with diagonal entries 1, and $\circ$ represents the Hadamard product. Then the difference of each player's token is $Q = R_0 - R$, where $R_0 = [M(t) \circ (E_n - g)] E_n^\top$.

A positive value in the entries of Q represents that the donating player has more tokens than the receiving player after viewpoint adjustments, thus it is normal that the donating player will donate some tokens to the receiving player. However, a negative value in the entries of Q requires some refinements because it is not allowed to have a negative value in the donation process. From the observational studies in [2], we still observe some donations from the donating players who have less tokens than the receiving players, but the amount is very small. Therefore, in order to reflect these small donations, we adjust our matrix $\hat{Q} = Q_m + (|Q_n|/5)$, where $Q \equiv Q_m + Q_n \equiv [q_{ij}]$, $Q_m = [q_{ij}] \geq 0$ and $Q_n = [q_{ij}] < 0$.

In addition, unless the network is a complete graph with all connections among players, there are some nonexist linkages between some players which are recorded in the tie matrix T. We do not need to consider these unconnected links in the token difference matrix, so $S_1 = \hat{Q}^\top \circ T$.

E. From the Token Difference Matrix to Transition Matrix

The values in every column of S_1 represent the unnormalized weights of the donation from the donating player to the receiving player. To change these weights into percentage scale, we have

$$S_2 \equiv \frac{(s_1)_j}{S_1 \bullet j}, \quad (2)$$

where $S_1 \equiv [(s_1)_{ij}]$ and $(s_1)_j$ is the j-th column vector of S_1, and the denominator $S_1 \bullet j \equiv \Sigma_j (s_1)_{1j} + (s_1)_{2j} + \cdots + (s_1)_{nj}$, for $j = 1, 2, \ldots, n$. Each column of S_2 lists the proportion how a player divides his donating tokens to his neighbors. Finally, we combine S_2 with the token ratio that a player keeps his tokens for himself,

$$P = S_2 \circ (E_n g^\top) + (I_n - D), \quad (3)$$

where $D \equiv \mathrm{diag}(g_1, g_2, \ldots, g_n)$. Each column of P lists the proportion how a player divides his overall tokens in a round. The diagonal entries represent the proportion of tokens a player keeps while the off-diagonal entries represent the proportion of tokens a player donates to his connected neighbors. Therefore, P is the transition matrix and the iteration step in the algorithm continues via $M(t+1) = PM(t)$.

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IV. A General Attribute Diffusion Mechanism

A. The Complete Algorithm

We summarize the mathematical model as follows. We assume that there are n nodes connected to donation and the linkage is described as a tie matrix T. The game starts with an initial token vector $M(0) \in R^{n\times 1}$. All nodes have some attributes of generosity $g \in R^{n\times 1}$ and tolerance $r \in R^{n\times 1}$. The game will end at either $t = t_{max}$ or $M(t)$ reaches the steady state with respect to a threshold ϵ. Then the algorithm for our proposed mathematical model is stated as follows.

Algorithm 1 (A General Attribute Diffusion Mechanism)

Input: Token vector $M(0) \in R^{n\times 1}$, generosity vector $g \in R^{n\times 1}$, ϵ, tolerance vector $r \in R^{n\times 1}$ and tie matrix $T \in R^{n\times n}$;
Set $t = 0$;
Do until convergence:
  Step 1. Compute the amount of each nodes' tokens depending on oneself's and everyone's viewpoints step by step using (1) to get R and R_0;
  Step 2. Calculate the difference matrix Q;
  Step 3. Get the $\hat{Q}$ if negative value adjustment applies;
  Step 4. Consider T on $\hat{Q}$ to get S_1;
  Step 5. Columnwise normalize S_1 into S_2 using (2);
  Step 6. Combine g and S_2 to obtain the transition matrix P via (3);
  Step 7. Compute $M(t+1) = PM(t)$;
    Stop when $\|M(t+1) - M(t)\| \le \epsilon$,
    Set $t \leftarrow t+1$, $M(t) \leftarrow M(t+1)$ and $g \leftarrow ge^{-rt}$;
End Do
Output: Token vector M at steady state;

B. Demonstration via the Four-Player Example

We use the first step of the four-player example in section III to demonstrate how Algorithm 1 works. Recall that there are four players A, B, C and D. They all connect to each other except B and D. They have 70, 30, 25 and 75 initial tokens respectively. We assume that their generosities to one another are "measured" via sociological methods and they are 0.7, 0.5, 0.3 and 0.8 respectively. Moreover, their tolerances to the continuation of the game are also "measured" and they are 0.1, 0.5, 0.8 and 0.3 respectively. Mathematically speaking, we have $n = 4$, $M(0) = (70, 30, 25, 75)$, $g = (0.7, 0.5, 0.3, 0.8)$, $r = (0.1, 0.5, 0.8, 0.3)$ and T is given. Here are the outcomes of each step of Algorithm 1.

Step 1. Compute B and C firstly then calculate the amount of each person's tokens depending on someone's mind to get R such that

$$R = \begin{pmatrix} 21.0000 & 7.3727 & 5.8859 & 22.8684 \\ 44.9012 & 15.0000 & 11.8476 & 49.0943 \\ 67.4704 & 22.2238 & 17.5000 & 73.8559 \\ 13.7781 & 4.8527 & 3.8767 & 15.0000 \end{pmatrix}.$$

Step 2. Calculate the difference of each two people's tokens to get the difference matrix Q

$$Q = \begin{pmatrix} 0 & 13.6273 & 15.1141 & -1.8684 \\ -29.9012 & 0 & 3.1524 & -34.0943 \\ -49.9704 & -4.7238 & 0 & -56.3559 \\ 1.2219 & 10.1473 & 11.1233 & 0 \end{pmatrix}.$$

Step 3. There are six negative values in Q and negative value adjustments are made

$$\hat{Q} = \begin{pmatrix} 0 & 13.6273 & 15.1141 & 0.3737 \\ 5.9802 & 0 & 3.1524 & 6.8189 \\ 9.9941 & 0.9448 & 0 & 11.2712 \\ 1.2219 & 10.1473 & 11.1233 & 0 \end{pmatrix}.$$

Step 4. Consider T on $\hat{Q}$ to obtain S_1:

$$S_1 = \begin{pmatrix} 0 & 5.9802 & 9.9941 & 1.2219 \\ 13.6273 & 0 & 0.9448 & 0 \\ 15.1141 & 3.1524 & 0 & 11.1233 \\ 0.3737 & 0 & 11.2712 & 0 \end{pmatrix}.$$

Step 5. Columnwise normalize S_1 into S_2

$$S_2 = \begin{pmatrix} 0 & 0.6548 & 0.4500 & 0.0990 \\ 0.4680 & 0 & 0.0425 & 0 \\ 0.5191 & 0.3452 & 0 & 0.9010 \\ 0.0128 & 0 & 0.5075 & 0 \end{pmatrix}.$$

Step 6. Combine g and S_2 to obtain the transition matrix P

$$P = \begin{pmatrix} 0.3000 & 0.3274 & 0.1350 & 0.0792 \\ 0.3276 & 0.5000 & 0.0128 & 0 \\ 0.3634 & 0.1726 & 0.7000 & 0.7208 \\ 0.0090 & 0 & 0.1522 & 0.2000 \end{pmatrix}.$$

Step 7. Calculate the next token vector $M(t+1)$

$$M(t+1) = \begin{pmatrix} 40.1360 \\ 38.2534 \\ 102.1756 \\ 19.4350 \end{pmatrix}.$$

C. Operation and Memory Counts

The operation and memory counts of Algorithm 1 for the tth iteration are summarized in Table I below. In the second column, we assume that $X \in R^{n\times n}$ is a matrix and $Y \in R^{n\times 1}$ is a vector, requiring $c_f n$ flops to evaluate $X^{-1}Y$, where c_f is a constant independent of n. In the third column, the memory requirement in terms of the number of variables is recorded. c_m is a constant independent of n. At iteration $t+1$, most of the work is done in the computation of R and P, for which R has to be computed using operation of the logarithm and exponential functions and P is to do proportion of token distribution.

We execute the operation and memory counts for the four-player example as a demonstration. We consider a social network with four nodes and the connection is described in a tie matrix T. Following Algorithm 1, the operation counts are: (1) the flops of B and C are 16 each, so the total flops in step 1 is 32; (2) the operation counts of R_0 and R are 24 and $144 + 4c_f$ respectively, and the flops of $Q = R_0 - R$ is 16, so the total flops in step 2 is $184 + 4c_f$; (3) the operation counts of $\hat{Q}$ is the number of negative elements of Q, so the total flops in step 3 is 6; (4) Since S_1 is from the Hadamard product, the total flops in step 4 is 16; (5) the total flops to obtain S_2 via columnwise normalization in step 5 is 28; (6) To obtain P from S_2, it involves an addition on two Hadamard products, so the total flops in step 6 is 96; (7) The total flops in the last step is 28. Thus the total number of operation counts for the four-player example is $390 + 4c_f$. We also have the memory counts of Algorithm 1 to be $16c_m$, which are all from the calculation of R.

TABLE I
OPERATION AND MEMORY COUNTS

Computation	Flops	Memory
B, C	$2n^2$	$-$
R_0,R, Q	$11n^2 + (2 + c_f)n$	$c_m n^2$
$\hat{Q}$	$n(n-1)/2$	$-$
S_1	n^2	$-$
S_2	$2n^2 - n$	$-$
P	$6n^2$	$-$
$M(t+1)$	$2n^2 - n$	$-$
Total	$(49n^2 + (2c_f - 1)n)/2$	$c_m n^2$

V. CONCLUSION

Inspired by a token donation game performed in [2], this paper suggests a mathematical algorithm for the general attribute diffusion mechanism on a network. It consists of an initialization and an iterative loop, where the attributes of the nodes flow within a network of defined structure under some mechanisms. In this paper, the token donation game is served as an example. It consists of a seven-step iterative algorithm that calculates the token amount of each player in every round of the game. Initially, the game assigns a network structure of attribute diffusion among players, the initial amount of tokens for each player and a threshold for game termination if the game continues to proceed overtime. Sociological methods are used to measure every player's generosity to share tokens to their connected neighbors and tolerance to the continuation of the game. The algorithm starts with calculating the token amount adjusted from other players' viewpoints. The token difference matrix is then obtained and columnwise normalized, and the columns of the resulting matrix show the weights of how each player shares their tokens. A transition matrix is obtained afterwards when the generosity vector and the normalized token difference matrix are combined. Such matrix is then used to generate the next token vectors as the game goes on. In addition to the algorithm itself, this paper also provides the operation and memory counts of the algorithm in terms of the number of the players.

Although our algorithm successfully simulates the diffusion behavior of the token owned by each player in the token donation game, it may require some potential improvements that are still under investigation. For example, it is obvious that unlike a player's generosity, the tolerance of the game continuation of a player is not a constant, but a baseline constant plus a function through iterations. This function is affected by some potential factors including environmental factors (temperature, humidity, etc), human factors (willing to win the game, worth of the presents, etc), psychological factors (winning/losing condition, etc) and many others. It remains a research question to social scientists to quantify this functional. From our own experience, the estimate from our algorithm works in the first few rounds when every player plays seriously, but the estimates become erratic when some players start to play randomly. Mostly due to the lack of wish to continue at an almost guaranteed losing situation.

Token donation is just an illustrative example to the general attribute diffusion. In fact, the similar algorithm can be used in other scientific researches such as information spread in the social media analysis, investment flows in the stock market analysis, and so on. With a well-defined mechanism proposed by expert opinions, such computer experiment greatly reduces the experimental resource in terms of time and cost because a "near-to-free" computer node is used to substitute a relatively high-paid human subject to conduct the experiment. In addition, it is sometimes quite unrealistic and inefficient to gather a large group of human subjects to perform an experiment that lasts for several minutes, when it can certainly be done in a computer experiment. Therefore, the computer experiment opens a new door for the practitioners to investigate in some phenomena in social and economical sciences when their relative experiments are too expensive in resources or even unrealistic to perform.

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