Proceedings of the World Congress on Engineering 2015 Vol II
WCE 2015, July 1 - 3, 2015, London, U.K.

Stochastic Models for Assets Allocation under the Framework of Prospect and Cumulative Prospect Theory

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***Abstract*—This paper examines the problem of choosing the optimal portfolio for an investor with asymmetric attitude to gains and losses described in the prospect theory of A. Tversky and D. Kahneman. We consider the portfolio optimization problem for an investor who follows the assumptions of the prospect theory and the cumulative prospect theory under conditions on the stochastic behavior both of the portfolio price and the discount factor.**

***Index Terms*—optimal portfolio selection; prospect theory; geometric brownian motion**

I. Introduction

One of the classic problems of the portfolio investment theory is the following one: for a given set of assets with the known prices and distribution function of returns, to find an optimal portfolio. Portfolio is a set of assets with weights, the sum of which is equal to 1 (the budget constraint).

The classical theory of portfolio investment considers an investor with a concave utility function u. Let x_T be the (random) price of the portfolio at time $t = T$, and w be the wealth of the investor at time $t = 0$. Then the problem of finding the optimal portfolio can be represented as follows:

$$E_0(u(x_T)) \to \max \tag{1}$$

under the constraint

$$E_0(mx_T) = w, \tag{2}$$

where $E_0(u(x_T))$ is the expected value (at the time $t = 0$) of utility $u(x_T)$, the maximum is taken over all state of nature at time T, m is a discount factor.

In this paper we consider the problem of finding the optimal portfolio for an investor with asymmetric attitudes to gains and losses described in the prospect theory of A. Tversky and D. Kahneman [1]. Their paper contains a number of examples and demonstrations showing that under the conditions of laboratory experiments people systematically violate the predictions of expected utility theory. Moreover, they proposed a new theory — the prospect theory which can explain the behavior of people in decision-making under risk in those experiments in which the traditional theory of expected utility failed. Cumulative prospect theory (CPT) was proposed in [2] and is the further development of prospect theory. The difference between this version and the original version of prospect theory is that cumulative probabilities are transformed, rather than the probabilities themselves. Modern economic literature considers the cumulative prospect theory as one of the best models explaining the behavior of the players, the investors in the experiment and in decision-making under risk.

The paper [3] shows that the prospect theory can resolve a number of decision making paradoxes, but the author notes that it is not a ready-made model for economic applications. Nevertheless, recent years show increasing interest in the problems lying in the intersection of prospect theory and portfolio optimization theory. It should be noted, that due to the computational difficulties connected to the complexity of the numerical evaluation of the CPT-utility, there are not so many works devoted to the portfolio optimization problem under the framework of both prospect theory [4], [5] and cumulative prospect theory [6], [7], [8], [9], [10]. While the papers contain some numerical results, only simple cases (2-3 artificially created assets) of the portfolio selection problem are considered. Besides, most of the papers are based on the assumption that testing data are normally distributed. However, it is well known that many asset allocation problems involve non-normally distributed returns since commodities typically have fat tails and are skewed.

The paper [5] tries to select the portfolio with the highest prospect theory utility amongst the other portfolios in the mean variance efficient frontier. Developing this idea, the work [6] shows that an analytical solution of the problem is mostly equivalent to maximising the CPT-utility function along the mean-variance efficient frontier.

First, we briefly present the main ideas of this theory, and then proceed to the problem of finding the function for the assessment of the prospects under some assumptions on the stochastic behavior of the discount factor m and the portfolio price.

II. EU-, PT- and CPT- investors

A. Expected Utility Theory

The most popular approach to the problem of portfolio choice under risk and uncertainty is the expected utility hypothesis. For an introduction to utility theory, see [11].

Bernoulli [12] and later Von Neumann and Morgenstern [13] suggested a theory for choosing an outcome from a set of risky or uncertain outcomes by comparing the expected utility values defined on final asset position. It later came to be known as Expected Utility Theory (EUT). It has been used as a reference model to find the optimal solution in many areas of economics.

Von Neumann and Morgenstern [13] formulated the utility in terms of a function. Let X is the set of all possible

Manuscript received February 19, 2015; revised March 4, 2015. This work was supported in the Russian Fund for Basic Research under Grant 14-01-00140.

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outcomes. x refers to the element of X. Let $G = \{X, F, P\}$ is probability space over X and let $U : X \to \mathbb{R}$ denote a utility function such that the value of $U(x)$ is a measure of the decision makers preference derived from the outcome x:

$$x \succeq y \;\Leftrightarrow\; U(x) \geq U(y),$$

where $x \succeq y$ means the outcome x is preferred at least as much as the outcome y. Thus, the relationship between wealth and the utility of consuming this wealth is described by a utility function, $U(\cdot)$. In general, each investor will have a different $U(\cdot)$. In expected utility theory, decision makers attitudes towards uncertainty are wholly modeled by the value of utility functions defined on final asset positions. Let $f_\xi(x)$ be the probability density function of a random variable ξ.

Definition 1. *The expected utility (EU) of the game G is the expected value of the utility functions of possible outcomes weighted by the corresponding probabilities:*

$$U_{EU}(G) = \int_X U(x) f_\xi(x) dx.$$

The expected utility hypothesis states that the individual (EU-investor) will make decisions following the principle of maximizing the value of his expected utility.

An important property of an expected utility function is that it is unique up to affine transformations. That is, if $U(\cdot)$ describes the preferences of an investor, then so does $U^*(\cdot) = c_1 U(\cdot) + c_2$, where $c_1 > 0$.

The range of reasonable utility functions should be restricted by economic reasoning. The expected utility function has the following properties:

1) positive marginal utility, i.e. $U'(x) > 0$ for all x.
2) risk aversion; a necessary and sufficient condition for risk aversion is that the expected utility function is concave, i.e. $U''(x) < 0$ for all x.

The most exploited type of utility functions is the power utility function defined by

$$U(x) = \frac{x^{1-\gamma}}{1-\gamma},$$

where $\gamma \in (0, 1)$. Marginal utility is $U'(x) = x^{-\gamma} > 0$ for all $x > 0$. We have $U''(x) = -\gamma x^{-\gamma-1} < 0$ for all $x > 0$.

B. Prospect Theory

Prospect theory (PT) has three essential distinctions from Expected Utility Theory:

- investor makes investment decisions based on deviation of his/her final wealth from a reference point and not according to his/her final wealth, i.e. PT-investor concerned with deviation of his/her final wealth from a reference level, whereas Expected Utility maximizing investor takes into account only the final value of his/her wealth.
- utility function is S-shaped with turning point in the origin, i.e. investor reacts asymmetrical towards gains and losses; moreover, he/she dislikes losses with a factor of $\lambda > 1$ as compared to his/hers liking of gains.
- investor evaluates gains and losses not according to the real probability distribution per ce but on the basis of the transformation of this real probability distribution, so that investor's estimates of probability are transformed in the way that small probability (close to 0) is overvalued and high probability (close to 1) is undervalued.

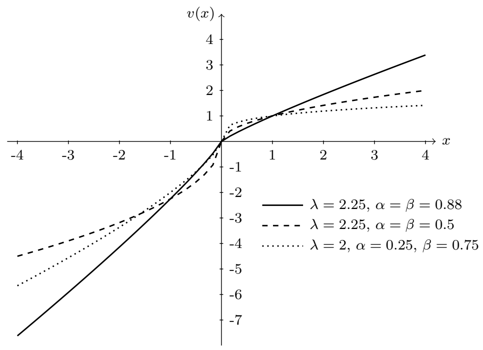

Figure 1. The plot of the value function $u(x)$ for different α, β, λ

CPT includes three important parts:

- a value function over outcomes, $v(\cdot)$;
- a weighting function over probabilities, $\omega(\cdot)$;
- PT-utility as unconditional expectation of the value function v under probability distortion ω.

Definition 2. *The value function derives utility from gains and losses and is defined as follows [2]:*

$$v(x) = \begin{cases} x^\alpha, & \text{if } x \geq 0, \\ -\lambda(-x)^\beta, & \text{if } x < 0. \end{cases} \tag{3}$$

The fig 1. plots the value function for different values of α, β, λ. Note that the value function is convex over losses if $0 \leq \beta \leq 1$ and it is strictly convex if $0 < \beta < 1$. Moreover, the value function reflects loss aversion when $\lambda > 1$. It follows from the fact that individual investors are more sensitive to losses than to gains. D. Kahneman and A. Tversky estimated [1] the parameters of the value function $\alpha = \beta = 0.88$, $\lambda = 2.25$ based on experiments with gamblers.

Definition 3. *Let $f_\xi(x)$ be the probability density function of a random variable ξ. The PT-probability weighting function $w : [0, 1] \to [0, 1]$ is defined by*

$$w\left(f_\xi(x)\right) = \frac{\left(f_\xi(x)\right)^\delta}{\left((f_\xi(x))^\delta + (1 - f_\xi(x))^\delta\right)^{1/\delta}}, \quad \delta \leq 1 \tag{4}$$

It is easy to verify that

1) $w : [0, 1] \to [0, 1]$ is differentiable on [0,1];
2) $w(0) = 0$, $w(1) = 1$;
3) if $\delta > 0.28$ then w is increasing on [0,1];
4) if $\delta = 1$ then $w(f_\xi(x)) = f_\xi(x)$.

In the following we will assume that $0.28 < \delta \leq 1$

Fig. 2 presents the plots of the probability weighting function for different values of δ.

Definition 4. *The PT-utility of a gamble G with stochastic return ξ is defined as [1]*

$$U_{PT}(G) = \int_{-\infty}^{\infty} v(x) w(f_\xi(x)) dx, \tag{5}$$

where $f_\xi(x)$ is the probability density function of ξ.

ISBN: 978-988-14047-0-1
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

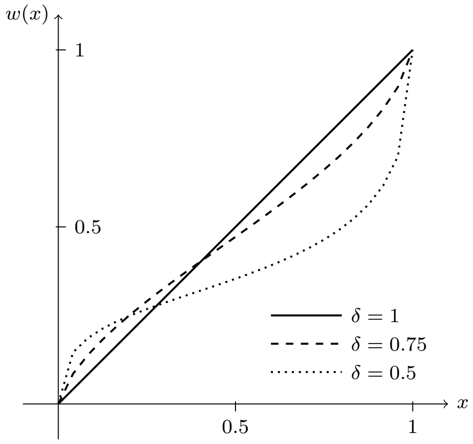

Figure 2. The plot of the probability weighting function $w(x)$ for different δ

C. Cumulative Prospect Theory

We will consider the development of the prospect theory, Cumulative Prospect Theory, published in 1992 [2]. The description of CPT includes three important parts:

- a value function over outcomes, $v(\cdot)$;
- a weighting function over *cumulative* probabilities, $w(\cdot)$;
- CPT-utility as unconditional expectation of the value function v under probability distortion w.

Definition 5. *Let $F_\xi(x)$ be cumulative distribution function (cdf) of a random variable ξ. The probability weighting function $w : [0,1] \to [0,1]$ is defined by*

$$w\left(F_\xi(x)\right) = \frac{\left(F_\xi(x)\right)^\delta}{\left((F_\xi(x))^\delta + (1 - F_\xi(x))^\delta\right)^{1/\delta}}, \quad \delta \leq 1 \qquad (6)$$

Definition 6. *The CPT-utility of a gamble G with stochastic return ξ is defined as [3]*

$$U_{CPT}(G) = \int_{-\infty}^{0} v(x) dw(F_\xi(x)) - \int_{0}^{\infty} v(x) dw(1 - F_\xi(x)), \qquad (7)$$

where $F_\xi(x)$ is cumulative distribution function of ξ.

If we apply integration by part, then CPT-utility of G defined in (7) can be rewritten as

$$U_{CPT}(G) = \int_{0}^{\infty} w(1 - F_\xi(x)) dv(x) - \int_{-\infty}^{0} w(F_\xi(x)) dv(x). \qquad (8)$$

III. Simple stochastic model

In this section we will suppose that the financial market consists of one risk-free and one risky assets. We examine the natural problem of how an investor optimizes her portfolio holding in a risky asset under PT and CPT with one risk-free asset and one risky asset with some assumption on the stochastic behavior of the discount factor and the risky asset price. The main goal is to compare solutions of this problem under PT and CPT assumptions with solution of the problem under Expected Utility Theory.

We will assume that the price S of risky asset follows the standard lognormal diffusion process given by the stochastic differential equation known as Geometric Brownian Motion:

$$\frac{dS}{S} = \mu dt + \sigma dz, \qquad (9)$$

where μ is a drift, σ is a standard deviation, $dz = \epsilon\sqrt{dt}$, the random variable ϵ is a standard normal, $\epsilon \sim N(0, 1)$.

We will assume that there is also a money market security that pays the real interest rate rdt (risk-free asset):

$$\frac{dB}{B} = rdt. \qquad (10)$$

Following [14] we will assume that the discount factor Λ follows the process

$$\frac{d\Lambda}{\Lambda} = -rdt - \frac{\mu - r}{\sigma} dz, \qquad (11)$$

where S is the price of the risky asset, r is risk-free rate.

It is well-known [14] that the solution of (9) is

$$\ln S_T = \ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right) T + \sigma\sqrt{T}\epsilon, \qquad (12)$$

where S_0 is the price of the risky asset at the moment 0, S_T is the asset price on the date T. The solutions of (11) is

$$\ln \Lambda_T = \ln \Lambda_0 - \left(r + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2\right) T - \frac{\mu - r}{\sigma}\sqrt{T}\varepsilon, \qquad (13)$$

where $\varepsilon \sim N(0, 1)$, and S_0 is the price of the portfolio at $t = 0$. It follows from (13) that

$$m_T = m_T(\epsilon) = \frac{\Lambda_T}{\Lambda} = \exp\left[-\left(r + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2\right) T - \frac{\mu - r}{\sigma}\sqrt{T}\varepsilon\right]. \qquad (14)$$

Let W_0 denote the investor's wealth at the time $t = 0$. Let V denote the amount of money invested in the risky asset. Then $W_0 - V$ is the wealth invested in the risk-free asset. It follows from (12) that the investor wealth W_T on the date $t = T$ is given by

$$W_T = (W_0 - V)e^{rT} + Ve^{(\mu - \frac{\sigma^2}{2})T + \sigma\sqrt{T}\epsilon}. \qquad (15)$$

Then the discounted value of the investor's wealth is

$$\widetilde{W}_T = m_T W_T. \qquad (16)$$

Lemma 1. *Let $a \in \mathbb{R}$. Then*

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{a\epsilon - \frac{1}{2}\epsilon^2} d\epsilon = e^{\frac{1}{2}a^2}.$$

Proof: We have

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{a\epsilon - \frac{1}{2}\epsilon^2} d\epsilon = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}(\epsilon^2 - 2a\epsilon + a^2) + \frac{1}{2}a^2} d\epsilon = e^{\frac{1}{2}a^2} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}(\epsilon - a)^2} d\epsilon = e^{\frac{1}{2}a^2}.$$

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ISBN: 978-988-14047-0-1
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

The expected value at the moment $t = 0$ of the discounted value of the investor's wealth at the moment $t = T$ is equal to

$$E_0(\widetilde{W_T}) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\left(r+\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2\right)T - \frac{\mu-r}{\sigma}\sqrt{T}\varepsilon} \times$$
$$((W_0 - V)e^{rT} + Ve^{(\mu-\frac{\sigma^2}{2})T+\sigma\sqrt{T}\epsilon})e^{-\frac{1}{2}\epsilon^2} d\epsilon =$$
$$= (W_0 - V)e^{-\left(\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2\right)T} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{\mu-r}{\sigma}\sqrt{T}\varepsilon - \frac{1}{2}\epsilon^2} d\epsilon +$$
$$+ Ve^{-\left(r+\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2\right)T + (\mu-\frac{\sigma^2}{2})T} \times$$
$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{\mu-r}{\sigma}\sqrt{T}\varepsilon + \sigma\sqrt{T}\epsilon - \frac{1}{2}\epsilon^2} d\epsilon = W_0.$$

In the next subsections we examine the problem of finding the optimal portfolio (1)-(2) for three different types of investors: EU-investor with a power utility function, PT-investor, CPT-investor.

A. EU-investor with a power utility function

Suppose that EU-investor is maximizing the expected value of a power utility function. Then

$$U_{EU}(V) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{W_T^{1-\gamma}}{1-\gamma} e^{-\frac{1}{2}\epsilon^2} d\epsilon, \tag{17}$$

where W_T is defined in (15).

Denote

$$f(V) := \frac{((W_0 - V)e^{rT} + Ve^{(\mu-\frac{\sigma^2}{2})T+\sigma\sqrt{T}\epsilon})^{1-\gamma}}{1-\gamma}. \tag{18}$$

The second order Taylor expansion of function $f(V)$ about 0 is

$$\tilde{f}(V) = f(0) + \frac{f'(0)}{1!} V + \frac{f''(0)}{2!} V^2. \tag{19}$$

Let us denote

$$\tilde{U}_{EU}(V) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \tilde{f}(V) e^{-\frac{1}{2}\epsilon^2} d\epsilon, \tag{20}$$

We will examine the problem for the maximization of $\tilde{U}_{EU}(V)$ over V,

$$\tilde{U}_{EU}(V) \to \max_{V\in[0,\infty)}. \tag{21}$$

Theorem 2. *Let $\mu \geq r$. Then there is a unique solution V^* of the problem* (21) *defined by*

$$V^* = \frac{W_0}{\gamma} \frac{-1 + e^{(\mu-r)T}}{1 - 2e^{(\mu-r)T} + e^{2(\mu-r)T+\sigma^2 T}},\ V^* \approx \frac{W_0}{\gamma} \frac{\mu - r}{\sigma^2}. \tag{22}$$

Proof: We have

$$f'(V) = ((W_0 - V)e^{rT} + Ve^{(\mu-\frac{\sigma^2}{2})T+\sigma\sqrt{T}\epsilon})^{-\gamma} \times$$
$$(-e^{rT} + e^{(\mu-\frac{\sigma^2}{2})T+\sigma\sqrt{T}\epsilon}) \tag{23}$$

and

$$f''(V) = -\gamma((W_0 - V)e^{rT} + Ve^{(\mu-\frac{\sigma^2}{2})T+\sigma\sqrt{T}\epsilon})^{-\gamma-1} \times$$
$$(-e^{rT} + e^{(\mu-\frac{\sigma^2}{2})T+\sigma\sqrt{T}\epsilon})^2. \tag{24}$$

Substituting (23) and (24) into (19) and (17) we get

$$\widetilde{U}_{EU}(V) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{(W_0 e^{rT})^{1-\gamma}}{1-\gamma} e^{-\frac{1}{2}\epsilon^2} d\epsilon +$$
$$\frac{V(W_0 e^{rT})^{-\gamma}}{\sqrt{2\pi}} \int_{\mathbb{R}} (-e^{rT} + e^{(\mu-\frac{\sigma^2}{2})T+\sigma\sqrt{T}\epsilon}) e^{-\frac{1}{2}\epsilon^2} d\epsilon -$$
$$\frac{V^2(W_0 e^{rT})^{-\gamma-1}}{\gamma\sqrt{2\pi}} \int_{\mathbb{R}} (-e^{rT} + e^{(\mu-\frac{\sigma^2}{2})T+\sigma\sqrt{T}\epsilon})^2 e^{-\frac{1}{2}\epsilon^2} d\epsilon \tag{25}$$

Then $\widetilde{U}'_{EU}(V) = 0$ is equivalent to

$$V = \frac{1}{\gamma} W_0 e^{rT} \frac{A}{B}, \tag{26}$$

where $A = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (-e^{rT} + e^{(\mu-\frac{\sigma^2}{2})T+\sigma\sqrt{T}\epsilon}) e^{-\frac{1}{2}\epsilon^2} d\epsilon$ and $B = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (-e^{rT} + e^{(\mu-\frac{\sigma^2}{2})T+\sigma\sqrt{T}\epsilon})^2 e^{-\frac{1}{2}\epsilon^2} d\epsilon$. Using Lemma 1 we have

$$A = -\frac{1}{\sqrt{2\pi}} e^{rT} \int_{\mathbb{R}} e^{-\frac{1}{2}\epsilon^2} d\epsilon +$$
$$e^{(\mu-\frac{\sigma^2}{2})T} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\sigma\sqrt{T}\epsilon} e^{-\frac{1}{2}\epsilon^2} d\epsilon =$$
$$= -e^{rT} + e^{(\mu-\frac{\sigma^2}{2})T+\frac{\sigma^2}{2}T} = -e^{rT} + e^{\mu T}. \tag{27}$$

and

$$B = e^{2rT} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}\epsilon^2} d\epsilon -$$
$$- 2e^{rT+(\mu-\frac{\sigma^2}{2})T} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\sigma\sqrt{T}\epsilon} e^{-\frac{1}{2}\epsilon^2} d\epsilon +$$
$$e^{2(\mu-\frac{\sigma^2}{2})T} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{2\sigma\sqrt{T}\epsilon} e^{-\frac{1}{2}\epsilon^2} d\epsilon =$$
$$= e^{2rT} - 2e^{rT+(\mu-\frac{\sigma^2}{2})T+\frac{\sigma^2}{2}T} + e^{2(\mu-\frac{\sigma^2}{2})T+2\sigma^2 T} =$$
$$e^{2rT} - 2e^{rT+\mu T} + e^{2\mu T+\sigma^2 T}. \tag{28}$$

It follows from (26), (27), (28) that

$$V^* = \frac{1}{\gamma} W_0 \frac{e^{rT}(-e^{rT} + e^{\mu T})}{e^{2rT} - 2e^{rT+\mu T} + e^{2\mu T+\sigma^2 T}} =$$
$$\frac{1}{\gamma} W_0 \frac{-1 + e^{(\mu-r)T}}{1 - 2e^{(\mu-r)T} + e^{2(\mu-r)T+\sigma^2 T}}.$$

Since $e^x \approx 1 + x$ for a small x we get

$$V^* \approx \frac{1}{\gamma} W_0 \frac{\mu - r}{\sigma^2}.$$

Note that $\widetilde{U}''_{EU}(V) \leq 0$ for all V and therefore V^* defined by (22) is the solution of the problem (21). ■

B. PT-investor

For both PT-investor and CPT-investor we will assume that reference level of wealth at the time moment $t = T$ is $X = W_0 e^{rT}$, i.e. X is the amount of wealth the investor would have received on the date $t = T$ after investing W_0 with the continuously compounding rate r. Then the deviation from reference point X on the date $t = T$ is equal to

$$D_T(V) = W_T - X = V \cdot (e^{(\mu-\frac{\sigma^2}{2})T+\sigma\sqrt{T}\epsilon} - e^{rT}). \tag{29}$$

The allocation problem for PT-investor can be stated as follows:

$$U_{PT}(D_T(V)) \to \max_{V\in[0,\infty)}. \tag{30}$$

Let ϵ^* be such that $e^{(\mu-\sigma^2/2)T+\sigma\sqrt{T}\epsilon^*} = e^{rT}$ i.e. $\epsilon^* = \frac{-(\mu-r-\sigma^2/2)T}{\sigma\sqrt{T}}$, or

$$\epsilon^* = \left(-\frac{\mu-r}{\sigma}+\frac{\sigma}{2}\right)\sqrt{T}. \tag{31}$$

Note that if $V \neq 0$ then $D(V) \geq 0$ if the random $\epsilon \geq \epsilon^*$, and $D(V) < 0$ otherwise.

Denote

$$A_1 = \int_0^{+\infty} (e^{\sigma\sqrt{T}x}-1)^\alpha w(f(x+\epsilon^*))dx,$$
$$A_2 = \int_0^{+\infty} (1-e^{-\sigma\sqrt{T}x})^\beta w(f(-x+\epsilon^*))dx,$$

where

$$f(\epsilon) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\epsilon^2}dt \tag{32}$$

is the normal density function.

Theorem 3. *If $\alpha < \beta$ then there exists a unique solution V^* of the allocation problem* (30)*,*

$$V^* = e^{-rT}\left(\frac{1}{\lambda}\frac{\alpha}{\beta}\frac{A_1}{A_2}\right)^{\frac{1}{\beta-\alpha}}. \tag{33}$$

Proof: It follows from definition 4 that PT-utility of decision V can be written as follows:

$$\begin{aligned} &U_{PT}(V) = \\ &V^\alpha \int_{\epsilon^*}^{+\infty} (e^{(\mu-\frac{\sigma^2}{2})T+\sigma\sqrt{T}\epsilon} - e^{rT})^\alpha w(f(\epsilon))d\epsilon - \\ &- V^\beta\lambda \int_{-\infty}^{\epsilon^*} (e^{rT} - e^{(\mu-\frac{\sigma^2}{2})T+\sigma\sqrt{T}\epsilon})^\beta w(f(\epsilon))d\epsilon. \end{aligned} \tag{34}$$

Changing variables by $\epsilon = x+\epsilon^*$ and $\epsilon = -x+\epsilon^*$ in the first and the second integrals respectively, we get

$$\begin{aligned} &U_{PT}(V) = \\ &V^\alpha e^{\alpha rT} \int_0^{+\infty} (e^{\sigma\sqrt{T}x}-1)^\alpha w(f(x+\epsilon^*))dx - \\ &- \lambda \cdot V^\beta e^{\beta rT} \int_0^{+\infty} (1-e^{-\sigma\sqrt{T}x})^\beta w(f(-x+\epsilon^*))dx. \end{aligned}$$

The allocation problem (30) can be rewritten as

$$V^\alpha e^{\alpha rT}A_1 - \lambda \cdot V^\beta e^{\beta rT}A_2 \to \max_{V\in[0,+\infty)}.$$

We have

$$U'_{PT}(V) = \alpha V^{\alpha-1}e^{\alpha rT}A_1 - \lambda\beta V^{\beta-1}e^{\beta rT}A_2 = 0$$

at point $V^* = e^{-rT}\left(\frac{1}{\lambda}\frac{\alpha}{\beta}\frac{A_1}{A_2}\right)^{\frac{1}{\beta-\alpha}}$. Moreover,

$$\begin{aligned} &U''_{PT}(V^*) = \\ &-\alpha(\alpha-1)V^{\alpha-2}e^{\alpha rT}A_1 + \beta(\beta-1)V^{\beta-2}e^{\beta rT}A_2 < 0 \end{aligned}$$

if and only if $\alpha < \beta$. ■

While the reference point $X = W_0e^{rT}$ depends on the initial wealth W_0, the deviation from the reference point $D(V)$ does not depend on W_0. Therefore, as it is shown in (33) the optimal amount to invest in the risky asset V^* does not depend on W_0.

Corollary 1. *Let $\alpha < \beta$. Then $\frac{dV^*}{d\lambda} < 0$ and if $\lambda \to \infty$ then $V^* \to 0$.*

The corollary shows that the weight of risky asset in the portfolio is decreasing with increasing loss aversion.

C. CPT-investor

The allocation problem for CPT-investor can be stated as follows:

$$U_{CPT}(D_T(V)) \to \max_{V\in[0,\infty)}, \tag{35}$$

where $D_T(V)$ is defined in (29).

Let ϵ^* be defined in (31). Denote

$$F_1 = -\int_0^{+\infty} (e^{\sigma\sqrt{T}x}-1)^\alpha dw(1-\Phi(x+\epsilon^*)),$$
$$F_2 = \int_0^{+\infty} (1-e^{-\sigma\sqrt{T}x})^\beta dw(\Phi(-x+\epsilon^*)),$$

where

$$\Phi(\epsilon) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\epsilon} e^{-\frac{1}{2}t^2}dt \tag{36}$$

is the normal cumulative distribution function.

Theorem 4. *If $\alpha < \beta$ then there exists a unique solution V^* of the allocation problem* (35)*,*

$$V^* = e^{-rT}\left(\frac{1}{\lambda}\frac{\alpha}{\beta}\frac{F_1}{F_2}\right)^{\frac{1}{\beta-\alpha}}.$$

Proof: It follows from definition 6 that CPT-utility of decision V can be written as follows:

$$\begin{aligned} &U_{CPT}(V) = \\ &-V^\alpha \int_{\epsilon^*}^{+\infty} (e^{(\mu-\frac{\sigma^2}{2})T+\sigma\sqrt{T}\epsilon} - e^{rT})^\alpha dw(1-\Phi(\epsilon)) - \\ &- V^\beta\lambda \int_{-\infty}^{\epsilon^*} (e^{rT} - e^{(\mu-\frac{\sigma^2}{2})T+\sigma\sqrt{T}\epsilon})^\beta dw(\Phi(\epsilon)), \end{aligned} \tag{37}$$

where $\Phi(\epsilon)$ is the normal cumulative distribution function.

Changing variables by $\epsilon = x+\epsilon^*$ and $\epsilon = -x+\epsilon^*$ in the first and the second integrals respectively, we get

$$\begin{aligned} &U_{CPT}(V) = \\ &-V^\alpha e^{\alpha rT} \int_0^{+\infty} (e^{\sigma\sqrt{T}x}-1)^\alpha dw(1-\Phi(x+\epsilon^*)) - \\ &- \lambda \cdot V^\beta e^{\beta rT} \int_0^{+\infty} (1-e^{-\sigma\sqrt{T}x})^\beta dw(\Phi(-x+\epsilon^*)). \end{aligned}$$

The allocation problem (35) can be rewritten as

$$V^\alpha e^{\alpha rT}F_1 - \lambda \cdot V^\beta e^{\beta rT}F_2 \to \max_{V\in[0,+\infty)}.$$

We have

$$U'_{CPT}(V) = \alpha V^{\alpha-1}e^{\alpha rT}F_1 - \lambda\beta V^{\beta-1}e^{\beta rT}F_2 = 0$$

at point $V^* = e^{-rT}\left(\frac{1}{\lambda}\frac{\alpha}{\beta}\frac{F_1}{F_2}\right)^{\frac{1}{\beta-\alpha}}$. Moreover,

$$\begin{aligned} &U''_{CPT}(V^*) = \\ &-\alpha(\alpha-1)V^{\alpha-2}e^{\alpha rT}F_1 - \beta(\beta-1)V^{\beta-2}e^{\beta rT}F_2 < 0 \end{aligned}$$

if and only if $\alpha < \beta$. ■

Corollary 2. *Let $\alpha < \beta$. Then $\frac{dV^*}{d\lambda} < 0$ and if $\lambda \to \infty$ then $V^* \to 0$;*

The corollary shows that the weight of risky asset in the portfolio is decreasing with increasing loss aversion.

It should be noted that in the framework of Expected Utility Theory there is a unique solution of the allocation problem for one risk-free and one risky assets.

ISBN: 978-988-14047-0-1
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

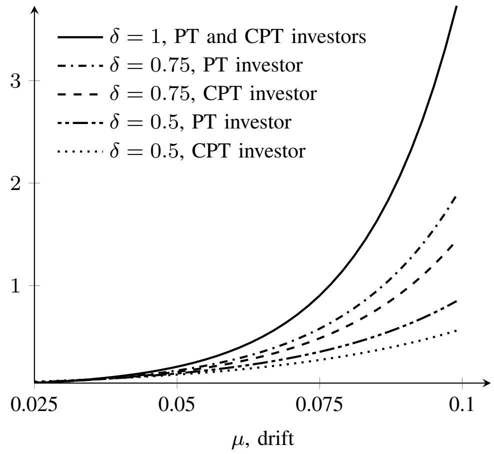

Figure 3. The dependence of μ on V for PT- and CPT- investors and different δ

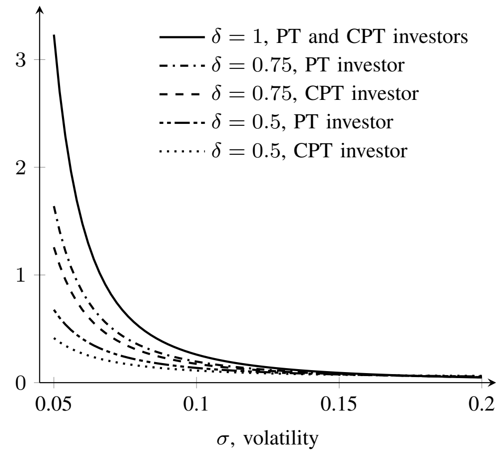

Figure 4. The dependtnce of σ on V for PT- and CPT- investors and different δ

IV. Comparative Analysis of EU-, PT- and CPT-investors

The problem solution for the EU-investor is similar to the solution of classical Merton portfolio choice problem where returns are assumed to be normally distributed, and the investor has the Constant Relative Risk Aversion utility (CRRA utility) [15]. This optimal portfolio has two main components: $\frac{\mu - r}{\sigma^2}$ is a measure of the investments in the risky asset (defined as the Sharpe ratio divided by the standard deviation σ), and λ is a measure of the curvature of the CRRA investor's utility function. Therefore, the PT- and CPT-optimal portfolios have 2 components that play approximately the same roles as the ones represented in the EU-solution: the value of $\left(\frac{1}{\lambda}\frac{\alpha}{\beta}\frac{A_1}{A_2}\right)^{\frac{1}{\beta-\alpha}}$ (or $\left(\frac{1}{\lambda}\frac{\alpha}{\beta}\frac{F_1}{F_2}\right)^{\frac{1}{\beta-\alpha}}$ for CPT-investor) can be seen as a risk-reward measure and the parameters α and β are related to the curvature of the value function on the positive and negative domains, respectively [9]. However, these two frameworks (as PT- and CPT- investors have the same basis) have significant differences based on the specific features of the mentioned components. Furthermore, taking into consideration all the components included in the final solutions it can be stated that the main difference between PT- (CPT-)investor and EU-investor is that the former is dependent on the investor's wealth W_0 at the time $t = 0$ but is not influenced by the time factor T. Vice versa is correct for the latter.

Figure 3 presents the dependance of μ on V for PT- and CPT- investors and different δ. We can see that V is increasing with increase of μ. The choices of optimal V for PT- and CPT-investors coincide in case $\delta = 1$. The CPT-investor is more cautious than the PT-investor, as his choice of the optimal value of V is always less than the optimal choice of V for the PT-investor in case $\delta < 1$.

Figure 4 presents the dependance of σ on V both for PT- and CPT- investors for different values of δ. We can see that V is decreasing with increase of σ. The choices of optimal V for PT- and CPT-investors coincide in case $\delta = 1$. Again, we can conclude that the CPT-investor is more cautious than the PT-investor, as his choice of the optimal value of V is always less than the optimal choice of V for the PT-investor in case $\delta < 1$.

V. Conclusion

The paper deals with the problem of optimal portfolio choice for PT- and CPT- investors described in [10]. We have considered a simple stochastic model with one risk-free asset and one risky asset that follows geometrical Brownian motion stochastic equation. It turned out that if the parameters α, β of the value function $u(\cdot)$ satisfy the inequality $\alpha < \beta$ then there exists a non-trivial optimal choice for the weights of the assets. It is consistent with the findings of [9].

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ISBN: 978-988-14047-0-1
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)