Mechanical Design of Viscoelastic Parts Fabricated Using Additive Manufacturing Technologies

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***Abstract*—Mechanical design of viscoelastic parts fabricated using additive manufacturing technologies are under consideration. Such a design is of great importance for obtaining final products with desired shape and strength which are determined by their stress-strain state. Mathematical model of model of viscoelastic solid which grows due to the process of additive manufacturing is proposed. Complete system of boundary value problem equations is obtained. A method for solving formulated boundary value problem is developed. Qualitative conclusions concerning the behavior of growing solids are presented.**

***Index Terms*—additive manufacturing technologies, growing solid, viscoelastic material, shape, strength**

INTRODUCTION

ADDITIVE manufacturing (AM) technologies include stereolithography, electrolytic deposition, laser and thermal 3D printing, production of 3D integrated circuits and a number of other technologies. Actually, there is a real boom in the development of AM technologies since they allow to reproduce a 3D object of arbitrarily complicated shape (in theory from any material) with high accuracy and low expenses in a short time. However, problems of deformation and strength of products fabricated using such technologies remain still unsolved. Mathematical models and methods developed in the paper allow one to study the stress-strain state of parts of devices, machines, and mechanisms created in AM processes from viscoelastic materials. This gives an opportunity to estimate their shape distortion, strength, stability and life time. This problem is of general interest for the modern technologies in engineering, medicine, electronics industry, aerospace industry, and other fields.

I. CHARACTERISTIC FEATURES OF GROWING SOLIDS

The process of accretion or deposition of new material to a solid is studied in the fundamental scientific area called Growing Solids Mechanics. This area deals with all sorts of solid materials including elastic, viscoelastic, plastic, composite and graded materials. Currently a great number of AM fabricated part are made from viscoelastic materials with complex properties so we consider just such materials (see [1–6]).

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By a (piecewise) continuously growing solid we mean a solid solid whose composition, mass or volume varies as a result of a (piecewise) continuous addition of material to its surface. The process of adding new material to the solid is called accretion or growth. For piecewise-continuous accretion the following basic stages of its deformation are strictly followed: before accretion, during the continuous growth, and after the accretion has ceased and growth has stopped. Each of these stages is characterized by the times when it starts and ends. The first is characterized by the time of application of a load to the solid and the time when growth starts. The second by the time when growth starts and the time when it ends. Conversely, the third is characterized by the time when growth ends and the time when it starts. The process under investigation is usually concluded by the third stage, for which the time when the next stage begins is taken to be as long as desired. The solid on whose surface new material is deposited starting from the time when accretion starts is called the basic or original solid. The solid consisting of the material pieces added to the basic solid over the time interval from the beginning of accretion up to a given instant of time is called the additional solid. The additional solid can have a complex structure and consist of a collection of solids formed over different time intervals of continuous accretion. We call them sub-solids. The additional solid is obviously the union of sub-solids. The domains occupied by the former and latter can be disconnected. The union of the basic and the additional solids will be called the accreted orgrowing solid. Note that accretion can also occur without the basic solid, starting from an infinitesimal material element. The part of the surface where infinitesimal pieces of the material are deposited at the actual instant is called the accretion or growth surface. The growth surface may be disconnected, in general. In particular, it can be the whole surface of the solid. Finally, the part of the surface of the original or the growing solid that coincides with the growth surface at the time when growth starts will be called the base surface. The base surface is clearly the part of the surface of the solid on which material is to be deposited during the next stage of continuous accretion. At different stages it coincides, as a rule, with the surface between the basic solid and the additional solid as well as with the surfaces between the sub-solids.

We assume that the basic solid, which is made from a viscoelastic ageing material, occupies a domain Ω_0 with the surface S_0 and is free of stresses up to the time τ_0 of application of the load. From τ_0 up to the time τ_1 when accretion starts the classical boundary conditions are given on S_0, the specific form of which is stated below. At τ_1 the continuous accretion of a solid begins due to the addition of material particles to the accretion surface $S^*(t)$. As it grows, the solid occupies a domain $\Omega(t)$ with surface $S(t)$. It is obvious that $S^*(t) \subseteq S(t)$. The time when a particle characterized by a position vector x is deposited on the solid will be denoted by $\tau^*(\mathbf{x})$ and called the time of deposition of the particle on the growing solid. The configuration of the accreted solid is completely defined by the function $\tau^*(\mathbf{x})$ depending on the spatial coordinates. Boundedness and piecewise-continuity are the general conditions usually imposed on $\tau^*(\mathbf{x})$.

We denote by $\tau_1^*(\mathbf{x})$ the time when an element of the growing solid is formed and by $\tau_0(\mathbf{x})$ the time when a load is applied to it. Naturally, $\tau_1^*(\mathbf{x}) \leq \tau_0(\mathbf{x}) = \tau_0$ for the elements of the basic solid ($\mathbf{x} \in \Omega_0$).

To simplify the problem we consider the case of small deformations and zero volumetric force.

The vector equilibrium equation is obviously satisfied in the domain occupied by the growing solid at each instant of time. For quasistatic processes it has the form

$$\mathbf{x} \in \Omega(t)\colon \nabla \cdot \mathbf{T} = \mathbf{O}, \tag{1}$$

where T is the stress tensor, and ∇ is the Hamilton operator (here and henceforth we use the conventional notation of tensor calculus).

The Cauchy conditions and the compatibility equations for deformations are always satisfied in the domain occupied by the basic solid

$$\mathbf{x} \in \Omega_0\colon \mathbf{E} = \frac{1}{2}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T], \quad \nabla \times (\nabla \times \mathbf{E})^T = \mathbf{O}, \tag{2}$$

where $\mathbf{E}$ is the strain tensor and $\mathbf{u}$ is the displacement vector. But in the domain $\Omega^*(t)$ occupied by the additional solid ($\Omega^*(t) = \Omega(t)/\Omega_0$) only their analogues involving the rates of change of the corresponding variables are satisfied

$$\mathbf{x} \in \Omega^*(t)\colon \mathbf{D} = \frac{1}{2}[\nabla \mathbf{v} + (\nabla \mathbf{v})^T], \quad \nabla \times (\nabla \times \mathbf{D})^T = \mathbf{O},$$
$$\mathbf{D} = \frac{\partial \mathbf{E}}{\partial t}, \quad \mathbf{v} = \frac{\partial \mathbf{u}}{\partial t},$$

i.e. the strains are incompatible, in general.

The latter reflects the fact that deposited elements may be subject to deforming actions prior to the time of deposition on a basic solid independently of the processes taking place in the solid itself.

To study the stress strain state (SSS) of a growing solid one must know the laws of deformation of the basic solid from the instant τ_0 when the load is applied up to the instant τ_1 when accretion starts and of the deposited material from the instant $\tau_0(\mathbf{x})$ when a load is applied to this material up to the instant $\tau^*(\mathbf{x})$ of their deposition on the growing solid. The state of the original solid is determined from the solution of the problem with fixed boundary. The initial state of new elements which represent deposited surfaces as well as the boundary condition on the moving surface of a growing body can be determined by solving an additional contact problem of interaction between a solid and a surface.

Furthermore, we observe that

$$\tau^*(\mathbf{x}) = t \tag{3}$$

is the equation of the growth surface and, by (4)

$$s_n = |\nabla \tau^*(\mathbf{x})|^{-1} \tag{4}$$

is the velocity of motion of the surface $S^*(t)$ in the normal direction.

The traditional boundary conditions for the displacement vector and the vector of surface forces are given on the stationary sections of the surface of the growing solid.

To describe the behaviour of the material of the growing solid we use the constitutive equations for an nonuniform ageing solid. Extending the definition of $\tau_0(x)$ by a constant τ_0 to the domain occupied by the original solid, we write

$$dev\mathbf{E}(\mathbf{x}, t) = \frac{dev\mathbf{T}(\mathbf{x}, t)}{2G(t - \tau_1^*(\mathbf{x}), \mathbf{x})} - \int_{\tau_0(\mathbf{x})}^{t} \frac{dev\mathbf{T}(\mathbf{x}, \tau)}{2G(\tau - \tau_1^*(\mathbf{x}), \mathbf{x})} \times K_1(t - \tau_1^*(\mathbf{x}), \tau - \tau_1^*(\mathbf{x}), \mathbf{x})\, d\tau, \tag{5}$$

$$I_1[\mathbf{E}(\mathbf{x}, t)] = \frac{I_1[\mathbf{T}(\mathbf{x}, t)]}{E^*(t - \tau_1^*(\mathbf{x}), \mathbf{x})} - \int_{\tau_0(\mathbf{x})}^{t} \frac{I_1[\mathbf{T}(\mathbf{x}, \tau)]}{E^*(\tau - \tau_1^*(\mathbf{x}), \mathbf{x})} \times K_2(t - \tau_1^*(\mathbf{x}), \tau - \tau_1^*(\mathbf{x}), \mathbf{x})\, d\tau, \tag{6}$$

where $G(t)$, $E^*(t)$ and $K_1(t, \mathbf{x})$, $K_2(t, \mathbf{x})$ are the instantaneous elastic strain moduli and creep functions for pure shear and uniform compression, respectively, $I_1(\mathbf{K})$ denotes the first invariant of a tensor $\mathbf{K}$, and $dev\mathbf{K}$ is the deviator of $\mathbf{K}$.

The description of the process of continuous accretion of a viscoelastic ageing solid involves three characteristic instants: the instant $\tau_1^*(\mathbf{x})$ when the element with coordinate $\mathbf{x}$ is formed, the instant $\tau_0(\mathbf{x})$ when a load is applied to this element, and the instant $\tau^*(\mathbf{x})$ when the element is deposited on the growing solid. These three instants are different, in general.

The deposition process is largely determined by specifying these three instants. If the processes of continuous concrete casting, ice formation, crystal growth, etc. are studied, then $\tau_1^*(\mathbf{x}) = \tau_0(\mathbf{x}) = \tau^*(\mathbf{x})$, i.e. the elements are deposited at the same instant as they are formed and a load is applied to them. If spray deposition or erection of a structure from a large number of blocks is modelled by a continuous growth process, then, as a rule, $\tau_0(\mathbf{x}) = \tau^*(\mathbf{x})$ and the instant $\tau_1^*(\mathbf{x})$ when the elements are formed is arbitrary. If the deformatiort of elements begins as soon as they are formed and they are being added to the basic solid only over some time interval, then $\tau_1^*(\mathbf{x}) = \tau_0(\mathbf{x}) \neq \tau^*(\mathbf{x})$ and so on.

Before formulating the problem considered in the present paper, we emphasize that the problem of the growth of a solid differs in a major way from that involving the removal of material. The latter is characterized solely by the fact that the domain occupied by the solid is reduced, subject to the standard equations and boundary conditions.

Suppose that a homogeneous viscoelastic ageing solid occupying a domain Ω_0 with surface S_0 ($\mathbf{x} \in \Omega_0$) is formed at instant $\tau_1^*(\mathbf{x}) = 0$ and is free of stresses up to the instant $\tau_0 \geq 0$ when a load is applied. Starting from the latter instant, we consider two kinds of boundary conditions on the surface of the solid (surface forces on $S_1(t)$ and displacements on $S_2(t)$).

The sections of the surface on which different boundary conditions are given do not intersect one another and cover the whole surface of the solid. The dependence of S_i on t enables us to take into account the possible evolution of the system of loads, punches, etc. on S_0, and is assumed to be

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piecewise constant. Unless the solid surface is closed, the behaviour of stresses or strains at infinity is prescribed.

Continuous deposition of material formed simultaneously with the solid ($\tau_1^*(\mathbf{x}) = 0$) starts at $\tau_1 \geq \tau_0$. The solid occupies a domain $\Omega(t)$ with surface $S(t)$ during its growth. The growth surface $S^*(t)$ ($S^*(\tau_1) \subset S_0$) moves in space. The sections $S_i(t)$ ($i = 1, 2$) on which the common boundary conditions are given can vary because of the loading of the stationary surface of the additional solid. We assume that the growing surface is always free of outer loads and new deposited surfaces are loaded at the instant of their deposition.

At the instant $\tau_2 > \tau_1$ the accretion of the solid ceases, and starting from this instant four kinds of boundary conditions are given on the sections $S_i(t)$ of the surface $S_1 = S(\tau_2)$ of the solid occupying the domain $\Omega_1 = \Omega(\tau_2)$.

After some time, at the instant $\tau_3 > \tau_2$ the solid accretion may start again. An accretion surface may appear which is not related in any way to the previous one. Then the accretion may stop at instant τ_4, and so on, leading to the problem of piecewise-continuous accretion of a solid with n instants at which the growth starts and, respectively, n instants when it stops.

Proceeding to the study of the basic stages of the process of piecewise-continuous accretion of a viscoelastic solid, we note that fairly slow processes will be considered everywhere below, so that the inertial terms can be neglected in the equilibrium equations.

II. BVP for a Solid Prior to the Growth

We consider the SSS of a viscoelastic ageing solid Ω_0 in the time interval $[\tau_0, \tau_1]$. We write the equilibrium equation in the form (1)

$$\nabla \cdot \mathbf{T} = \mathbf{O}. \tag{7}$$

We represent the boundary conditions described above as follows

$$\begin{aligned} &\mathbf{x} \in S_1(t): \quad \mathbf{n} \cdot \mathbf{T} = \mathbf{p}_0, \\ &\mathbf{x} \in S_2(t): \quad \mathbf{u} = \mathbf{u}_0, \end{aligned} \tag{8}$$

where $\mathbf{p}_0$ and $\mathbf{u}_0$ are given vectors of surface forces and strains and $\mathbf{n}$ is the unit vector normal to the solid surface. The Cauchy conditions are written as follows (see (2))

$$\mathbf{E} = \frac{1}{2}[\nabla\mathbf{u} + (\nabla\mathbf{u})^T]. \tag{9}$$

We take the constitutive equations in the form (6) and (7), assuming that the transverse contraction (Poisson's) ratio of the instantaneous elastic strain and the creep strain of the ageing material are identical and are equal to ν. Then we have (see [1])

$$\mathbf{T} = G(\mathbf{I} + \mathbf{N}(\tau_0, t))[2\mathbf{E} + (K-1)I_1(\mathbf{E})\mathbf{1}], \tag{10}$$

where

$$\begin{aligned} &(\mathbf{I} + \mathbf{N}(\tau_0, t))^{-1} = (\mathbf{I} - \mathbf{L}(\tau_0, t)), \\ &2G = E(1+\nu)^{-1}, \quad K = (1-2\nu)^{-1}, \\ &\mathbf{L}(\tau_0, t)f(t) = \int_{\tau_0}^{t} f(\tau)K(t,\tau)\,d\tau, \end{aligned}$$

$$\begin{aligned} &\omega(t,\tau) = 2C(t,\tau)(1+\nu), \\ &K(t,\tau) = E(\tau)\frac{\partial}{\partial\tau}[E^{-1}(\tau) + C(t,\tau)] = \\ &\qquad = K_1(t,\tau) = G(\tau)\frac{\partial}{\partial\tau}[G^{-1}(\tau) + \omega(t,\tau)], \end{aligned}$$

where $E = E(t)$ and $G = G(t)$ are the elastic moduli under tension and shear, $C(t,\tau)$ and $\omega(t,\tau)$ are the creep measures under tension and shear, $K(t,\tau)$ is the creep function under tension, and $\mathbf{1}$ is the unit tensor. The arguments are omitted in a number of obvious cases above. They will also be omitted in what follows and will be used only in those cases when their absence may be misleading.

Thus (8)–(11) constitute the boundary-value problem (BVP) of the linear theory of elasticity for a homogeneous ageing basic solid, the SSS of which can be described by the solution of the system for $t \in [\tau_0, \tau_1]$.

We transform the BVP for the basic solid. Let us introduce the notation

$$\begin{aligned} &\mathbf{N}^0 = \mathbf{H}(\tau_0, t)\mathbf{N}G^{-1}, \mathbf{a}^0 = \mathbf{H}(\tau_0, t)\mathbf{a}G^{-1}, \\ &\mathbf{H}(\psi, t) = (\mathbf{I} - \mathbf{L}(\psi, t)), \end{aligned} \tag{11}$$

where $\mathbf{N}$ and $\mathbf{a}$ are an arbitrary tensor and arbitrary vector, respectively. We apply the operator $\mathbf{H}(\tau_0, t)$ to the relations in (8)–(11) containing $\mathbf{T}$ after dividing them by G. Then, since $\mathbf{H}(\tau_0, t)$ commutes with the Hamilton operator, we obtain the following BVP using (12) ($\tau_0 \leq t \leq \tau_1$)

$$\begin{aligned} &\nabla \cdot \mathbf{T}^0 = \mathbf{O}, \\ &\mathbf{x} \in S_1(t): \quad \mathbf{n} \cdot \mathbf{T}^0 = \mathbf{p}_0^0, \\ &\mathbf{x} \in S_2(t): \quad \mathbf{u} = \mathbf{u}_0, \\ &\mathbf{E} = \frac{1}{2}[\nabla\mathbf{u} + (\nabla\mathbf{u})^T], \\ &\mathbf{T}^0 = 2\mathbf{E} + (K-1)I_1(\mathbf{E})\mathbf{1}. \end{aligned} \tag{12}$$

Unlike (8)–(11), time occurs in the BVP (13) as a parameter. The latter is mathematically equivalent to the BVP of the theory of elasticity with a parameter t. All analytic and numerical methods of the theory of elasticity can be used when constructing the solution of such a problem, which undoubtedly lends itself better to investigation than the problem (10)–(13) of the theory of viscoelasticity.

In order that $\mathbf{T}$, $\mathbf{E}$, and $\mathbf{u}$ be a solution of the BVP (8)–(11) it is necessary and sufficient that $\mathbf{T}^0$, $\mathbf{E}$, and $\mathbf{u}$ form a solution of the BVP (13) and the relation

$$\mathbf{T}(\mathbf{x}, t) = G(t)[\mathbf{T}^0(\mathbf{x}, t) + \int_{\tau_0}^{t} \mathbf{T}^0(\mathbf{x}, \tau)R(t,\tau)\,d\tau], \tag{13}$$

be satisfied ($\tau_0 \leq t \leq \tau_1$). Here $R(t,\tau)$ is the resolvent of the kernel $K(t,\tau)$.

Therefore, solving the BVP (12) with t as a parameter, one can reconstruct the true characteristics of the SSS of the original viscoelastic ageing solid (10)–(13) from using (14).

III. BVP for a Continuously Growing Solid

We now consider the process of continuous accretion of a solid ($\tau_1 \leq t \leq \tau_2$). For a growing solid we have:
the equilibrium equation

$$\nabla \cdot \mathbf{T} = \mathbf{O}, \tag{14}$$

the boundary conditions on the stationary part of the surface

$$\begin{aligned} \mathbf{x} \in S_1(t)&: \quad \mathbf{n}\cdot\mathbf{T} = \mathbf{p}_0, \\ \mathbf{x} \in S_2(t)&: \quad \mathbf{u} = \mathbf{u}_0, \end{aligned} \tag{15}$$

the condition on the growing surface which can be obtained from the solution of a contact problem for solid and surface provided that they interact without friction (smooth contact)

$$\begin{aligned} &\mathbf{x} \in S^*: \mathbf{n}\cdot\frac{\partial \mathbf{T}}{\partial t} = p\,\mathbf{n}, \quad p = -s_n(\mathcal{T}_s : \mathbf{L}), \\ &\mathbf{v} = \frac{\partial \mathbf{u}}{\partial t}, \quad s_n = \mathbf{n}\cdot\mathbf{v}, \quad (t = \tau^*(\mathbf{x})), \end{aligned} \tag{16}$$

where $\mathcal{T}_s$ is the 2D tensor of the deposited elastic surface tension, $\mathbf{L}$ is the 2D tensor of this surface curvature, the relation between the rates of strain and displacement

$$\mathbf{D} = \frac{1}{2}[\nabla\mathbf{v} + (\nabla\mathbf{v})^T], \tag{17}$$

and the constitutive equation in the form

$$\mathbf{T} = G(\mathbf{I} + \mathbf{N}(\tau_0(\mathbf{x}), t))[2\mathbf{E} + (K-1)I_1(\mathbf{E})\mathbf{1}], \tag{18}$$

$$\tau_0(\mathbf{x}) = \begin{cases} \tau_0, & \mathbf{x} \in \Omega_0, \\ \tau^*(\mathbf{x}), & \mathbf{x} \in \Omega^*(t). \end{cases}$$

Relations (15)–(19) form a general non-inertial initial BVP (IVBP) for a continuous growing solid, where the operator $(\mathbf{I} - \mathbf{L}(\tau_0(\mathbf{x}), t)) = \mathbf{H}(\tau_0(\mathbf{x}), t))$ and its inverse $(\mathbf{I} + \mathbf{N}(\tau_0(\mathbf{x}), t))$ can be determined from (11) and (12) with τ_0 replaced by $\tau_0(\mathbf{x})$. We observe that the process of continuous deposition of new elements on the basic solid under investigation leads, in general, to governing relations containing discontinuities on the interface between the original and the additional solids.

Let us transform the IBVP for a continuously accreted viscoelastic ageing solid into a problem with the time parameter that has the same form as the BVP of the theory of elasticity. We omit technical details and obtain the final result in the form of BVP as follows

$$\begin{aligned} &\nabla\cdot\mathbf{S} = \mathbf{O}, \\ &\mathbf{x} \in S_1(t): \quad \mathbf{n}\cdot\mathbf{S} = \mathbf{R}\mathbf{p}_0, \\ &\mathbf{x} \in S_2(t): \quad \mathbf{v} = \mathbf{v}_0, \\ &\mathbf{x} \in S^*(t): \quad \mathbf{n}\cdot\mathbf{S} = \mathbf{R}(p\,\mathbf{n}) \quad (t = \tau^*(\mathbf{x})), \\ &\mathbf{D} = \frac{1}{2}[\nabla\mathbf{v} + (\nabla\mathbf{v}^T], \\ &\mathbf{S} = 2\mathbf{D} + (K-1)I_1(\mathbf{D})\mathbf{1}, \end{aligned} \tag{19}$$

where $\mathbf{R}$ acts on an arbitrary vector $\mathbf{a}(\mathbf{x}, t)$ by the rule

$$\begin{aligned} \mathbf{R}\mathbf{a}(\mathbf{x}, t) = \frac{1}{G(t)}\frac{\partial \mathbf{a}(\mathbf{x}, t)}{\partial t} + \int_{\tau_0(\mathbf{x})}^{t} \frac{\partial \mathbf{a}(\mathbf{x}, \tau)}{\partial \tau}\frac{\partial \omega(t, \tau)}{\partial t} d\tau \\ + \mathbf{a}(\mathbf{x}, \tau_0(\mathbf{x}))\frac{\partial \omega(t, \tau_0(\mathbf{x}))}{\partial t}, \quad \mathbf{S} = \frac{\partial(\mathbf{HT})}{\partial t}, \end{aligned} \tag{20}$$

Note that the conditions on $S_1(t)$ and $S^*(t)$ are identical.

Relations (20) supplemented with the initial conditions for the basic solid at $t = \tau_1$ form an BVP with t as a parameter.

For $\mathbf{T}$, $\mathbf{E}$, and $\mathbf{u}$ to be solutions of IBVP (15)–(19) it is necessary and sufficient that $\mathbf{S}$, $\mathbf{D}$, and $\mathbf{v}$ form the solution of (20) and that the following relations be satisfied

$$\begin{aligned} \mathbf{T}(\mathbf{x}, t) = G(t)\Bigg\{ \frac{\mathbf{T}(\mathbf{x}, \tau_0(\mathbf{x}))}{G(\tau_0(\mathbf{x}))}\left[1 + \int_{\tau_0(\mathbf{x})}^{t} R(t, \tau)\,d\tau\right] \\ + \int_{\tau_0(\mathbf{x})}^{t}\left[\mathbf{S}(\mathbf{x}, \tau) + \int_{\tau_0(\mathbf{x})}^{\tau} \mathbf{S}(\mathbf{x}, \zeta)\,d\zeta R(t, \tau)\right] d\tau \Bigg\}, \\ \mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}, \tau_0(\mathbf{x})) + \int_{\tau_0(\mathbf{x})}^{t} \mathbf{v}(\mathbf{x}, \tau)\,d\tau. \end{aligned} \tag{21}$$

Hence, the solution of the problem of the accretion of a viscoelastic ageing solid can be obtained by the solution of the mathematically identical problems with a parameter t, which have the same form as the BVP of the clacssical theory of elasticity. Then the true stresses and displacements in the growing solid can be reconstructed using (22).

Relations (22) indicate that the SSS for a growing viscoelastic solid depends on the whole history of loading and accretion of the solid. The initial values of the displacements $\mathbf{u}(\mathbf{x}, \tau^*(\mathbf{x}))$ of the deposited elements in (22) are usually set to be zero (since the SSS of the growing solid does not depend on them).

IV. BVP for Solid After the Growth Stop

Suppose that the solid ceases to grow at instant τ_2. At this instant it occupies a domain Ω_1 with surface S_1, on which two kinds of boundary conditions are specified, as in the case of the problem for the basic solid. Moreover $S^*(\tau_2) = S_1^* \subseteq \cup_i S_i(t)$ $(i = 1, 2)$. In this case the problem for the invariable solid occupying Ω_1 is similar to (15)–(19) without the initial-boundary condition on $S^*(t)$

$$\begin{aligned} &\nabla\cdot\mathbf{T} + \mathbf{f} = \mathbf{O}, \\ &\mathbf{x} \in S_1(t): \quad \mathbf{n}\cdot\mathbf{T} = \mathbf{p}_0, \\ &\mathbf{x} \in S_2(t): \quad \mathbf{u} = \mathbf{u}_0, \\ &\mathbf{D} = \frac{1}{2}[\nabla\mathbf{v} + (\nabla\mathbf{v})^T], \\ &\mathbf{T} = G(\mathbf{I} + \mathbf{N}(\tau_0(\mathbf{x}), t))[2\mathbf{E} + (K-1)I_1(\mathbf{E})\mathbf{1}], \end{aligned} \tag{22}$$

with $\tau^*(\mathbf{x}) = \tau_2$ for $\mathbf{x} \in S_1^*$. The stresses, strains and displacements at $t = \tau_2$ found by solving the growth problem at the previous step serve as the initial conditions.

One can obtain the following BVP (see (20))

$$\begin{aligned} &\nabla\cdot\mathbf{S} = \mathbf{O}, \\ &\mathbf{x} \in S_1(t): \quad \mathbf{n}\cdot\mathbf{S} = \mathbf{R}\mathbf{p}_0, \\ &\mathbf{x} \in S_2(t): \quad \mathbf{v} = \mathbf{v}_0, \\ &\mathbf{D} = \frac{1}{2}[\nabla\mathbf{v} + (\nabla\mathbf{v}^T], \\ &\mathbf{S} = 2\mathbf{D} + (K-1)I_1(\mathbf{D})\mathbf{1}, \end{aligned} \tag{23}$$

where the initial conditions remain as before.

For $\mathbf{T}$, $\mathbf{E}$, and $\mathbf{u}$ to be solutions of the IBVP (23) it is necessary and sufficient that $\mathbf{S}$, $\mathbf{D}$, and $\mathbf{v}$ form a solution of BVP (24) and that relations (22) be satisfied.

Thus, to construct the solution of the problem over the time interval $[\tau_0, \tau_3]$ one has to construct the solutions of the following tree identical problems (having the same form as the BVP of the theory of elasticity with a parameter t): problem (13) for $t = \tau_0$ as well as problems (20), and (24). The SSS of the growing solid can then be reconstructed for any $t \in [\tau_0, \tau_3]$ from (14) and (22).

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V. The Case of Piecewise-Continuous Growth

Suppose that the growth process restarts at instant τ_3 and deposition of new elements begin on the surface S_1 of the solid (or part of it) occupying the domain Ω_1. Then, by analogy with Section III, one can obtaina problem in the form (20) describing the behaviour of the growing viscoelastic solid up to the instant τ_4 when the accretion stops again. Naturally, the new growth surface may not be related in any way to the previous one, the functions and parameters in (20) may take new values. Once the problem is solved, the SSS of the growing solid can be determined using (22).

For $t \geq \tau_4$, when the solid does not grow, the problem can be reduced in the same way to the form (24) and then (22) can be used.

The following step-wise scheme can be used to solve the problem of arbitrary piecewise-continuous accretion. First (13) is solved. Then solutions of (20) and (24) are constructed at each stage involving either continuous accretion or, respectively, no growth at all. The final results can be obtained using (22).

Hence it follows that the process of piecewise-continuous accretion of a viscoelastic ageing solid with any finite number of instants when the growth starts and stops can be considered using the method proposed. The problem with n instants when growth starts (and, naturally, n instants when it stops) can be reduced to the study of $2n + 1$ problems of one type, which have the same form as the BVP of the theory of elasticity containing t as a parameter. Once these $2n + 1$ problems are solved, the SSS of the viscoelastic ageing solid under consideration can be easily reconstructed for any time from the above formulas.

The one-to-one correspondence between the solutions of the problem of piecewise continuous accretion of a viscoelastic ageing solid and the BVP of the theory of elasticity established in the present section enables us to conclude that a unique solution of the IBVP exists that describes the piecewise continuous accretion of a viscoelastic solid because a unique solution of the BVP of the theory of elasticity exists.

VI. Conclusions

One can obtain a number of interesting results from (13), (20), (24), (14) and (22) using the property of limited creep of a viscoelastic material. If one assumes that only the surface of the basic solid is subject to a load, the actions are stationary, and accretion does not involve pretension, then the interaction between newly deposited particles and the solid already formed can be neglected starting from some instant t^0. In other words, starting from t^0 the growth process has little effect on the state of the part of the solid formed prior to to and the part formed for $t > t^0$ is practically stress-free. In particular, if the instant when a stationary load is applied to the basic solid is much earlier than the instant when the accretion starts, all other conditions being equal, then the effect of accretion on the state of the basic solid will be quite small and practically the whole additional solid will be strain-free. Similar conclusions can be drawn when considering a load regime of the original solid under which the actions remain constant for a prolonged period of time prior to the beginning of growth, irrespective of their variation at earlier times.

The effects considered have a clear mechanical meaning. Indeed, the deformation of a viscoelastic solid will practically cease after a period of time under limited creep conditions and stationary actions. Subsequent deposition of stress-free elements leads to a situation when the interaction between the parts of the solid already formed and those being created during the growth process is negligible.

Relations (13), (20), (24), (14) and (22) also enable us to predict such phenomena inherent in growing solids as the presence of residual stresses after the loads are removed, the presence of surfaces of stress discontinuity in the growing solid, and the dependence of the SSS of a viscoelastic solid on the growth rate (only the order of the acts of deposition and loading matters in the elastic case).

Finally, we discuss one more important aspect of the problem of accretion of a solid. It is concerned with the correspondence between the solution of the accretion IBVP and the viscoelasticity BVP for a variable boundary. The question is as follows: when will the solution of the non-classical accretion problem be the same as that of the classical problem of solid mechanics in a domain which varies with time? It turns out that the solutions are the same only when the strains in the growing solid and the deposited elements can be made compatible . Being a degenerate case of the IBVP describing the accretion of a solid, such an accretion regime clearly cannot be realized in practice.

Unlike the degenerate case when the strains in the whole solid are compatible during the accretion process in the case of stress-free elements being deposited on the solid which is a completely relevant version of the accretion process the problem will fail to become much simpler. It provides a brilliant demonstration of the effects related to accretion in model examples and is often encountered in applications. Here we have a situation when some inhomogeneous condition, rather than the homogeneous one, is trivial in a certain sense, unlike the traditional formulation of the BVP in solid mechanics.

Thus, using the presented approach for mechanical design of AM fabricated parts from viscoelastic materials one can determine the strength and the shape of final products. Moreover, on the basis of this mechanical analysis one can work out effective recommendations for improving the technological process.

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