The Partial Order on Category of Semigroups and Endo-Cayley Digraphs

C. Promsakon

***Abstract*—Let $\mathbb{C}$ be a set of all structures (S_f, A) where S is a semigroup, f is an endomorphism on S and A is a subset of S. We define operation $\otimes$ on $\mathbb{C}$ and also prove $(\mathbb{C}, \otimes)$ is a semigroup. Similarly, we show $\mathbb{D}$, the set of all endo-Caylay sigraphs, is a semigroup under the tensor product. Moreover, we find partial order relations in both $\mathbb{C}$ and $\mathbb{D}$.**

***Index Terms*—Endo-Cayley digraph, Partial order, Semigroups.**

I. Introduction

Let $(S_1, \cdot)$ and $(S_2, \circ)$ be semigroups. We can create a new semigroup from S_1 and S_2 as $(S_1 \times S_2, \times)$ where $(a, b) \times (x, y) = (a \cdot x, b \circ y)$ for all $(a, b), (x, y) \in S_1 \times S_2$. This semigroup is called as **product semigroup**. Clearly that $\times$ is well-defined. Let $(a, b), (s, t)$ and (x, y) be elements in $S_1 \times S_2$. We have that $[(a, b) \times (s, t)] \times (x, y) = (a \cdot s, b \circ t) \times (x, y) = (a \cdot s \cdot x, b \circ t \circ y) = (a, b) \times (s \cdot x, t \circ y) = (a, b) \times [(s, t) \times (x, y)]$. Hence $S_1 \times S_2$ with operator $\times$ is a semigroup.

For any semigroup S, we denote $End(S)$ for the set of all endomorphisms on S. Let S be a semigroup and $f_1, f_2 \in End(S)$. We will show that $f_1 \circ f_2 \in End(S)$. Let x and y be elements in S. Then $f_1 \circ f_2(xy) = f_1(f_2(xy)) = f_1(f_2(x)f_2(y)) = f_1(f_2(x))f_1(f_2(y)) = (f_1 \circ f_2(x))(f_1 \circ f_2(y))$. Hence $f_1 \circ f_2 \in End(S)$. As we know associate property holds for functions composition, therefore $End(S)$ is a semigroup under function composition. In Theorem 1, we show an example of an endomorphism on the product semigroup.

Theorem 1. *Let S_1 and S_2 be semigroups and f_1 and f_2 be endomorphisms on S_1 and S_2, respectively. Define $f_1 \times f_2 : S_1 \times S_2 \to S_1 \times S_2$ by $f_1 \times f_2(s_1, s_2) = (f_1(s_1), f_2(s_2))$. Then $f_1 \times f_2 \in End(S_1 \times S_2)$.*

Proof: Let $(a, b), (x, y) \in S_1 \times S_2$. Then $f_1 \times f_2((a, b)(x, y)) = f_1 \times f_2((ax, by)) = (f_1(ax), f_2(by)) = (f_1(a)f_1(x), f_2(b)f_2(y)) = (f_1(a), f_2(b))(f_1(x), f_2(y)) = (f_1 \times f_2(a, b))(f_1 \times f_2(x, y))$. Hence $f_1 \times f_2 \in End(S_1 \times S_2)$. ■

Let $\mathbb{C} = \{(S_f, A) | S$ is a semigroup, $A \subseteq S, f \in End(S)\}$. We say $(S_{f_1}, A) = (T_{f_2}, B)$ if there exists a function $f : S \to T$ such that $S \cong^f T$, $f(A) = B$ and $f \circ f_1 = f_2 \circ f$. We will show that $=$ is an equivalent relation on $\mathbb{C}$.

Proposition 2. *A relation $=$ is an equivalent relation on $\mathbb{C}$.*

Proof: We show $=$ preserve reflexive, symmetric and transitive properties here.

- To show $=$ is reflexive, let $(S_f, A) \in \mathbb{C}$. Clearly that $S \cong^I S$, $I(A) = A$ and $If_1 = f_1 = f_1 I$ where I is an identity function. So $=$ is reflexive.
- To show $=$ is symmetric, let (S_{f_1}, A) and (T_{f_2}, B) be in $\mathbb{C}$ such that $(S_{f_1}, A) = (T_{f_2}, B)$. Then there exists a function $f : S \to T$ such that $S \cong^f T$, $f(A) = B$ and $f \circ f_1 = f_2 \circ f$. So f is a bijection and a bijection $f^{-1} : T \to S$ exists. We obtain that $T \cong^{f^{-1}} S$, $f^{-1}(B) = A$ and $f^{-1}f_2 = f_1 f^{-1}$. Therefore $(T_{f_2}, B) = (S_{f_1}, A)$.
- To show $=$ is transitive, let (S_{f_1}, A), (T_{f_2}, B) and (U_{f_3}, C) be elements in $\mathbb{C}$ such that $(S_{f_1}, A) = (T_{f_2}, B)$ and $(T_{f_2}, B) = (U_{f_3}, C)$. Then there exist bijections $f : S \to T$ and $g : T \to U$ such that $S \cong^f T$ $f(A) = B$, $ff_1 = f_2 f$ and $T \cong^g U$ $g(B) = C$, $gf_2 = f_3 g$. So $g \circ f$ is a bijection and also $S \cong^{g \circ f} U$. Next, $g \circ f(A) = g(f(A)) = g(B) = C$ and $(g \circ f) \circ f_1 = g \circ (f \circ f_1) = g \circ (f_2 \circ f) = (g \circ f_2) \circ f = (f_3 \circ g) \circ f = f_3 \circ (g \circ f)$. Hence $(S_{f_1}, A) = (U_{f_3}, C)$.

Therefore $=$ is an equivalent relation on $\mathbb{C}$. ■

Next, we define an operator on $\mathbb{C}$, called $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, as $(S_{f_1}, A) \otimes (T_{f_2}, B) = (S \times T_{f_1 \times f_2}, A \times B)$ for all (S_{f_1}, A) and (T_{f_2}, B) be elements in $\mathbb{C}$. Let $(S_{f_1}, A), (T_{f_2}, B) \in \mathbb{C}$. It is clear by Theorem 1 that $S \times T$ is a semigroup and $f_1 \times f_2 \in End(S \times T)$. Since A and B are subset of S and T, respectively, we have $A \times B \subseteq S \times T$. Therefore $\otimes$ is well-defined. Now, we show $\mathbb{C}$ with operator $\otimes$ is a semigroup.

Theorem 3. *$(\mathbb{C}, \otimes)$ is a commutative semigroup.*

Proof: We already show $\otimes$ has a closed property. It remains to show that $\otimes$ preserves an associate property. Let $(S_{f_1}, A), (T_{f_2}, B)$ and (U_{f_3}, C) be in $\mathbb{C}$. Then

$$\begin{aligned}
&[(S_{f_1}, A) \otimes (T_{f_2}, B)] \otimes (U_{f_3}, C) \\
&= (S \times T_{f_1 \times f_2}, A \times B) \otimes (U_{f_3}, C) \\
&= (S \times T \times U_{f_1 \times f_2 \times f_3}, A \times B \times C) \\
&= (S_{f_1}, A) \otimes (T \times U_{f_2 \times f_3}, B \times C) \\
&= (S_{f_1}, A) \otimes [(T_{f_2}, B) \otimes (U_{f_3}, C)].
\end{aligned}$$

Therefore $(\mathbb{C}, \otimes)$ is a semigroup. Next, we show $(\mathbb{C}, \otimes)$ is commutative. Let (S_{f_1}, A) and (T_{f_2}, B) be elements in $\mathbb{C}$. Define $\phi : S \times T \to T \times S$ by $\phi(s, t) = (t, s)$ for all $(s, t) \in S \times T$. Let $(s_1, t_1), (s_2, t_2) \in S \times T$. Then

$$\begin{aligned}
\phi((s_1, t_1) \times (s_2, t_2)) &= \phi(s_1 s_2, t_1 t_2) \\
&= (t_1 t_2, s_1 s_2) \\
&= (t_1, s_1)(t_2, s_2) \\
&= \phi(s_1, t_1) \times \phi(s_2, t_2).
\end{aligned}$$

So ϕ is homomorphism. Clearly that ϕ is bijective. Hence ϕ is isomorphism and also $S \times T \cong^{\phi} T \times S$. Since ϕ is bijective,

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C. Promsakon are with the Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok,Bangkok 10800, Thailand.
E-mail: chanon.p@sci.kmutnb.ac.th.

we have that $\phi(A \times B) = B \times A$. Let $(s,t) \in S \times T$. Then

$$\begin{aligned}(\phi \circ f_1 \times f_2)(s,t) &= \phi(f_1(s), f_2(t)) \\ &= (f_2(t), f_1(s)) \\ &= f_2 \times f_1(t,s) \\ &= (f_2 \times f_1)(\phi(s,t)) \\ &= ((f_2 \times f_1) \circ \phi)(s,t).\end{aligned}$$

Therefore $(S_{f_1}, A) \otimes (T_{f_2}, B) = (T_{f_2}, B) \otimes (S_{f_1}, A)$. ■

Next, we present idempotent elements of $\mathbb{C}$. We use $E(\mathbb{C})$ for the set of idempotent elemaent of $\mathbb{C}$

Theorem 4. *Let $\{e\}$ be a trivial semigroup. Then $\{(\{e\}_{Id}, \{e\}), (\{e\}_{Id}, \phi)\} \in E(\mathbb{C})$. Moreover, $(\{e\}_{Id}, \{e\})$ is an identity of $\mathbb{C}$.*

Proof: Clearly that

$$(\{e\}_{Id}, \{e\}) \otimes (\{e\}_{Id}, \{e\}) = (\{(e,e)\}_{Id\times Id}, \{(e,e)\}).$$

We will show that $(\{(e,e)\}_{Id\times Id}, \{(e,e)\}) = (\{e\}_{Id}, \{e\})$. Define $f : \{(e,e)\} \to \{e\}$ by $f((e,e)) = e$. We obtain that $\{(e,e)\} \cong^f \{e\}$, $f((e,e)) = e$ and $f \circ (Id \times Id)(e,e) = f((e,e,)) = e = Id(e) = Id(f((e,e))) = Id \circ f((e,e))$.

Hence $(\{(e,e)\}_{Id\times Id}, \{(e,e)\}) = (\{e\}_{Id}, \{e\})$. We can prove that $(\{e\}_{Id}, \phi) \otimes (\{e\}_{Id}, \phi) = (\{e\}_{Id}, \phi)$. So $(\{e\}_{Id}, \{e\}), (\{e\}_{Id}, \phi) \in E(\mathbb{C})$.

Let (S_f, A) be an element in $\mathbb{C}$. Define $g : S \to S \times \{e\}$ by $g(s) = (s,e)$ for all $s \in S$. It is easy too see that g is an isomorphism and $g(A) = A \times \{e\}$. Let $s \in S$. Then $g \circ f(s) = g(f(s)) = (f(s), e) = f \times Id(s,e) = (f \times Id) \circ g(s)$. Hence $(S_f, A) = (S_f, A) \otimes (\{e\}_{Id}, \{e\})$. Because $\mathbb{C}$ is commutative, so $(\{e\}_{Id}, \{e\})$ is an identity. ■

Now, we have $\mathbb{C}$ is a commutative monoid. Our goal is to define a partial order on $\mathbb{C}$. We present that results in section 2. In [1], we see a relation between $\mathbb{C}$ and $\mathbb{D}$ where $\mathbb{D}$ is a category of digraph. So we will find a partial order on $\mathbb{D}$ with respond to a partial order on $\mathbb{C}$ showed in section 3.

II. Partial order on $\mathbb{C}$

This section, we present a partial order on $\mathbb{C}$ and also show a compatible property of that partial order.

Theorem 5. *Let $\leq$ be a relation on $\mathbb{C} \times \mathbb{C}$ defined by $(S_{f_1}, A) \leq (T_{f_2}, B)$ if and only if there is a subsemigroup H of T such that $S \cong^f H$, $f(A) \subseteq B$ and $f \circ f_1 = f_2 \circ f$. Then $\leq$ is a partial order on $\mathbb{C}$.*

Proof: We will show that $\leq$ preserves reflexive, anti-symmetric and transitive properties.

- To show $\leq$ is reflexive, let (S_{f_1}, A) be an element in $\mathbb{C}$. Clearly that $(S_{f_1}, A) \cong^{id_S} (S_{f_1}, A)$, $id_S(A) = A$ and $id_S \circ f_1 = f_1 \circ id_S$. Hence $(S_{f_1}, A) \leq (S_{f_1}, A)$.
- To show $\leq$ is anti-symmetric, let (S_{f_1}, A) and (T_{f_2}, B) be elements in $\mathbb{C}$ such that $(S_{f_1}, A) \leq (T_{f_2}, B)$ and $(T_{f_2}, B) \leq (S_{f_1}, A)$. Then there exist subsemigroups H_1 and H_2 of S and T, respectively, such that $S \cong^f H_2$ and $T \cong^g H_1$. So $|S| = |H_2| \leq |T|$ and $|T| = |H_1| \leq |S|$. Hence $H_1 = S$ and $H_2 = T$. We have $S \cong^f T$ and $f = g^{-1}$. Since $f(A) \subseteq B$ and $f^{-1}(A) = g(A) \subseteq A$, we have $f(A) = B$. Finally, we have $ff_1 = f_2 f$, because $(S_{f_1}, A) \leq (T_{f_2}, B)$. Hence $(S_{f_1}, A) = (T_{f_2}, B)$.
- To show $\leq$ is transitive, let (S_{f_1}, A), (T_{f_2}, B) and (U_{f_3}, C) be elements in $\mathbb{C}$ such that $(S_{f_1}, A) \leq (T_{f_2}, B)$ and $(T_{f_2}, B) \leq (U_{f_3}, C)$. Then we have $S \cong^f H_1$, $f(A) \subseteq B$ and $ff_1 = f_2 f$ for all $H_1 < T$ and $T \cong^g H_2$, $g(B) \subseteq C$ and $gf_1 = f_2 g$ for all $H_2 < U$. So $g \circ f : S \to g(H_1)$ is bijective. Since $H_1 < T$ and g is a semigroup homomorphism, we have $g(H_1) < U$. Let $s_1, s_2 \in S$. Then we have $g \circ f(s_1 s_2) = g(f(s_1 s_2)) = g(f(s_1)f(s_2)) = g(f_1(s_1))g(f_1(s_2)) = (g \circ f)(s_1)(g \circ f)(s_2)$. Hence $g \circ f$ is a homomorphism. Therefore $S \cong^{g\circ f} g(H_1)$. Since f and g are bijective, we have $f(A) \subseteq B$ imply that $g \circ f(A) \subseteq g(B) \subseteq C$. Next, let $s \in S$. Then we have $((g \circ f)f_1)(s) = g(f(f_1(s))) = g((ff_1)(s)) = g((f_2 f)(s) = g(f_2(f(s))) = (gf_2)(f(s)) = (f_3 g)(f(s)) = f_3(g(f(s)) = f_3((g \circ f)(s)) = (f_3(g \circ f))(s)$. Hence $(g \circ f)f_1 = f_3(g \circ f)$. Therefore $(S_{f_1}, A) \leq (U_{f_3}, C)$.

So $\leq$ is a partial order relation on $\mathbb{C}$. ■

Example 6. Let (S_{f_1}, A) and $(T_{f_2}, \{e\})$ be elements in $\mathbb{C}$ where e is an idempotent of T. We have $(S_{f_1}, A) \leq (S_{f_1}, A) \otimes (T_{f_2}, \{e\})$. To prove this statement, define $\beta : S \to S \times \{e\}$ by $\beta(s) = (s,e)$ for all $s \in S$. Clearly that $S \times \{e\} < S \times T$ and β is bijective. Hence $S \cong^\beta S \times \{e\}$. For any $a \in A$, $\beta(a) = (a,e) \in A \times \{e\}$. Thus $\beta(A) \subseteq A \times \{e\}$. Let $s \in S$. Then $\beta f_1(s) = (f_1(s), e) = (f_1 \times f_2)(s,e) = (f_1 \times f_2)\beta(s)$. Hence $\beta f_1 = (f_1 \times f_2)\beta$. Therefore $(S_{f_1}, A) \leq (S_{f_1}, A) \otimes (T_{f_2}, \{e\})$.

Suppose that $((\mathbb{N}, +)_{id_\mathbb{N}}, \{5\}) \leq (([5,\infty), \cdot)_{id_\mathbb{R}}, \{5\})$. Then there is an isomorphism $f : \mathbb{N} \to H$ for some $H < [5,\infty)$ and $f(5) = 5$. Since $5 = f(5) = f(1+1+1+1+1) = f(1)^5$, we have $f(1) = \sqrt[5]{5} \notin [5,\infty)$. This is a contradiction. Hence $((\mathbb{N}, +)_{id_\mathbb{N}}, \{5\}) \nleq (([5,\infty), \cdot)_{id_\mathbb{R}}, \{5\})$.

Proposition 7. *For any (S_f, A) in $\mathbb{C}$, $(S_f, A) \leq (S_f, A) \otimes (S_f, A)$.*

Proof: Let (S_f, A) be an element in $\mathbb{C}$. Clearly that $\bigcup_{s\in S} < (s,s) >$ is a subsemigroup of $S \times S$. Define $g : S \to S \times S$ by $g(s) = (s,s)$ for any $s \in S$. We leave to show g is a isomorphism. For any $s \in S$, we have $gf(s) = (f(s), f(s)) = f \times f(s,s) = ((f \times f)g)(s)$. Hence $gf = (f \times f)g$. Let $a \in A$. Then $g(a) = (a,a) \in A \times A$. So $g(A) \subseteq A \times A$. Therefore $(S_f, A) \leq (S_f, A) \otimes (S_f, A)$. ■

Theorem 8. *$\leq$ is compatible.*

Proof: We know $(\mathbb{C}, \otimes)$ is commutative by Theorem 3. It is sufficient to show $\leq$ is right compatible. Let (S_{f_1}, A), (T_{f_2}, B) and (U_{f_3}, C) be elements in $\mathbb{C}$ such that $(S_{f_1}, A) \leq (T_{f_2}, B)$. Then there is a isomorphism $f : S \to H$ for some $H < T$ such that $f(A) \subseteq B$ and $ff_1 = f_2 f$. We will show that $(S \times U_{f_1\times f_3}, A \times C) \leq (T \times U_{f_2\times f_3}, B \times C)$.

First, we define $\phi : S \times U \to H \times U$ as $\phi(s,u) = (f(s), u)$ for all $(s,u) \in S \times U$. Clearly $\phi(s,u) \in T \times U$, since $f(s) \in H$ for all $s \in S$. So ϕ is well-defined. Let (s_1, u_1) and (s_2, u_2) be elements in $S \times U$. Then

$$\begin{aligned}\phi((s_1,u_1)(s_2,u_2)) &= \phi(s_1 s_2, u_1 u_2) \\ &= (f(s_1 s_2), u_1 u_2) \\ &= (f(s_1)f(s_2), u_1 u_2) \\ &= (f(s_1)u_1)(f(s_2), u_2) \\ &= \phi(s_1,u_1)\phi(s_2,u_2).\end{aligned}$$

So ϕ is a homomorphism. Let $(h,u) \in H \times U$. Then $h \in H = Im(f)$. So there is $s \in S$ such that $f(s) = h$ and also $\phi(s,u) = (h,u)$. Hence ϕ is onto. To prove ϕ is one-to-one, let (s_1,u_1) and (s_2,u_2) be elements in $S \times U$ such that $(f(s_1),u) = \phi(s_1,u_1) = \phi(s_2,u_2) = (f(s_2),u)$. Then $f(s_1) = f(s_2)$ and $u_1 = u_2$. Since f is injective, we have $s_1 = s_2$. Thus $(s_1,u_1) = (s_2,u_2)$ and ϕ is injective. Therefore $S \times U \cong^{\phi} T \times U$.

Let (a,c) be an element in $A \times C$. Then $\phi(a,c) = (f(a),c) \in B \times C$. Hence $\phi(A \times C) \subseteq B \times C$.

The last part of proof, we need to show that $\phi(f_1 \times f_3) = (f_2 \times f_3)\phi$. Let $(s,u) \in S \times U$. Then

$$\begin{aligned}\phi(f_1 \times f_3)(s,u) &= \phi(f_1(s), f_3(u)) \\ &= (f(f_1(s)), f_3(u)) \\ &= (ff_1(s), f_3(u)) \\ &= (f_2f(s), f_3(u)) \\ &= (f_2 \times f_3)(f(s), u) \\ &= (f_2 \times f_3)(\phi(s,u)) \\ &= (f_2 \times f_3)\phi(s,u).\end{aligned}$$

Hence $\phi(f_1 \times f_3) = (f_2 \times f_3)\phi$.

By all of above, we can conclude that $(S \times U_{f_1 \times f_3}, A \times C) \leq (T \times U_{f_2 \times f_3}, B \times C)$. Therefore $\leq$ is compatible. ■

Corollary 9. *For any (S_f, A) in $\mathbb{C}$, $(S_f, A) \leq (S_f, A)^n$ for all integer n.*

For any endomorphism f on $\mathbb{Z}_4$, $f(0) = 0$. So $(\mathbb{Z}_{4_{Id}}, \{0\}) \not\leq (\mathbb{Z}_{4_{Id}}, \{1\})$. Next, $(\mathbb{Z}_{4_{Id}}, \{0\}) \otimes (\mathbb{Z}_{4_{Id}}, \phi) = ((\mathbb{Z}_4 \times \mathbb{Z}_4)_{Id \times Id}, \phi) = (\mathbb{Z}_{4_{Id}}, \{1\}) \otimes (\mathbb{Z}_{4_{Id}}, \phi)$. Hence converse of Theorem 8 is not true.

III. Partial order on the set of endo-Cayley graphs

We begin this section by giving some definitions used in this paper. Let G_1 and G_2 be digraphs. We call H as **subgraph** of G_1, if $V(H) \subset V(G_1)$ and $E(H) \subseteq E(G_1)$. A subgraph H is **induce**, if whenever vertices u and v in $V(H)$ are joined by arc e in G, then e is in $E(H)$. Assume $V \subseteq V(G_1)$. We refer $G_1[V]$ as a induce subgraph with vertex set is V. A **homomorphism** between G_1 and G_2 is a function $f : V(G_1) \to V(G_2)$ such that if $(x,y) \in E(G_1)$, then $(f(x), f(y)) \in E(G_2)$. We call a homomorphism which is bijective as **isomorphism**. If there is an isomorphaim between G_1 and G_2 denoted by $G_1 \cong G_2$, we refer as $G_1 = G_2$. We call G_1 is **embedded** in G_2, if $G_1 \cong H$ for some subgraph H of G_2.

The **tensor product** of G_1 and G_2, denoted by $G_1 \otimes G_2$, is a graph with vertex set $V(G_1) \times V(G_2)$ where (u_1,v_1) is adjacent to (u_2,v_2) if and only if u_1 is adjacent to u_2 and v_1 is adjacent to v_2.

We note here that $G_1 \otimes G_2 = G_2 \otimes G_1$ for any graphs G_1 and G_2, because $G_1 \otimes G_2 \cong G_2 \otimes G_1$.

Theorem 10. *Let G_1, G_2 and G_3 be graphs. Then $G_1 \otimes (G_2 \otimes G_3) = (G_1 \otimes G_2) \otimes G_3$.*

Proof: Clearly that $V(G_1 \otimes (G_2 \otimes G_3) = G_1 \times G_2 \times G_3 = V((G_1 \otimes G_2) \otimes G_3)$. Let $(u_1,v_1,w_1), (u_2,v_2,w_2) \in V(G_1 \times G_2 \times G_3)$ be adjacent called as arec e. Then

$e \in E(G_1 \otimes (G_2 \otimes G_3))$
$\leftrightarrow (u_1,u_2) \in E(G_1), ((v_1,w_1),(v_2,w_2)) \in E(G_2 \times G_3)$
$\leftrightarrow (u_1,u_2) \in E(G_1), (v_1,v_2) \in E(G_2), (w_1,w_2)) \in E(G_3)$
$\leftrightarrow ((u_1,v_1),(u_1,v_2)) \in E(G_1 \otimes G_2), (w_1,w_2)) \in E(G_3)$
$\leftrightarrow e \in E((G_1 \otimes G_2) \otimes G_3)$.

Hence $G_1 \otimes (G_2 \otimes G_3) = (G_1 \otimes G_2) \otimes G_3$. ■

Let S be semigroup, f endo-morphism on S and A a sebset of S. A **endo-Cayley graph of semigroup S on endomorphism f with connecting set** A, denoted by $endo - Cayley_f(s,A)$, is a graph whose vertex is element in S and there is an arc (u,v) if $v = f(u)a$ for some $a \in A$.

In [2], the theorem about relation between tensor product of endo-Cayley graphs and product of semigroups are shown. We post here.

Theorem 11. *Let S_1 and S_2 be semigroups, $A_1 \subseteq S_1$, $A_2 \subseteq S_2$ and endomorphisms f_1 and f_2 on S_1 and S_2, respectively. Then* $endo - Cay_{f_1 \times f_2}(S_1 \times S_2, A_1 \times A_2) = endo - Cay_{f_1}(S_1,A_1) \otimes endo - Cay_{f_2}(S_2,A_2)$.

Proof: For convenient, we let $G_1 = endo - Cay_{f_1 \times f_2}(S_1 \times S_2, A_1 \times A_2)$, $G_2 = endo - Cay_{f_1}(S_1,A_1)$ and $G_3 = endo - Cay_{f_2}(S_2,A_2)$. Let (u_1,v_1) be adjacent to (u_2,v_2) in G_1. Then

$((u_1,v_1),(u_2,v_2)) \in E(G_1)$
$\leftrightarrow (u_2,v_2) = (f_1(u_1)a_1, f_2(v_1)a_2)$, $a_1 \in A_1$ and $a_2 \in A_2$
$\leftrightarrow u_2 = f_1(u_1)a_1$ and $v_2 = f_2(v_1)a_2$
$\leftrightarrow (u_1,u_2) \in E(G_2)$ and $(v_1,v_2) \in E(G_3)$
$\leftrightarrow ((u_1,v_1),(u_2,v_2)) \in E(G_2 \otimes G_3)$.

Therefore $G_1 = G_2 \otimes G_3$. The proof was completed. ■

Let $\mathbb{D}$ be a category of endo-Cayley graphs. Clearly that $\mathbb{D}$ have a commutative property. So we have the following theorem.

Theorem 12. *$(\mathbb{D}, \otimes)$ is a commutative semigroup.*

Proof: It follows by Theorem 11 and 10. ■

Theorem 13. *Define a relation $\leq$ on $\mathbb{D}$ as $G_1 \leq G_2$ if G_1 is embedded in G_2. Then $\leq$ is partial order on $\mathbb{D}$. Moreover, $\leq$ is compatible.*

Proof: We ignore to show $\leq$ is partial order. We show $\leq$ is compatible by assuming G_1, G_2 and G_3 are graphs such that $G_1 \leq G_2$. Then $G_1 \cong^f G_2$. Define $g : G_1 \times G_2 \to G_2 \times G_3$ as $g(u,v) = (f(u),v)$. Clearly that g is bijective. Let vertices (u_1,v_1) and (u_2,v_2) in $G_1 \otimes G_3$ be adjacent. We have $(u_1,u_2) \in E(G_1)$ and $(v_1,v_2) \in E(G_3)$. So $(f(u_1),f(u_2)) \in E(G_2)$. Hence we have $((f(u_1),v_1),(f(u_2),v_2)) \in E(G_2 \otimes G_3)$. So g is a graph isomorphism. Therefore $G_1 \otimes G_3 \leq G_2 \otimes G_3$. Since $\mathbb{D}$ is commutative, we can conclude that $\leq$ is compatible. ■

Similarly to the converse of Theorem 8, if $G_1 \otimes G_3 \leq G_2 \otimes G_3$, it does not imply $G_1 \leq G_2$. We show here. We know that $k_n \otimes \bar{k_n} = \bar{k_{n^2}} = \bar{k_n} \otimes \bar{k_n}$ for any integer n but $k_n \not\leq \bar{k_n}$.

Theorem 14. *Let $\beta : \mathbb{C} \to \mathbb{D}$ by $\beta((S_f, A)) = endo - Cay_f(S,A)$ for any $(S_f,A) \in \mathbb{C}$. Then β is homomorphism.*

Proof: We show first that β is well defined. Let (S_{f_1}, A) and (T_{f_2}, B) be elements in $\mathbb{C}$ such that $(S_{f_1}, A) = (T_{f_2}, B)$.

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Then $S \cong^f T$, $f(A) = B$, and $ff_1 = f_2 f$. We show $endo - Cay_{f_1}(S, A) = endo - Cay_{f_2}(T, B)$ by proving that f is a graph isomorphism. Let $(s, f_1(s)a)$ be arc in $endo - Cay_{f_1}(S, A)$. Then $s \in S$ and $a \in A$. So $f(s) \in T$ and $f(a) \in B$. Hence $(f(s), f(f_1(s)a)) = (f(s), f(f_1(s))f(a)) = (f(s), (ff_1)(s)f(a)) = (f(s), (f_2 f)(s)f(a)) = (f(s), f_2(f(s))f(a)) \in E(endo - Cay_{f_1}(S, A))$. Since f is bijective, we have f is a graph homomorphism. Therefore $endo - Cay_{f_1}(S, A) = endo - Cay_{f_2}(T, B)$ and also β is well defined. It is easy to see that β is an onto function. Theorem 11 shows β is a homomorphism. ■

The homomorphism β is not injective showed in following example.

Example 15. *Let $S = (\mathbb{Z}_4, \cdot)$, $T = (\mathbb{Z}_2 \times \mathbb{Z}_2, \cdot)$, $A = \{0\}$ and $B = \{(0, 0)\}$. Then $Id_S \in End(S)$, $Id_T \in End(T)$ and $(S_{Id_S}, A), (T_{Id_T}, B) \in \mathbb{C}$. The endo-Cay graph of (S_{Id_S}, A) and (T_{Id_T}, B) are showed as below.*

$endo - Cay_{Id_S}(S, A) \quad endo - Cay_{Id_T}(T, B)$

It is clear that $endo - Cay_{Id_S}(S, A) = endo - Cay_{Id_T}(T, B)$ but obviously $(S_{Id_S}, A) \neq (T_{Id_T}, B)$ because $\mathbb{Z}_4 \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Theorem 16. *one vertex and one loop are exactly two idempotent element in $\mathbb{D}$. Moreover, an identity element in $\mathbb{D}$ is an one loop.*

Proof: By Theorem 4 and the existence of onto homomorphism between $\mathbb{C}$ and $\mathbb{D}$. ■

Now, we have a partial order on $\mathbb{C}$ and $\mathbb{D}$ and also know relation between both semigroup. So it is possible to compare that both partial order. The result is showed in Theorem 17.

Theorem 17. *Let (S_{f_1}, A) and (T_{f_2}, B) be in $\mathbb{C}$. If $(S_{f_1}, A) \leq (T_{f_2}, B)$, then $endo - Cay_{f_1}(S, A) \leq endo - Cay_{f_2}(T, B)$.*

Proof: Assume that $(S_{f_1}, A) \leq (T_{f_2}, B)$. Then $S \cong^f H$ for some $H < T$, $f(A) \subseteq B$ and $ff_1 = f_2 f$. We claim that $endo - Cay_{f_1}(S, A) \cong^f endo - Cay_{f_2}(T, B)[H]$.

We first prove that f is a graph homomorphism. Let $(s, f_1(s)a)$ be an arc in $endo - Cay_{f_1}(S, A)$ for some $s \in S$ and $s \in S$. Then $s \in S$ and $f_1(s)a \in S$. So $f(s) \in S$ and also $f_2(f(s))f(a) = (ff_1(s))f(a) = f(f_1(s)a) \in H$. By assumption, we have $f(a) \in B$ and arc $(f_2(f(s)), f_2(f(s))f(a))$ in $E(endo - Cay_{f_2}(T, B))$. Since both $f_2(f(s))$ and $f_2(f(s))f(a)$ are vertices in H, we have that $(f_2(f(s)), f_2(f(s))f(a)) \in E(endo - Cay_{f_2}(T, B)[H])$. We have f is bijective by assumption. So that claim is proved. Therefore $endo - Cay_{f_1}(S, A) \leq endo - Cay_{f_2}(T, B)$. ■

The converse of Theorem 17 is not true shown in Example 15.

We recall a natural partial order, Mitsch order. Let S be a semigroup and a, b elements in S. The natural partial order $\leq$ is defined by $a \leq b$ if and only if $a = xb = by$ and $xa = a$ for some $x, y \in S^1$. Next Theorem, we generalize elements in $\mathbb{C}$ with respect to a natural partial order.

Theorem 18. *Let (S_{f_1}, A) and (T_{f_2}, B) be in $\mathbb{C}$. $(S_{f_1}, A) \leq (T_{f_2}, B)$ where $\leq$ is a natural partial order if and only if $(S_{f_1}, A) = (T_{f_2}, B)$.*

Proof: Clearly that if $(S_{f_1}, A) = (T_{f_2}, B)$, then it follows that $(S_{f_1}, A) \leq (T_{f_2}, B)$. Suppose $(S_{f_1}, A) \leq (T_{f_2}, B)$. Then $(S_{f_1}, A) = (X_{g_1}, U) \otimes (T_{f_2}, B) = (T_{f_2}, B) \otimes (Y_{g_2}, V)$ and $(X_{g_1}, U) \otimes (S_{f_1}, A) = (S_{f_1}, A)$ for some $(X_{g_1}, U), (Y_{g_2}, V) \in \mathbb{C}$. So we have $|X| = 1$ which implies that (X_{g_1}, U) is an identity of $\mathbb{C}$. Hence $(S_{f_1}, A) = (X_{g_1}, U) \otimes (T_{f_2}, B) = (T_{f_2}, B)$. ■

Corollary 19. *A partial order relation on $\mathbb{C}$ and a partial order relation on $\mathbb{D}$ cover the Mitsch order.*

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