Interal Equations with Several Different Operators and Their Application to Mechanics

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***Abstract*—Integral equations with several different operators often arise in various fields of natural sciences. We call them mixed integral equations. One can find an integral equation with one Volterra and one compact operator in tribology and an equation with two Volterra operators and one compact operator in contact mechanics. Such equations are very specific and it is necessary to develop new approach for their solution. This paper deals with a new projection method for analytic solution of these equations. We consider the general case and, in particular,the contact problem for a viscoelastic foundation with a nonuniform coating. We write out the solution which can't be obtained by any other known method with sufficient accuracy.**

***Index Terms*—integral equation, Volterra operator, compact operator, contact mechanics, tribology**

INTRODUCTION

INTEGRAL equations with several different operators arose in the 70th–80th of XX century as a result of mathematical modeling of wear process (see, e.g., [1–2]) and contact interaction (see, e.g., [1–14]) for viscoelastic foundations with thin coatings. First attempts to obtain the solutions of these equations allowed to reduce them to solving of systems of infinite systems of Volterra integral equations which have no still analytical and numerical research methods. Moreover, contact problems for solids with nonuniform coatings and coatings with rough surface have posed another mathematical problem regarding adequate methods for equations with rapidly oscillating functions in initial data. We consider both problems and construct new efficient projection method as well as demonstrate its application to contact problem in mechanics.

I. PROJECTION METHOD FOR SOLVING MIXED EQUATIONS ON A BOUNDED SET

A. Mixed Operator Equation with a Given Right-Hand Side

Consider a mixed multi-dimensional equation with integral operators of Volterra and Schmidt types

$$\begin{aligned} &\sigma(t)(\mathcal{I}-\mathcal{V}_1)y(\mathbf{x},t)+(\mathcal{I}-\mathcal{V}_2)\mathcal{S}y(\mathbf{x},t)=\frac{f(\mathbf{x},t)}{h(\mathbf{x})},\\ &\mathcal{S}y(\mathbf{x},t)=\int_\Omega S(\mathbf{x},\boldsymbol{\xi})y(\boldsymbol{\xi},t)\,d\Omega_\xi,\quad S(\mathbf{x},\boldsymbol{\xi})=\frac{F(\mathbf{x},\boldsymbol{\xi})}{h(\mathbf{x})},\\ &\mathcal{V}_p y(\mathbf{x},t)=\int_{\tau_0}^t V_p(t,\tau)y(\mathbf{x},\tau)\,d\tau,\quad \mathbf{x}\in\Omega,\quad \tau_0\le t\le T. \end{aligned} \tag{1}$$

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In this section, we consider some general questions of the theory of mixed equations.

Let the right-hand side $f(\mathbf{x},t)/h(\mathbf{x})$ of equation (1) be known. It is required to find the function $y(\mathbf{x},t)$. Here, $f(\mathbf{x},t)$ and $y(\mathbf{x},t)$ are continuous functions of $t\in[1,T]$ with values in $L_2(\Omega)$; $\sigma(t)$ is a given positive continuous function; $h(\mathbf{x})>0$ is a given function of class $L_2(\Omega)$; $F(\mathbf{x},\boldsymbol{\xi})$ is a symmetric positive definite Fredholm kernel; $\mathcal{V}_1$ and $\mathcal{V}_2$ are Volterra operators; $\mathcal{S}$ is a Schmidt operator.

Let us transform the equation with the Schmidt operator to an equation with a Hilbert–Schmidt operator. To this end, we multiply (1) by $\sqrt{h(\mathbf{x})}$ and change the variables as follows

$$\begin{aligned} &q(\mathbf{x},t)=\sqrt{h(\mathbf{x})}y(\mathbf{x},t),\\ &F^h(\mathbf{x},\boldsymbol{\xi})=\frac{S(\mathbf{x},\boldsymbol{\xi})\sqrt{h(\mathbf{x})}}{\sqrt{h(\boldsymbol{\xi})}}=\frac{F(\mathbf{x},\boldsymbol{\xi})}{\sqrt{h(\mathbf{x})h(\boldsymbol{\xi})}}. \end{aligned} \tag{2}$$

Then

$$\begin{aligned} &\sigma(t)(\mathcal{I}-\mathcal{V}_1)q(\mathbf{x},t)+(\mathcal{I}-\mathcal{V}_2)\mathcal{F}^h q(\mathbf{x},t)=\frac{f(\mathbf{x},t)}{\sqrt{h(\mathbf{x})}},\\ &\mathcal{F}^h q(\mathbf{x},t)=\int_\Omega F^h(\mathbf{x},\boldsymbol{\xi})q(\boldsymbol{\xi},t)\,d\Omega_\xi,\\ &\mathbf{x}\in\Omega,\quad \tau_0\le t\le T. \end{aligned} \tag{3}$$

where $q(\mathbf{x},t)$ and $f(\mathbf{x},t)/\sqrt{h(\mathbf{x})}$ are continuous functions of $t\in[\tau_0,T]$ with values in the Hilbert space $L_2(\Omega)$; $\mathcal{F}^h$ is a Hilbert–Schmidt operator, and the other functions have been specified above.

Suppose that the right-hand side of equation (3) is known and we have to find the function $q(\mathbf{x},t)$.

Let us seek a solution of the mixed equation (3) in the form of a series

$$q(\mathbf{x},t)=\sum_{k=1}^\infty q_k(t)\varphi_k^h(\mathbf{x}), \tag{4}$$

where $\varphi_k^h(\mathbf{x})$ are eigenfunctions of the operator $\mathcal{F}^h$ corresponding to eigenvalues $\mu_k^h>0$, i.e.,

$$\mathcal{F}^h\varphi_k^h(\mathbf{x})\,d\xi=\mu_k^h\varphi_k^h(\mathbf{x}),\quad k=1,2,\ldots \tag{5}$$

The representation (4) is possible, since the system of eigenfunctions of the operator $\mathcal{F}^h$ forms a basis in $L_2(\Omega)$.

Let us construct the functions of the basis in the form

$$\varphi_k^h(\mathbf{x})=\frac{\Phi_k^h(\mathbf{x})}{\sqrt{h(\mathbf{x})}},\quad k=1,2,\ldots \tag{6}$$

with explicit dependence on the function $h(\mathbf{x})$, where

$$\begin{aligned} \int_\Omega \varphi_i^h(\boldsymbol{\xi})\varphi_j^h(\boldsymbol{\xi})\,d\Omega_\xi&=\int_\Omega \frac{\Phi_i^h(\boldsymbol{\xi})\Phi_j^h(\boldsymbol{\xi})}{h(\boldsymbol{\xi})}\,d\Omega_\xi\\ &=\delta_{ij}=\begin{cases}1, & \text{for } i=j,\\ 0, & \text{for } i\ne j.\end{cases} \end{aligned} \tag{7}$$

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In order to construct such eigenfunctions, we first construct a basis $p_n^h(\mathbf{x})$ in $L_2(\Omega)$ for which

$$\int_\Omega p_i^h(\boldsymbol{\xi}) p_j^h(\boldsymbol{\xi})\, d\Omega_\xi = \delta_{ij}, \quad p_n^h(\mathbf{x}) = \frac{P_n^h(\mathbf{x})}{\sqrt{h(\mathbf{x})}}, \qquad (8)$$
$$n = 1, 2, \dots$$

Such a basis can be constructed by the formulas

$$P_1^h(\mathbf{x}) = \frac{f_1(\mathbf{x})}{\sqrt{H_{11}}}, \quad \Delta_0 = 1, \quad \Delta_1 = H_{11},$$
$$\Delta_n = \begin{vmatrix} H_{11} & H_{12} & \dots & H_{1n} \\ H_{21} & H_{22} & \dots & H_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ H_{n1} & H_{n2} & \dots & H_{nn} \end{vmatrix}, \quad H_{ij} = \int_\Omega \frac{f_i(\boldsymbol{\xi}) f_j(\boldsymbol{\xi})}{h(\boldsymbol{\xi})}\, d\Omega_\xi, \qquad (9)$$
$$P_n^h(\mathbf{x}) = \frac{1}{\sqrt{\Delta_{n-1}\Delta_n}} = \begin{vmatrix} H_{11} & H_{12} & \dots & H_{1n} \\ H_{21} & H_{22} & \dots & H_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_1(\mathbf{x}), & f_2(\mathbf{x}) & \dots & f_n(\mathbf{x}) \end{vmatrix},$$

where $f_i(\mathbf{x})$ is an arbitrary complete system of linearly independent function in $L_2(\Omega)$.

Let us represent the kth eigenfunction in the form of a series with respect to the basis $p_i^h(\mathbf{x})$ of $L_2(\Omega)$. We have

$$\varphi_k^h(\mathbf{x}) = \sum_{i=1}^\infty \varphi_{i(k)}^h p_i^h(\mathbf{x}), \quad p_i^h(\mathbf{x}) = \frac{P_i^h(\mathbf{x})}{\sqrt{h(\mathbf{x})}},$$
$$\Phi_k^h(\mathbf{x}) = \sum_{i=1}^\infty \varphi_{i(k)}^h P_i^h(\mathbf{x}). \qquad (10)$$

The Hilbert–Schmidt kernel $F^h(\mathbf{x}, \boldsymbol{\xi})$ can be expanded into double series with respect to the chosen basis

$$F^h(\mathbf{x}, \boldsymbol{\xi}) = \sum_{m=1}^\infty \sum_{n=1}^\infty F_{mn}^h p_m^h(\mathbf{x}) p_n^h(\boldsymbol{\xi}),$$
$$F_{mn}^h = \int_\Omega\int_\Omega F^h(\mathbf{x}, \boldsymbol{\xi}) p_m^h(\mathbf{x}) p_n^h(\boldsymbol{\xi})\, d\Omega_x\, d\Omega_\xi, \quad F_{mn}^h = F_{nm}^h. \qquad (11)$$

Substituting (10) and (11) into (5), we obtain an infinite system of linear algebraic equations for the determination of the eigenvalues and the eigenfunction expansion coefficients. This system has a symmetric matrix and can be written as follows

$$\sum_{n=1}^\infty F_{mn}^h \varphi_{n(k)}^h = \mu_k^h \varphi_{m(k)}^h, \quad m = 1, 2, \dots \qquad (12)$$

In order to calculate approximations for N eigenvalues and eigenfunctions of the Hilbert–Schmidt operator $\mathcal{F}^h$, it is necessary to find the eigenvalues and orthonormal eigenvectors of the matrix

$$[F_{NN}^h] = \begin{pmatrix} F_{11}^h & F_{12}^h & \cdots & F_{1N}^h \\ F_{12}^h & F_{22}^h & \cdots & F_{2N}^h \\ \vdots & \vdots & \vdots & \vdots \\ F_{1N}^h & F_{2N}^h & \cdots & F_{NN}^h \end{pmatrix}. \qquad (13)$$

The eigenvalues of the matrix (13) give approximations for the first N eigenvalues of the Hilbert–Schmidt operator, and the components of its orthonormal eigenvectors give approximate values of the expansion coefficients for the first N eigenfunctions of that series with N orthonormal functions of the basis.

Let us write the right-hand side of equation (3) in the form

$$\frac{f(\mathbf{x}, t)}{\sqrt{h(\mathbf{x})}} = \sum_{k=1}^\infty f_k^h(t) \varphi_k^h(\mathbf{x}) = \sum_{k=1}^\infty f_k^h(t) \frac{\Phi_k^h(\mathbf{x})}{\sqrt{h(\mathbf{x})}},$$
$$f_k^h(t) = \int_\Omega \frac{f(\boldsymbol{\xi}, t)}{\sqrt{h(\boldsymbol{\xi})}} \varphi_k^h(\boldsymbol{\xi})\, d\Omega_\xi = \int_{-1}^1 \frac{f(\mathbf{x}, t)}{h(\mathbf{x})} \Phi_k^h(\mathbf{x})\, dx. \qquad (14)$$

Substituting (4), (14) into (3) and taking into account (5), we obtain the following sequence of Volterra equations for the unknown functions $q_k(t)$:

$$(\mathcal{I} - \mathcal{V}_k^h) q_k(t) = \delta_k^h(t), \quad \delta_k^h(t) = \frac{f_k^h(t)}{\sigma(t) + \mu_k^h},$$
$$\mathcal{V}_k^h = \frac{\sigma(t)\mathcal{V}_1 + \mu_k^h \mathcal{V}_2}{\sigma(t) + \mu_k^h}, \quad \mathcal{V}_k^h f(t) = \int_{\tau_0}^t V_k^h(t, \tau) f(\tau)\, d\tau, \qquad (15)$$
$$V_k^h = \frac{\sigma(t) V_1(t, \tau) + \mu_k^h V_2(t, \tau)}{\sigma(t) + \mu_k^h}, \quad k = 1, 2, \dots$$

where all operators $\mathcal{V}_k^h$ are of Volterra type, just as the operators $\mathcal{V}_1$ and $\mathcal{V}_2$, since $\mu_k^h \to 0$ as $k \to \infty$.

A solution of the sequence of Volterra equations (15) can be constructed by any known analytical and numerical method. This solution can be written in the form

$$q_k(t) = (\mathcal{I} + \mathcal{R}_k^h)\, \delta_k^h(t), \quad (\mathcal{I} - \mathcal{V}_k^h)^{-1} = (\mathcal{I} + \mathcal{R}_k^h),$$
$$\mathcal{R}_k^h f(t) = \int_{\tau_0}^t R_k^h(t, \tau) f(\tau)\, d\tau, \qquad (16)$$

where $\mathcal{R}_k^h$ is the resolvent operator for $\mathcal{V}_k^h$, and $R_k(t, \tau)$ is the resolvent of the kernel $V_k^h(t, \tau)$. The series (13) converges in $L_2(\Omega)$ uniformly in $t \in [\tau_0, t]$, and its sum is a continuous function of t with values in $L_2(\Omega)$.

Finally, taking into account (2), (4), and (6), (16), we find that

$$y(\mathbf{x}, t) = \frac{1}{h(\mathbf{x})} \sum_{k=1}^\infty (\mathcal{I} + \mathcal{R}_k^h) \delta_k^h(t) \Phi_k^h(\mathbf{x}). \qquad (17)$$

Note that the function $h(\mathbf{x})$ enters the solution (17) in explicit form, which allows us to solve equation (1) with high accuracy, even for a rapidly oscillating function $h(\mathbf{x})$.

B. Mixed Operator Equation with Auxiliary Conditions

Consider equation (1) with the right-hand side $f(\mathbf{x}, t) = \sum_{i=1}^N \alpha_i(t) f_i(\mathbf{x}) - g(\mathbf{x}, t)$ and N auxiliary integral conditions on the unknown function $y(x, t)$. The problem is to find a solution of the operator equation

$$\sigma(t)(\mathcal{I} - \mathcal{V}_1) y(\mathbf{x}, t) + (\mathcal{I} - \mathcal{V}_2)\mathcal{S} y(\mathbf{x}, t)$$
$$= \sum_{i=1}^N \alpha_i(t) \frac{f_i(\mathbf{x})}{h(\mathbf{x})} - \frac{g(\mathbf{x}, t)}{h(\mathbf{x})}, \qquad (18)$$
$$\mathbf{x} \in \Omega, \quad \tau_0 \le t \le T.$$

with the auxiliary conditions

$$\int_\Omega y(\boldsymbol{\xi}, t) f_i(\boldsymbol{\xi}) d\Omega_\xi = M_i(t), \quad i = 1, \dots, N, \qquad (19)$$

regarding $y(\mathbf{x}, t)$ and $\alpha_1(t), \dots, \alpha_N(t)$ as unknown functions. All other functions in (18) are assumed given, and $g(\mathbf{x}, t)$ is a continuous function of t with values in $L_2(\Omega)$; $f_i(\boldsymbol{\xi})$ is a system of N linearly independent functions in $L_2(\Omega)$.

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Let us transform the equation with the Schmidt operator to an equation with a Hilbert–Schmidt operator by changing the variables as in (2). Then, equation (18) and the auxiliary conditions (19) become

$$\begin{aligned} &\sigma(t)(\mathcal{I}-\mathcal{V}_1)q(\mathbf{x},t)+(\mathcal{I}-\mathcal{V}_2)\mathcal{F}^h q(\mathbf{x},t)\\ &\qquad=\sum_{i=1}^{N}\alpha_i(t)\frac{f_i(\mathbf{x})}{\sqrt{h(\mathbf{x})}}-\frac{g(\mathbf{x},t)}{\sqrt{h(\mathbf{x})}}, \qquad (20)\\ &\mathbf{x}\in\Omega,\quad \tau_0\le t\le T,\\ &\int_\Omega q(\boldsymbol{\xi},t)\frac{f_i(\boldsymbol{\xi})}{\sqrt{h(\boldsymbol{\xi})}}\,d\Omega_\xi=M_i(t),\quad i=1,\ldots,N. \qquad (21)\end{aligned}$$

In order to construct a solution of the mixed integral equation (18) with the auxiliary conditions (19), we construct a special basis in $L_2(\Omega)$ with explicit dependence on the function $1/\sqrt{h(\mathbf{x})}$. To this end, we complement the system of N linearly independent functions $f_i(\mathbf{x})$, so as to obtain a complete system in $L_2(\Omega)$, and then use formulas (8) and (9). As a result, we obtain a basis $p_n^h(\mathbf{x})$ in $L_2(\Omega)$ for which (in view of (8)and (9)) the following expansion holds

$$p_i^h(\mathbf{x})=\frac{P_i^h(\mathbf{x})}{\sqrt{h(\mathbf{x})}}=\sum_{k=1}^{i}a_{ik}\frac{f_k(\mathbf{x})}{\sqrt{h(\mathbf{x})}},\quad i=1,\ldots,N. \qquad (22)$$

Resolving the system of algebraic equations (22), we obtain

$$\frac{f_i(\mathbf{x})}{\sqrt{h(\mathbf{x})}}=\sum_{k=1}^{i}b_{ik}p_k^h(\mathbf{x}),\quad i=1,\ldots,N, \qquad (23)$$

the matrix of system (23) being the inverse of the matrix corresponding to system (22).

Let us represent the Hilbert space $L_2(\Omega)$ as a direct sum of its orthogonal subspaces

$$L_2(\Omega)=L_2^\circ(\Omega)\oplus L_2^*(\Omega), \qquad (24)$$

where $L_2^\circ(\Omega)$ is the Euclidean space with the basis $p_1(\mathbf{x}),\ldots,p_N(\mathbf{x})$, and $L_2^*(\Omega)$ is the Hilbert space with the basis $\{p_k(\mathbf{x})\}$ $(k=N+1,N+2,\ldots)$.

Note that any continuous function of t with values in $L_2(\Omega)$ can be represented as a sum of continuous functions of t with values in $L_2^\circ(\Omega)$ and $L_2^*(\Omega)$. Let us write such a representation for the integrand

$$q(\mathbf{x},t)=q^\circ(\mathbf{x},t)+q^*(\mathbf{x},t),\quad q^\circ(\mathbf{x},t)=\sum_{n=1}^{N}q_n^\circ(t)p_n^h(\mathbf{x}). \qquad (25)$$

Using the auxiliary conditions (21), together with (23) and (25), we obtain the following system of equations

$$\sum_{k=1}^{i}b_{ik}q_n^\circ(t)=M_i(t),\quad i=1,\ldots,N. \qquad (26)$$

The solution of this system determines the coefficients of the first term in the expansion (25) of $q^\circ(\mathbf{x},t)$:

$$q_i^\circ(t)=\sum_{k=1}^{i}a_{ik}M_k(t),\quad i=1,\ldots,N. \qquad (27)$$

In view of (23), the right-hand side of the equation can be written in the form

$$\begin{aligned} &\frac{f(\mathbf{x},t)}{\sqrt{h(\mathbf{x})}}=\sum_{i=1}^{N}\alpha_i(t)\frac{f_i(\mathbf{x})}{\sqrt{h(\mathbf{x})}}-\frac{g(\mathbf{x},t)}{\sqrt{h(\mathbf{x})}}=f_h^\circ(\mathbf{x},t)+f_h^*(\mathbf{x},t),\\ &f_h^\circ(\mathbf{x},t)=\sum_{k=1}^{N}\sum_{i=k}^{N}[\alpha_i(t)b_{ik}-g_k^{h\circ}(t)]p_k^h(\mathbf{x}),\\ &f_h^*(\mathbf{x},t)=-g_h^*(\mathbf{x},t), \qquad (28)\\ &\frac{g(\mathbf{x},t)}{\sqrt{h(\mathbf{x})}}=g_h^\circ(\mathbf{x},t)+g_h^*(\mathbf{x},t),\quad g_h^\circ(\mathbf{x},t)=\sum_{k=1}^{N}g_k^{h\circ}(t)p_k^h(\mathbf{x}),\\ &g_k^{h\circ}(t)=\int_\Omega\frac{g(\boldsymbol{\xi},t)}{\sqrt{h(\boldsymbol{\xi})}}p_k^h(\boldsymbol{\xi})\,d\Omega_\xi,\quad k=1,\ldots,N.\end{aligned}$$

Note that in the representations (25)–(28) for $q(\mathbf{x},t)$, the function $q^\circ(\mathbf{x},t)$ is known (as determined by the auxiliary conditions), and the term $q^*(\mathbf{x},t)$ is to be found. Conversely, for the right-hand side, we should find $f_h^\circ(\mathbf{x},t)$, and $f_h^*(\mathbf{x},t)$ is given by $g(\mathbf{x},t)/\sqrt{h(\mathbf{x})}$.

One can introduce an operator of orthogonal projection that maps the space $L_2(\Omega)$ onto $L_2^{h\circ}(\Omega)$:

$$\mathcal{P}_h^\circ f(\mathbf{x})=\int_\Omega f(\boldsymbol{\xi})\sum_{i=1}^{N}p_i^h(\mathbf{x})p_i^h(\boldsymbol{\xi})\,d\Omega_\xi. \qquad (29)$$

Obviously, the orthogonal projector $\mathcal{P}_h^*=\mathcal{I}-\mathcal{P}_h^\circ$ maps $L_2(\Omega)$ onto $L_2^{h*}(\Omega)$. Moreover, the following relations hold:

$$\begin{aligned} &\mathcal{P}_h^\circ q(\mathbf{x},t)=q^\circ(\mathbf{x},t),\quad \mathcal{P}_h^* q(\mathbf{x},t)=q^*(\mathbf{x},t),\\ &\mathcal{P}_h^\circ\frac{f(\mathbf{x},t)}{\sqrt{h(\mathbf{x})}}=f_h^\circ(\mathbf{x},t),\quad \mathcal{P}_h^*\frac{f(\mathbf{x},t)}{\sqrt{h(\mathbf{x})}}=f_h^*(\mathbf{x},t). \end{aligned} \qquad (30)$$

We apply the projection operator $\mathcal{P}_h^*$ to equation (20) and obtain an integral equation in $L_2^{h*}(\Omega)$ (with a known right-hand side) for the determination of $q^*(\mathbf{x},t)$:

$$\begin{aligned} &\sigma(t)(\mathcal{I}-\mathcal{V}_1)q^*(\mathbf{x},t)+(\mathcal{I}-\mathcal{V}_2)\mathcal{P}_h^*\mathcal{F}^h q^*(\mathbf{x},t)\\ &\qquad=-g^*(\mathbf{x},t)-(\mathcal{I}-\mathcal{V}_2)\mathcal{P}_h^*\mathcal{F}^h q^\circ(\mathbf{x},t),\\ &\mathcal{P}_h^*\mathcal{F}^h\phi(\mathbf{x},t)=\int_\Omega F_h^*(\mathbf{x},\boldsymbol{\xi})\phi(\boldsymbol{\xi},t)\,d\Omega_\xi, \qquad (31)\\ &\mathbf{x}\in\Omega,\quad \tau_0\le t\le T,\\ &F_h^*(\mathbf{x},\boldsymbol{\xi})=F^h(\mathbf{x},\boldsymbol{\xi})-\int_\Omega F^h(\mathbf{s},\boldsymbol{\xi})\sum_{i=1}^{N}p_i^h(\mathbf{x})p_i^h(\mathbf{s})\,d\Omega_s.\end{aligned}$$

The operator $\mathcal{P}_h^*\mathcal{F}^h$ is a Hilbert–Schmidt operator from $L_2^{h*}(\Omega)$ to $L_2^{h*}(\Omega)$. Let us construct a solution of equation (31) in the form of a series with respect to its eigenfunctions that form a basis in $L_2^{h*}(\Omega)$. Let us construct the system of these functions.

Let $\varphi_k^{h*}(\mathbf{x})$ be eigenfunctions of the operator $\mathcal{P}_h^*\mathcal{F}^h$ and μ_k^{h*} the corresponding eigenvalues. We have

$$\mathcal{P}_h^*\mathcal{F}^h\varphi_k^{h*}(\mathbf{x})=\mu_k^{h*}\varphi_k^{h*}(\mathbf{x}),\quad k=N+1,N+2,\ldots \qquad (32)$$

Let us represent the eigenfunction $\varphi_i^{h*}(\mathbf{x})$ as a series with respect to the basis $p_i^h(\mathbf{x})$ $(i\ge N+1)$

$$\begin{aligned} &\varphi_k^{h*}(\mathbf{x})=\sum_{i=N+1}^{\infty}\varphi_{i(k)}^{h*}p_i^h(\mathbf{x}),\quad p_i^h(\mathbf{x})=\frac{P_i^h(\mathbf{x})}{\sqrt{h(\mathbf{x})}},\\ &\Phi_k^{h*}(\mathbf{x})=\sum_{i=N+1}^{\infty}\varphi_{i(k)}^{h*}P_i^h(\mathbf{x}). \end{aligned} \qquad (33)$$

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Using (11) and (31), we obtain the following double series expansion for the kernel $F_h^*(\mathbf{x}, \boldsymbol{\xi})$:

$$F_h^*(\mathbf{x}, \boldsymbol{\xi}) = \sum_{m=N+1}^{\infty} \sum_{n=N+1}^{\infty} F_{mn}^h p_m^h(\mathbf{x}) p_n^h(\boldsymbol{\xi}) + \sum_{i=1}^{N} \sum_{n=N+1}^{\infty} F_{in}^h p_n^h(\mathbf{x}) p_i^h(\boldsymbol{\xi}). \tag{34}$$

Note that the coefficients in the expansion of the kernel $F_h^*(\mathbf{x}, \boldsymbol{\xi})$ in (34) coincide with those in the expansion of the kernel $F^h(\mathbf{x}, \boldsymbol{\xi})$, and this allows us to use the available data instead of recalculating the coefficients of the new problem.

Substituting (33) and (34) into (32), we obtain an infinite system of linear algebraic equations for the determination of the eigenvalues and the eigenfunction expansion coefficients. This system has a symmetric matrix and can be written in the form

$$\sum_{n=N+1}^{\infty} F_{mn}^h \varphi_{n(k)}^{h*} = \mu_k^{h*} \varphi_{m(k)}^{h*}, \quad m = N+1, N+2, \ldots \tag{35}$$

Now, let us construct a solution of equation (31). To this end, we represent the functions $q^*(\mathbf{x}, t)$ and $g_h^*(\mathbf{x}, t)$ in the form of series with eigenfunctions of the operator $\mathcal{P}_h^* \mathcal{F}^h$:

$$\begin{aligned} q^*(\mathbf{x}, t) &= \sum_{k=N+1}^{\infty} q_k^*(t) \varphi_k^{h*}(\mathbf{x}), \\ g_h^*(\mathbf{x}, t) &= \sum_{k=N+1}^{\infty} g_k^{h*}(t) \varphi_k^{h*}(\mathbf{x}), \\ g_k^{h*}(t) &= \int_{-1}^{1} g_h^*(\boldsymbol{\xi}, t) \varphi_k^{h*}(\boldsymbol{\xi}) \, d\Omega_\xi, \end{aligned} \tag{36}$$

and substitute these into (31). Then, taking into account (25)–(28) and (30)–(34), we obtain the following sequence of independent Volterra equations of the second kind:

$$\begin{aligned} &(\mathcal{I} - \mathcal{V}_k^{h*}) q_k^*(t) = \delta_k^{h*}(t), \quad \mathcal{V}_k^{h*} = \frac{\sigma(t)\mathcal{V}_1 + \mu_k^{h*}\mathcal{V}_2}{\sigma(t) + \mu_k^{h*}}, \\ &\mathcal{V}_k^{h*} f(t) = \int_{\tau_0}^{t} V_k^{h*}(t, \tau) f(\tau) \, d\tau, \\ &V_k^{h*}(t, \tau) = \frac{\sigma(t) V_1(t, \tau) + \mu_k^{h*} V_2(t, \tau)}{\sigma(t) + \mu_k^{h*}}, \\ &\delta_k^{h*}(t) = -\frac{1}{\sigma(t) + \mu_k^{h*}} \left[g_k^{h*}(t) + (\mathcal{I} - \mathcal{V}_2) \sum_{i=1}^{N} F_{k(i)}^h q_i^\circ(t) \right], \\ &F_{k(i)}^h = \sum_{n=N+1}^{\infty} F_{in}^h \varphi_{n(k)}^{h*}, \\ &i = 1, \ldots, N, \quad k = N+1, N+2, \ldots \end{aligned} \tag{37}$$

Resolving (37) with respect to $q_k^*(t)$, we get

$$\begin{aligned} &q_k^*(t) = (\mathcal{I} + \mathcal{R}_k^{h*}) \delta_k^{h*}(t), \quad (\mathcal{I} - \mathcal{V}_k^{h*})^{-1} = (\mathcal{I} + \mathcal{R}_k^{h*}), \\ &\mathcal{R}_k^{h*} f(t) = \int_{\tau_0}^{t} R_k^{h*}(t, \tau) f(\tau) \, d\tau, \end{aligned} \tag{38}$$

where $R_k^{h*}(t, \tau)$ is the resolvent of the kernel $V_k^{h*}(t, \tau)$.

We see that in view of (35)–(38) the function $q^*(\mathbf{x}, t)$ has been determined, and it is easy to find $q(\mathbf{x}, t)$, since $q^\circ(\mathbf{x}, t)$ is known by assumption (see (25)–(27)). Hence, taking into account the transformation of the variables (2), we finally obtain

$$y(\mathbf{x}, t) = \frac{1}{h(\mathbf{x})} \left[\sum_{n=1}^{N} \sum_{k=1}^{n} a_{nk} M_k(t) P_n^h(\mathbf{x}) + \sum_{k=N+1}^{\infty} q_k^*(t) \Phi_k^{h*}(\mathbf{x}) \right], \tag{39}$$

The solution (39) depends on the function $h(x)$ in explicit manner, and this allows us to solve equation (48) with high accuracy by keeping a relatively small number of terms in the series even for a rapidly oscillating $h(\mathbf{x})$.

In practical calculations, the number of terms in the expansions has to be limited. For instance, taking the basis functions $p_k^h(\mathbf{x})$ with $k = N+1, \ldots, M$, we obtain the Mth approximation of the desired solution. In this case, for the construction of eigenvalues and eigenfunctions of the Hilbert–Schmidt operator $\mathcal{P}_h^* \mathcal{F}^h$ one should find the eigenvalues and orthonormal eigenvectors of the matrix

$$[F_{MM}^h] = \begin{pmatrix} F_{N+1N+1}^h & F_{N+1N+2}^h & \cdots & F_{N+1M}^h \\ F_{N+1N+2}^h & F_{N+2N+2}^h & \cdots & F_{N+2M}^h \\ \vdots & \vdots & \ddots & \vdots \\ F_{N+1M}^h & F_{N+2M}^h & \cdots & F_{MM}^h \end{pmatrix}. \tag{40}$$

The eigenvalues of the matrix (65) give approximations of the first $M - N$ eigenvalues of the Hilbert–Schmidt operator, and the components of its orthonormal eigenvectors approximate the coefficients in the expansion of the first $M - N$ eigenfunctions of this operator. Recall that the first N terms of the expansion (25) of $q^\circ(\mathbf{x}, t)$ of the function $q(\mathbf{x}, t)$ are known by assumption. Therefore, constructing the next $M - N$ terms of the expansion (36) of $q^*(\mathbf{x}, t)$, we obtain the Mth approximation of the solution $q(\mathbf{x}, t)$ (see (25)).

It is important to observe that the matrix (40) can be obtained from the matrix (13) by deleting its first N rows and columns. This allows us to construct an expansion of the original kernel only once and then use these data for the examination of the new kernel arising in the problem with auxiliary conditions.

Now, in order to find the functions $\alpha_i(t)$ ($i = 1, \ldots, N$), we apply the projection operator $\mathcal{P}_h^\circ$ to equation (31). As a result, we get

$$\begin{aligned} \alpha_k(t) = \sum_{i=k}^{N} a_{ik} \Bigg\{ & g_i^{h\circ}(t) + \sigma(t)(\mathcal{I} - \mathcal{V}_1) \sum_{m=1}^{i} a_{im} M_m(t) \\ & + (\mathcal{I} - \mathcal{V}_2) \left[\sum_{j=1}^{N} F_{ji}^h \sum_{m=1}^{j} a_{jm} M_m(t) + \sum_{j=N+1}^{\infty} F_{j(i)}^h q_j^*(t) \right] \Bigg\}. \end{aligned} \tag{41}$$

Note that relations (41) form a system of N linear algebraic equations (with a triangular matrix) for the determination of the unknown quantities $\alpha_1(t), \ldots, \alpha_N(t)$.

Thus, we have constructed a complete solution of the integral equation (18) with the auxiliary conditions (19).

II. Application to Contact mechanics

Let us solve a plane problem of the interaction between a rigid punch and a viscoelastic foundation with rough surface

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which is described by the rapidly oscillating function $h(x)$. The surface of the punch is described by comparatively smooth function $g(x)$. Viscoelastic behavior of the foundation and the coating are given by continuous function $\sigma(t)$ and continuous or polar kernels $V_1(t,\tau)$ and $V_2(t,\tau)$ respectively.

Consider equation (1) with the right-hand side in the form $f(x,t) = \alpha_1(t) + \alpha_2(t)x - g(x,t)$ and two auxiliary integral conditions on the unknown function $y(x,t)$. The problem is to find the solution of the integral equation ($-1 \le x \le 1$, $1 \le t \le T$)

$$\sigma(t)\left[y(x,t) - \int_1^t V_1(t,\tau)y(x,\tau)\,d\tau\right] + \int_{-1}^1 S(x,\xi)y(\xi,t)\,d\xi - \int_1^t V_2(t,\tau)\int_{-1}^1 S(x,\xi)y(\xi,\tau)\,d\xi\,d\tau = \frac{\alpha_1(t)}{h(x)} + \frac{\alpha_2(t)x}{h(x)} - \frac{g(x,t)}{h(x)}, \quad S(x,\xi) = \frac{F(x,\xi)}{h(x)}. \tag{42}$$

with the auxiliary conditions

$$\int_{-1}^1 y(\xi,t)\,d\xi = M_1(t), \quad \int_{-1}^1 \xi\, y(\xi,t)\,d\xi = M_2(t), \tag{43}$$

where $y(x,t)$ is the contact pressure, $\alpha_1(t)$ and $\alpha_2(t)$ are the unknown settlement and tilt angle of a punch, $F(x,\xi)$ is the known kernel of the plane contact problem compact operator, x is the longitudinal coordinate, and t is the time variable.

Let us transform the equation with the Schmidt kernel to an equation with a Hilbert–Schmidt kernel by changing the variables in (42) and (43) (see (2)). As a result, the integral equation and the auxiliary conditions become ($-1 \le x \le 1$, $1 \le t \le T$)

$$\sigma(t)\left[q(x,t) - \int_1^t V_1(t,\tau)q(x,\tau)\,d\tau\right] + \int_{-1}^1 F^h(x,\xi)q(\xi,t)\,d\xi - \int_{\tau_0}^t V_2(t,\tau)\int_{-1}^1 F^h(x,\xi)q(\xi,\tau)\,d\xi\,d\tau = \frac{\alpha_1(t)}{\sqrt{h(x)}} + \frac{\alpha_2(t)x}{\sqrt{h(x)}} - \frac{g(x,t)}{\sqrt{h(x)}}, \tag{44}$$

$$\int_{-1}^1 \frac{q(\xi,t)}{\sqrt{h(\xi)}}\,d\xi = M_1(t), \quad \int_{-1}^1 \frac{q(\xi,t)}{\sqrt{h(\xi)}}\xi\,d\xi = M_2(t). \tag{45}$$

In order to construct a solution of the mixed integral equation (44) with the auxiliary conditions (45), we use the basis $p_n^h(x)$ in $L_2[-1,1]$ and note that the space $L_2[-1,1]$ can be represented as a direct sum of its orthogonal subspaces, $L_2[-1,1] = L_2^{h\circ}[-1,1] \oplus L_2^{h*}[-1,1]$, where $L_2^{h\circ}[-1,1]$ is the Euclidean space with the basis $p_1^h(x)$ and $p_2^h(x)$, and $L_2^{h*}[-1,1]$ is the Hilbert space with the basis $p_k^h(x)$ ($k = 3,4,\ldots$). It can be seen that the integrand and the right-hand side can be represented as a sum of continuous functions of $t \in [1,T]$ with values in $L_2^{h\circ}[-1,1]$ and $L_2^{h*}[-1,1]$, respectively, i.e,

$$q(x,t) = q^\circ(x,t) + q^*(x,t), \qquad \frac{f(x,t)}{\sqrt{h(x)}} = f_h^\circ(x,t) + f_h^*(x,t), \tag{46}$$

and the following representations hold:

$$\begin{aligned}
&q^\circ(x,t) = q_1^\circ(t)p_1^h(x) + q_2^\circ(t)p_2^h(x),\\
&q_1^\circ(t) = \frac{M_1(t)}{\sqrt{J_0}}, \quad q_2^\circ(t) = \frac{J_0M_2(t) - J_1M_1(t)}{\sqrt{J_0(J_0J_2 - J_1^2)}},\\
&\frac{f(x,t)}{\sqrt{h(x)}} = \frac{\alpha_1(t)}{\sqrt{h(x)}} + \frac{\alpha_2(t)x}{\sqrt{h(x)}} - \frac{g(x,t)}{\sqrt{h(x)}},\\
&\frac{g(x,t)}{\sqrt{h(x)}} = g_h^\circ(x,t) + g_h^*(x,t),\\
&f_h^\circ(x,t) = \left[\sqrt{J_0}\alpha_1(t) + \frac{J_1}{\sqrt{J_0}}\alpha_2(t) - g_1(t)\right]p_1^h(x)\\
&\qquad + \left[\frac{\sqrt{J_0J_2 - J_1^2}}{\sqrt{J_0}}\alpha(t) - g_1\right]p_2^h(x),\\
&f_h^*(x,t) = -g_h^*(x), \quad g_h^\circ(x,t) = g_1^{h\circ}(t)p_1^h(x) + g_2^{h\circ}(t)p_2^h(x),\\
&g_1^\circ(t) = \int_{-1}^1 \frac{g(x,t)}{\sqrt{h(x)}}p_1^h(x)\,dx, \quad g_2^\circ(t) = \int_{-1}^1 \frac{g(x,t)}{\sqrt{h(x)}}p_2^h(x)\,dx.
\end{aligned} \tag{47}$$

Note that in the representation (46) for $q(x,t)$, the function $q^\circ(x,t)$ is known as determined by the auxiliary conditions, and the term $q^*(x,t)$ should be found. Conversely, for the right-hand side, $f_h^\circ(x,t)$ should be found and $f_h^*(x,t)$ is determined by the function $g(x,t)/\sqrt{h(x)}$. The facts mentioned above allow us to classify the resulting problem as a special case of the general projection problem.

According to the general method, in the present case one can introduce an operator of orthogonal projection that maps the space $L_2[-1,1]$ onto $L_2^{h\circ}[-1,1]$:

$$\mathcal{P}_h^\circ\phi(x,t) = \int_{-1}^1 \phi(\xi,t)[p_1^h(x)p_1^h(\xi) + p_2^h(x)p_2^h(\xi)]\,d\xi. \tag{48}$$

Obviously, the projector $\mathcal{P}_h^* = \mathcal{I} - \mathcal{P}_h^\circ$ maps $L_2[-1,1]$ onto $L_2^{h*}[-1,1]$. Moreover, the following relations hold:

$$\begin{aligned}
&\mathcal{P}_h^\circ q(x,t) = q^\circ(x,t), \quad \mathcal{P}_h^* q(x,t) = q^*(x,t),\\
&\mathcal{P}_h^\circ \frac{f(x,t)}{\sqrt{h(x)}} = f_h^\circ(x,t), \quad \mathcal{P}_h^* \frac{f(x,t)}{\sqrt{h(x)}} = f_h^*(x,t),
\end{aligned} \tag{49}$$

Let us apply the projection operator $\mathcal{P}_h^*$ to equation (44). Then, for the determination of $q^*(x,t)$, we obtain the following integral equation in $L_2^{h*}[-1,1]$ with a known right-hand side:

$$\begin{aligned}
&\sigma(t)\left[q^*(x,t) - \int_1^t V_1(t,\tau)q^*(x,\tau)\,d\tau\right]\\
&\quad + \int_{-1}^1 F_h^*(x,\xi)q^*(\xi,t)\,d\xi\\
&\quad - \int_{\tau_0}^t V_2(t,\tau)\int_{-1}^1 F_h^*(x,\xi)q^*(\xi,\tau)\,d\xi\,d\tau\\
&\quad = -g_h^*(x,t) - \int_{-1}^1 F_h^*(x,\xi)q^\circ(\xi,t)\,d\xi\\
&\quad + \int_1^t V_2(t,\tau)\int_{-1}^1 F_h^*(x,\xi)q^\circ(\xi,\tau)\,d\xi\,d\tau,\\
&\qquad -1 \le x \le 1, \quad 1 \le t \le T,
\end{aligned} \tag{50}$$

where the kernel of the integral equation

$$F_h^*(x,\xi) = F^h(x,\xi) - \int_{-1}^1 F^h(s,\xi)[p_1^h(x)p_1^h(s) + p_2^h(x)p_2^h(s)]\,ds \tag{51}$$

is a Hilbert–Schmidt kernel. A solution of equation (50) can be constructed in the form of a series in terms of eigenfunctions of the kernel (51). These form a basis in the Hilbert space $L_2^{h*}[-1,1]$. Let us construct a system of these eigenfunctions.

For an eigenfunction $\varphi_k^{h*}(x)$, let μ_k^{h*} be the corresponding eigenvalue of the kernel $F_p^*(x,\xi)$. Then

$$\int_{-1}^{1} F_h^*(x,\xi)\varphi_k^{h*}(\xi)\,d\xi = \mu_k^{h*}\varphi_k^{h*}(x), \quad k=3,4,\dots \tag{52}$$

Let us represent the eigenfunction $\varphi_i^{h*}(x)$ as a series in terms of the basis functions $p_i^h(x)$ ($i \geq 3$):

$$\varphi_k^{h*}(x) = \sum_{i=3}^{\infty} \varphi_{i(k)}^{h*} p_i^h(x), \quad k=3,4,\dots \tag{53}$$

The double series expansion of the kernel $F_h^*(x,\xi)$ is obtained with the help of (31) and (34)

$$F_h^*(x,\xi) = \sum_{m=3}^{\infty}\sum_{n=3}^{\infty} F_{mn}^h p_m^h(x) p_n^h(\xi) + \sum_{n=3}^{\infty} F_{1n}^h p_n^h(x) p_1^h(\xi) + \sum_{n=3}^{\infty} F_{2n}^h p_n^h(x) p_2^h(\xi). \tag{54}$$

Substituting (53) and (54) into (52), we obtain the following infinite system of linear algebraic equations (with a symmetric matrix) for the eigenvalues and the eigenfunction expansion coefficients:

$$\sum_{n=3}^{\infty} F_{mn}^h \varphi_{n(k)}^{h*} = \mu_k^{h*}\varphi_{m(k)}^{h*}, \quad m=3,4,\dots \tag{55}$$

Now, let us construct a solution of equation (50). To that end, we represent the functions $q^*(x,t)$ and $g_h^*(x,t)$ as series in terms of eigenfunctions of the kernel $F_h^*(x,\xi)$:

$$\begin{aligned} q^*(x,t) &= \sum_{k=3}^{\infty} q_k^*(t)\varphi_k^{h*}(x), \\ g_h^*(x,t) &= \sum_{k=3}^{\infty} g_k^{h*}(t)\varphi_k^{h*}(x), \\ g_k^{h*}(t) &= \int_{-1}^{1} g_h^*(x,t)\varphi_k^{h*}(x)\,dx. \end{aligned} \tag{56}$$

Substituting these into (50) and taking into account (47), (53)–(56), we obtain the following sequence of independent Volterra equations of the second kind:

$$\begin{aligned} & q_k^*(t) - \int_1^t V_k^{h*}(t,\tau) q_k^*(\tau)\,d\tau = \delta_k^{h*}(t), \\ & V_k^{h*}(t,\tau) = \frac{\sigma(t)V_1(t,\tau) + \mu_k^{h*}V_2(t,\tau)}{\sigma(t)+\mu_k^{h*}}, \\ & \delta_k^{h*}(t) = -\frac{1}{\sigma(t)+\mu_k^{h*}}\Bigg[g_k^{h*}(t) + \sum_{i=1}^{2} F_{k(i)}^h q_i^\circ(t) \\ & - \int_1^t V_2(t,\tau)\sum_{i=1}^{2} F_{k(i)}^h q_i^\circ(\tau)\,d\tau \Bigg], \\ & F_{k(i)}^h = \sum_{n=3}^{\infty} F_{in}^h \varphi_{n(k)}^{h*}, \qquad i=1,2, \quad k=3,4,\dots \end{aligned} \tag{57}$$

Resolving (57) with respect to $q_k^*(t)$ by any known method, we obtain

$$q_k^*(t) = f_k^{h*}(t) + \int_1^t R_k^{h*}(t,\tau) f_k^{h*}(\tau)\,d\tau, \tag{58}$$

where $R_k^{h*}(t,\tau)$ is the resolvent of the kernel $V_k^{h*}(t,\tau)$.

Now, in view of (56)–(58), the function $q^*(x,t)$ has been determined, and therefore, we easily find $q(x,t)$, since $q^\circ(x,t)$ is known (see (46) and (47)).

Now, let us find the functions $\alpha_1(t)$ and $\alpha_2(t)$. To this end, we apply the projection operator $\mathcal{P}_h^\circ$ to equation (50). As a result, we obtain the system of two linear algebraic equations (with a triangular matrix) for the determination of the unknown quantities $\alpha_1(t)$ and $\alpha_2(t)$.

III. Conclusions

Thus, we have constructed the solution of integral equation with two Volterra operators and one compact operator. New proposed projection method has allowed us

- to avoid the solution of infinite system of integral Volterra equations to which the classical method of separation of variables leads,
- to obtain such a form of the solution where function h enters the solution in explicit form,
- to solve mixed integral equation with high accuracy even for a rapidly oscillating function h when the classical method gives up to 100% mistake.

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