On Fuzzy Multi-objective Linear Fractional Programming Problems

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***Abstract*—Optimization of the ratio of two functions is called fractional programming or ratio optimization problem. If one can optimize simultaneously collection of fractional objective functions then the problem is called multi-objective fractional programming. This paper presents a survey on fractional programming problems. In contrast, this survey excludes many of the technical details and provides a road map of currently available studies in the literature. The general concepts are briefly described and references are included for further investigations. The main contribution of authors is review and arrange the available literature in a systematic way.**

***Index Terms*—Fractional programming, fuzzy multi-objective fractional programming, sum of ratio, fuzzy goal programming.**

I. Introduction

Linear fractional programming is one of the most popular fields today in nonlinear optimization theory. The optimization of several linear ratios is a special class optimization problems. From the last three decades, it has attracted the interest of practitioners and researchers. During the last decade, interest in ratio optimization problems has been increased especially because it has wide variety of application, specially financial sector, inventory management, production planning, banking sector and others. Basically, it is used for modelling real life problems with one or more objectives such as debt/equity, profit/cost, inventory/sales, actual cost/ standard cost, output/employees, nurses/patients ratios etc. with respect to some constraints. Another reason for the strong interest in the ratios optimization problems is that from a research point of view these problems pose significant theoretical and computational challenges. This is mainly because these problems are global optimization problems, i.e. they are known to generally having multiple local optimum solutions, not optimal solutions. The details of MOLFP can be seen in references [1-14].

A general linear fractional programming problem (LFP) can be formulated as follows

$$\begin{aligned}
&\text{Optimize } F(x) = \frac{c^T x + \alpha}{d^T x + \beta}\\
&\text{subject to}\\
&Ax = b,\\
&x \geq 0,\\
&\text{where } x, c^T, d^T, \in R^n, b \in R^m, \alpha, \beta \in R.
\end{aligned} \tag{1}$$

For some values of x, $d^T x + \beta$ may be equal to zero but here only the case $d^T x+\beta > 0$ is considered. If we take more than one objectives in general linear fractional programming problem (1), then, the problem is known as multi-objective linear fractional programming problem (MOLFP), mathematically it can be written as

$$\begin{aligned}
&\text{Optimize } F(x) = [\,F_1(x),\, F_2(x),\, \ldots F_k(x)],\\
&\text{where}\quad F_i(x) = \frac{f_i(x)}{m_i(x)} = \frac{c_i^T x + \alpha_i}{d_i^T x + \beta_i}.
\end{aligned} \tag{2}$$

Also, $f_i(x) = c_i^T x+\alpha_i$, $m_i(x) = d_i^T x+\beta_i$, are real valued function on X, where $S = \{x : Ax\,(\leq, =, \geq)\, b,\, x \geq 0,\, x \in R^n,\, b \in R^m,\, A = (a_{ij})_{m\times n},\, \alpha_i,\, \beta_i \in R\}$ and $c_i,\, d_i \in R^n$, $c_i^T x + \beta_i > 0 \quad (i = 1, 2, \ldots, k),\ \forall \ \ x \in S$ and S is assumed to be non- empty convex bounded set in R^n.

A solution $x^o \in S$ is said to be a Pareto optimal solution to MOFP problem (maximization case), if and only if there is no other solution $y \in S$ such that $\dfrac{f_i(y)}{g_i(y)} \geq \dfrac{f_i(x^o)}{g_j(x^o)} \quad \forall \quad i$ and $\dfrac{f_i(y)}{g_i(y)} > \dfrac{f_i(x^o)}{g_j(x^o)}$ for at least one j. All Pareto optimal points lie on the boundary of a feasible region S and the algorithms provide solution that may not be Pareto - optimal, but may satisfy other criteria, making them significant for practical applications. A solution point, $x^o \in S$, is said to be weak Pareto optimal, if there does not exist another solution point, $y \in S$, such that $\dfrac{f_i(y)}{g_i(y)} > \dfrac{f_i(x^o)}{g_i(x^o)} \quad \forall \quad i$. A point is weakly Pareto optimal if there is no other point that improves all of the objectives simultaneously. In comparison, a point is Pareto optimal if there is no other point that improves at least one objective without determining another function. A solution $x^o \in S$ is called fuzzy efficient solution to the fuzzy multi-objective linear fractional programming (FMOFP) problem, if and only if there is no other solution $y \in S$ such that $\mu_i\left(\dfrac{f_i(y)}{g_i(y)}\right) \geq \mu_i\left(\dfrac{f_i(x^o)}{f_i(x^o)}\right) \quad \forall \quad i$ and $\mu_j\left(\dfrac{f_j(y)}{f_j(y)}\right) > \mu_j\left(\dfrac{f_j(x^o)}{g_j(x^o)}\right)$ for at least one j.

It is true that any fuzzy efficient solution x^o to the FMOFP problem, such that $\dfrac{f_i(x^o)}{g_i(x^o)} \in (g_i,\, g_i + t_i) \quad \forall \quad i$, is a Pareto - optimal solution to the MOFP problem. A solution point, $x^o \in S$, is said to be a properly Pareto optimal, if it is pareto optimal and there exist alar $M > 0$ such that for each i,

$$\frac{\dfrac{f_i(x^o)}{g_i(x^o)} - \dfrac{f_i(y)}{g_i(y)}}{\dfrac{f_j(x^o)}{g_j(x^o)} - \dfrac{f_j(y)}{g_j(y)}} \leq M,$$

for some j such that $\dfrac{f_j(y)}{g_j(y)} > \dfrac{f_j(x^o)}{g_j(x^o)}$ whenever y is feasible for S and $\dfrac{f_i(y)}{g_i(y)} < \dfrac{f_i(x^o)}{g_i(x^o)}$.

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II. Fuzzy set theoretic method for MOLFP

If an imprecise aspiration level is introduced to each of the objectives of MOFP, then these fuzzy objectives are called fuzzy goals. This can be defined as follows

$$\begin{aligned}
&\text{Find} \quad X(x_1, x_2, ...x_n)\\
&\text{such that}\\
&F_i(x) \lessapprox g_i,\\
&\text{or}\\
&F_i(x) \gtrapprox g_i, \qquad\qquad (3)\\
&\text{subject to}\\
&x \in S = \{x \in R^n,\, Ax \le b, x \ge 0 \text{ with}\\
&b \in R^m,\, A \in R^{m\times n}\},
\end{aligned}$$

where g_i is the aspiration level of the i^{th} objective $F_i(x)$, also upper tolerance limit $\overline{t_i}$ and lower tolerance limit $\underline{t_i}$, respectively are given for the i^{th} fuzzy goal. The membership functions for each objective defined as below:
If $F_i(x) \lessapprox g_i$, then

$$\mu_i(x) = \begin{cases} 1, & if \quad F_i(x) \le g_i \\ \frac{\overline{t_i} - F_i(x)}{\overline{t_i} - g_i}, & if \quad g_i \le F_i(x) \le \overline{t_i} \\ 0 & if \quad F_i(x) \ge \overline{t_i}. \end{cases} \qquad (4)$$

If $\quad F_i(x) \ge g_i,$ then

$$\mu_i(x) = \begin{cases} 1, & if \quad F_i(x) \ge g_i \\ \frac{F_i(x) - \underline{t_i}}{g_i - \underline{t_i}}, & if \quad \underline{t_i} \le F_i(x) \le g_i \\ 0 & if \quad F_i(x) \le \underline{t_i}. \end{cases} \qquad (5)$$

Each expression $F_i(x) \lessapprox g_i$ is represented by a fuzzy set called fuzzy goal, whose membership function $\mu_i(x)$ provides the satisfaction degree λ_i to which the i^{th} fuzzy inequality is satisfied. The problem (3) is called fuzzy multi-objective linear fractional programming problem (FMOFP). Using Zimmermann (Max - Min) operator approach, the following formulation adopts

$$\begin{aligned}
&\text{Max} \quad \lambda\\
&\text{subject to}\\
&\lambda \le \mu_i(x), \qquad i = 1, 2, ...k \qquad\qquad (6)\\
&\lambda \in [0, 1], \quad x \in X = \{x \in R^n,\, Ax \le b, x \ge 0 \,\text{with}\\
&b \in R^m,\, A \in R^{m\times n}\},
\end{aligned}$$

and the second phase model is given by

$$\begin{aligned}
&\text{Max} \quad \lambda_i\\
&\text{subject to}\\
&\lambda_i \le \mu_i(x), \qquad i = 1, 2, ...k\\
&\lambda_i \in [0, 1], \quad x \in X = \{x \in R^n,\, Ax \le b, x \ge 0 \text{ with}\\
&b \in R^m,\, A \in R^{m\times n}\}.
\end{aligned}$$

If the optimal solution of (6) is unique, then it is fuzzy efficient solution for FMOFP. However, if the solution is not unique, the fuzzy efficiency is not guaranteed for all solutions of problem (6), but at least one of the optimal solutions is fuzzy efficient from all optimal solutions. In order to produce fuzzy efficient solution, various methods have been developed in the literature for MOFP.

A. *Fuzzy Goal Programming*

In fuzzy goal programming approaches, the highest degree of membership function is 1. So, for the defined membership function, the flexible membership goals with aspiration levels 1 can be expressed as

$$\begin{aligned}
&\frac{F_i(x) - \underline{t_i}}{g_i - \underline{t_i}} + d_i^- - d_i^+ = 1 \qquad \text{or}\\
&\frac{\overline{t_i} - F_i(x)}{\overline{t_i} - g_i} + d_i^- - d_i^+ = 1 \qquad\qquad (7)
\end{aligned}$$

Where $d_i^- \ge 0$, $d_i^+ \ge 0$ with $d_i^+.d_i^- = 0$ are, respectively under- and -over -deviations from the aspiration levels.
In conventional GP, the under- and over-deviational variables are included in the achievement function or minimizing them and that depend upon the type of the objective functions to be optimized.
In this approach, only the under-deviational variable d_i^- is required to be achieve the aspired levels of the fuzzy goals. It may be noted that any over - deviation from fuzzy goal indicates the full achievement of the membership value. Recently, Pal. et.al [15] proposed an efficient goal programming (GP)method for solving Fuzzy multi-objective linear fractional programming problems.

III. Review of Multi-objective Linear Fractional Programming

If we take more than one objectives in the general linear fractional programming problem (2), then, the problem is known as multi-objective linear fractional programming problem (MOLFP), mathematically it can be written as:

$$\begin{aligned}
&\text{Optimize}\, F(x) = [\,F_1(x),\, F_2(x),\, \dots F_k(x)],\\
&\text{subject to}\\
&x \in S = \{x : Ax\,(\le, =, \ge)\, b,\; x \ge\; 0,\; x \in \mathbb{R}^n,\; b \in \mathbb{R}^m,\\
&A = (a_{ij})_{m\times n},\; \alpha_i,\; \beta_i \in \mathbb{R}\} \qquad\qquad (8)\\
&\text{where} \quad F_i(x) = \frac{f_i(x)}{m_i(x)} := \frac{c_i^T x + \alpha_i}{d_i^T x + \beta_i}.
\end{aligned}$$

Also, $f_i(x) = c_i^T x + \alpha_i$, $m_i(x) = d_i^T x + \beta_i$, are real valued function on S, where and $c_i,\, d_i \in \mathbb{R}^n,\; c_i^T x + \beta_i > 0,$
$(i = 1, 2, \dots, k),\; \forall \quad x \in S$ and S is assumed to be non - empty convex bounded set in $\mathbb{R}^n$.

A. *Studies of optimality and duality conditions*

Coladas, Li and Wang [16] introduced two types of duality to a more general multi-objective fractional programming problem and proved weak, strong duality theorem for each type of duality. Dutta, Rao and Tiwari [17] noted a fallacy of the second order solution procedure for fuzzy multiple criteria linear fractional optimization described by Luhandjula [1984]. Dutta et al., [18], as a general one does not work

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efficiently if certain hypothesis are satisfied. B. Kheifam and K. Mirnia [19], studied multi-parametric sensitivity analysis under perturbations in multiple rows or columns of the constraint matrix in programming problems with piecewise linear fractional objective function using the concept of maximum volume in the tolerance region. They derived necessary and sufficient conditions to classify perturbation parameters as focal and non focal.

Lee et al., [20] gave the duality and the associated results on generalized linear fractional program. They extended the result on generalized linear fractional program by Jagannathan and Schaible [21] considered a class of non-differentiable multi-objective fractional programming problem in which every component of the objective functions contains a term involving the support functions of a compact convex set.

In [22], Metev and Gueoguieva gave a weakly efficient solution for multi-objective linear fractional programming problem using scalar optimization problem.

Saad, Biltogy and Farag [23]suggested an algorithm to solve fuzzy multi-objective integer linear fractional and algorithm based mainly upon a modified Isbell-Marlow method together with the branch and bound technique. Sakawa and Yano [24] focus on multi-objective linear factional programming problem with fuzzy parameters and presented a new interactive decision method.

B. *Studies of algorithmic and method development*

It may be pointed out that in most of the approaches to solve MOFP, the problems are converted into single objective fractional programming problems and then solved employing the well known available techniques of single objective programming [25-28]. To overcome the computational difficulties of using conventional approaches to MOLFP problems, the theory of fuzzy sets has been introduced in the field of linear fractional programming.

A Linguistic variable approach to fuzzy multi-objective linear fractional programming has been proposed by Luhandjula [29]. One drawback of this approach is that the aggregation of membership function is done with the compensatory operators which does not guarantee the efficiency of the optimal solution. While formulating the model, Luhandjula neither includes the feasible region nor formulates the constraints property. Therefore the model did not satisfy several MOLFP problems.

Luhandjulas [30] approach has been further modified by Dutta et al., [31] such as to obtain efficient solution to problem MOLFP. Dutta et al., [31] proved two lemmas which permit to define the concept of (z, ϵ)- proximity used in the larger frame work of linguistic variable domain. They used fuzzy membership functions $C^{N_i}(x)$ and $C^{D_i}(x)$ for numerator and denominator of MOLFP $\dfrac{N_i(x)}{D_i(x)}, \quad \forall \quad i = 1, 2, \ldots, k.$, respectively.

They used the "Simple Additive Weighting" (SAW) model to find efficient solution as follows:

Max $V(\mu) = \sum_{i=1}^{k}(w_i\mu_i^{N_i} + w_i^{'}\mu_i^{D_i})$

subject to $\mu_i^{N_i} = C^{N_i}(x), \quad \mu^{D_i} = C^{D_i}(x)$

$0 \leq \mu_i^{N_i} \leq 1, \quad 0 \leq \mu_i^{D_i} \leq 1, \quad \forall \quad i = 1,\, 2,\, ...k$

$X \geq 0,$ and $\sum_{i=1}^{k}(w_i + w_i^{'}) = 1,$ where, w_i and $w_i^{'}$ both positive, are the weights indicating the relative importance put by the decision maker.

The shortcoming in the work of Dutta et al., [31] is that the authors do not say any thing about the possibility of solution when a concrete problem MOLFP is formulated. Caballero and Hernandez [32] pointed out the difficulties in solving a linear fractional programming using goal programming method. They suggested a reliable efficiency restoration technique based on minimax philosophy. They shown that the linear programming can be used as search strategy for solutions that satisfy all goals, but it is unsuitable when search solutions do not exist.

Chakraborty and Gupta [33] gave a solution procedure for multi-objective linear fractional programming problem. They used the Charnes Cooper transformation $y = tx$ to linearize the fractional objective functions and Zimmermann approach is applied to find the fuzzy efficient of MOLFP. But they did not say anything about the other feasible point where the membership functions are fully achieved. Finally, they used the following program to find efficient solution:

Chergui and Moula [34] developed a branch and cut algorithm based on continuous fractional optimization for generating the whole integer efficient solutions of multi-objective integer linear fractional programming problem. The basic idea of computation phase of the algorithm is to optimize one of the fractional objective functions, then generate an integer feasible solution. Using the reduced gradients of the objective functions, an efficient cut is built and part of the feasible domain not containing efficient solutions is truncated by adding this cut. A sample problem is solved using this algorithm, and the main practical advantages of the algorithm are indicated.

Cheung, Yang and Hanzo [35] solved the joint power and sub carrier allocation problem in the context of maximizing the energy-efficiency of a multi-user, multi-realy orthogonal frequency division multiple access cellular network, where the objective function is formulated as the ratio of the spectral-efficiency over the total power dissipation.

Choo and Atkins [36] provided a thorough analysis of the bi-criteria case and using simplex algorithm used only one-dimensional parametric linear programming techniques is developed to evaluate the efficient frontier.

Chu and Yau [37] proposed to efficient methods based on the generalized Bender decomposition framework that take advantage of the special structures of the integrated problem.

Costa [38] gave a computational technique to compute the non-dominated solution of the MOLFP associated with a given weight for each objective function. The basic idea of this technique is to divide the non dominate region into two sub-regions and to analyze each of them in order to discard one of it if it can be proved that the maximum of the weighted sum is in the other. The same process is repeated for the remaining region and stop when the remaining region is so little that the difference among their non-dominated solution is lower than a predefined error.

In [39], Dutta, et al., developed a solution algorithm for a restricted class of multi-objective linear fractional programming problems in the sense that the denominators are identical.

In [40], Guzel and Sivri proposed a solution algorithm for solving multi-objective linear fractional programming problem (MOLFP). They made the objective functions ü positive for all $x \in X$ by translating objective functions that are negative for some $x \in X$. Then the problem is solved as a single objective function as the minimization of the deviations from maximum values of objective functions in the feasible region. Also in [78], same authors gave solution method for multi-objective linear fractional programming problem by expanding the each objective function using first order Taylor's theorem about the point at which the function attains the maximum value. Then multi-objective linear fractional programming problem (MOLFP) reduced to an equivalent MOLP. The resulting MOLP solved by assuming equal weighted problems.

Kato and Sakawa [41] discussed on multi-objective linear fractional programming problems with block angular structure. Lotif et al., [42] gave useful method to test the efficiency of the given feasible solution of MOLFP. They applied a geometric interpretation to test weak efficiency and for strong efficiency a linear programming approach is constructed. They developed two theorems for weak efficiency and strong efficiency for the solution of MOLFP.

Pal et al., [43] proposed a fuzzy goal programming method for solving MOLFP $Z_k(x) = \dfrac{c_k x + \alpha_k}{d_k x + \beta_k}$. They used the method of variable change on the under - and over deviational variables d_k^- and d_k^+ of the membership goal associated with the fuzzy goal of the model as follows:

$D_k^- = d_k^-(d_k x + \beta_k)$ and $D_k^+ = d_k^+(d_k x + \beta_k)$.

Pop and Dumitrescu [44] used the extended principle of Zadeh to aggregate fuzzy numbers and the kerres method to evaluate a fuzzy constraint.

In [45], Sadjadi et al., presented a multi-objective linear fractional inventory model using fuzzy programming. The resulted fuzzy model is transformed into an ordinary linear programming and then solved.

Toksari [46] developed an algorithm to solve MOLFP by Taylor series approach using fuzzy set theory. He linearized the membership function instead of objective function and the membership functions are linearized using following first order Taylor series approximation

$\mu_i(x) = \widetilde{\mu_i}(x) \cong \mu_i(x_i^*) + \sum_{j=1}^{n}[(x_j - x_{ij}^*)\frac{\partial \mu_i(x_i^*)}{\partial x_j}] + O(h^2)$.

The equal weights for membership functions are used to efficient solution for MOLFP.

In [47], Minasian and Pop pointed out certain shortcomings in the work of Dutta et al., [48] and they gave the correct proof of the theorem which validates obtaining of the efficient solutions. They noticed that the method presented by Instead of Simple Additive Weighting model of MOLFP, Ravi and Reddy [49] replace the addition operator by min operator as the aggregator and the model converted to be as follows:

Max $min(\mu_i^{N_i}, \mu_i^{D_i})$
subject to $\mu_i^{N_i} = C^{N_i}(x), \quad \mu^{D_i} = C^{D_i}(x)$
$0 \le \mu_i^{N_i} \le 1, \quad 0 \le \mu_i^{D_i} \le 1, \quad \forall \quad i = 1, 2, ...k$
$X \ge 0.$

$Max\ \lambda$
subject to
$\lambda \le \mu_i\ (tN_i\ (y/t)), \quad \forall \quad i \in I,$
$\lambda \le \mu_i(tD_i(y/t)), \quad \forall \quad i \in I^c,$
$tD_i(y/t) \le 1,$
$-\,tN_i(y/t) \le 1,$
$A(y/t) - b \le 0,$
$t \ge 0,\ y \ge 0.$

Valipaur, Yaghoobi and Mashinchi [50] suggested an iterative parametric approach for solving multi-objective linear fractional programming problems which only uses linear programming to obtain efficient solution and always converges to an efficient solution.

Wu [51] applied Taylor series method to linearize the auxiliary problem for efficiently computing and proposed a solution procedure for implementing the weight max-ordering approach.

IV. Conclusions

The works in this field are still not quite satisfactory and development of efficient computational procedures are one of the emerging areas of investigations. In this article, various types of techniques for multi-objective fractional programming problems are discussed. Recently published research articles in many research papers are also reviewed..

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