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Accretion of Spherical Viscoelastic Objects under Self-Gravity

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***Abstract*—Quasistatic deformation processes in aging viscoelastic spherical bodies formed by accretion in a central self-gravity field are considered. Inertial effects are neglected, and the strains are assumed to be small. The stress-strain state evolution in the growing ball before the beginning of growth, at the continuous growth stages and in the pauses between them, and after the end of growth is studied. The mathematical theory of growing solids is used to state initial–boundary value problems describing the deformation process at all of these stages. Closed-form analytic solutions of these problems are constructed. The stress-strain state evolution in a growing spherical body is computed numerically for various growth modes. Completely new mechanical effects specific to growing solids are discussed. One main application of this work is to estimate the gravitational stresses in the Earth.**

***Index Terms*—accretion, spherical solid, viscoelasticity, stress-strain state, self-gravity**

INTRODUCTION

IN many natural phenomena and technological processes, the bodies involved become larger and undergo shape changes owing to the addition of extra material. When studying such processes, one should simultaneously take into account the gradual influx of matter to the surface of the growing body and the loads applied to the body. This cannot in principle be done adequately in the framework of classical mechanics of solids even if one considers the traditional equations and boundary conditions in a time-varying field. The fact that problems concerning the mechanical behavior of growing solids generally have a number of peculiarities and form a specific class of problems in solid mechanics was understood in the pioneering paper [1], which was the first to indicate explicitly that it is wrong to use the strain compatibility conditions when calculating the stress state of a growing body and that this state is fundamentally different from the state of the same body loaded after the end of growth.

Bulk forces often occur as mechanical loads in the above-mentioned processes. These forces include those resulting from the action of external physical fields on the body (gravitational and Coulomb forces), inertia forces (primarily, the centrifugal force) caused by rigid motions of the body in space, and self-gravity forces of material particles.

It is well known that many artificial and natural materials (e.g., concrete, polymers, ice, rocks, soils, and wood) exhibit pronounced creep and aging properties. This means that their strain state may vary even under a constant load, and their mechanical characteristics experience changes caused by physicochemical mechanisms. We consider growth processes for such materials.

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I. STATEMENT OF THE PROBLEM AND DESCRIPTION OF THE APPROACH

A. Aim of research

Models of accretion formation of massive cosmic objects owing to the influx of additional external substance to the surface in general take into account the forces of interaction between material particles as well as the centrifugal forces of inertia due to rotation of the object. However, the latter are neglected compared with gravitational forces in the present paper. This is the right thing to do, at least in the first approximation, because we deal with quasistatic deformation processes in aging viscoelastic spherical bodies formed by accretion in a central self-gravity field; in this setting, the velocity of rotation of cosmic bodies is relatively low, while the influence of the gravitational forces on the body is enormous on a cosmic scale.

B. Applications to geomechanics and geophysics

The main application of our results is to the problem of estimating the self-gravitational stress distributions inside the Earth. A careful analysis shows that the classical mechanical approach, in which the Earth is treated as a ball of constant radius under self-gravity, leads to wrong results in this problem. Indeed, the corresponding solution exhibits strong circumferential contraction stresses on the surface of the self-gravitating body, which contradicts our everyday experience and motivates the development of a new approach.

This contradiction is caused by the assumption that there exists a “natural” (that is, stress-free) configuration of the whole body. This assumption underlies all classical statements of problems in mechanics of solids, but it is plainly not true in our setting. The absence of such a natural configuration for the Earth experiencing deformation under self-gravity is explained by the fact that the Earth has not been formed instantaneously. It has been formed by gradual surface accretion of additional matter from the outside over a long period of time. Owing to the force of gravity, the newly accreted matter exerts additional pressure on the earlier-formed central part of the body, hence causing a change in the stress-strain state of the latter.

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Thus, we arrive at the natural conclusion that the study of the gravitational stress state of the Earth should be based on a model in which the body is gradually growing under self-gravity rather than exists in final form from the very beginning. In other words, the problem should be considered in the framework of mechanics of growing solids. The studies carried out in the present paper lead to a geomechanical/geophysical model based on these principles.

Brown and Goodman [2] were apparently the first to solve the deformation problem for an accreted self-gravitating spherical shell. They assumed the shell material to be linearly elastic and considered a quasistatic growth process in the small strain approximation. The deformation problem for a rotating growing ball was solved in [3] under the same assumptions.

Note that, theoretically, the statement of geometrically linear quasistatic problems describing the growth (even nonlinear) of a continuous elastic body does not encounter any fundamental difficulties, because the rate of change in the stress–strain state of the body is completely determined by instantaneous characteristics of the growth and loading processes. The situation for hereditary materials is different in that the changes in the stress–strain state of the growing body at any time instant depend on the entire preceding deformation history of each material element (including the elements already existing in the body before the beginning of growth). Stating the problem is a real challenge from the mathematical viewpoint in this case.

A mathematical theory of growing solids in the framework of the linear theory of viscoelasticity was developed in [4]. In this theory, a general nonclassical boundary value problem is stated for a piecewise continuously growing viscoelastic aging body subjected to arbitrary surface and bulk forces. The problem is solved by reduction to a sequence of boundary value problems coinciding in form with classical boundary value problems of linear elasticity with a time parameter. After solving this sequence of problems, one can reconstruct the true stress–strain state of the growing solid by using the decoding formulas provided by the theory. We discuss the existence and uniqueness of the solution of the boundary value problem for a growing body and consider a model accretion problem for a body made of a viscoelastic aging material with self-gravity taken into account.

II. Boundary value problems and their solutions

A. Description and a mechanical model of the process

Consider a substance distributed in some region of space and subjected to the stationary gravitational field of a pre-existing ball-shaped core whose formation is assumed to have been stress-free. The force of gravity attracts the substance particles to the core. As a result, a solid spherical body begins to grow from the original core by continuous accretion of new substance particles attracted to its surface (Fig. 1).

All material elements of the growing body continually experience mutual gravitational attraction. This attraction results in the onset and development of time-varying stress and strain fields. The stress–strain state of this growing self-gravitating ball at any time instant is fundamentally different from the state that would occur in the same ball if it were formed by mechanical forces alone and only then placed in the self-gravity force field.

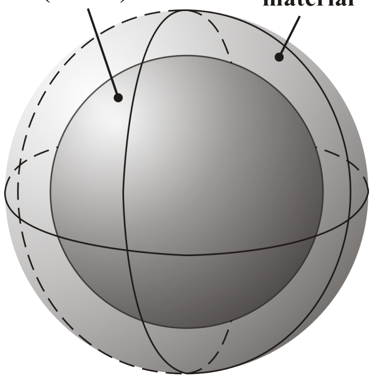

Fig. 1. Basic scheme of the accretion process

The response of a viscoelastic material to mechanical loads depends on the load duration and the material age. Therefore, the final state of a body gradually grown from such a material in a force field crucially depends on the nature and rate of the entire growth process.

Assume that the ball grows in its own gravitational field by influx of stress-free material to the surface. Consider the original homogeneous ball (core) of radius a_0 in the centrally symmetric force field [5]

$$\mathbf{f}(\mathbf{r}) = -\mathbf{e}_r cr, \quad c = \frac{4}{3}\pi\gamma_{\text{gr}} d^2, \quad r = |\mathbf{r}|, \quad \mathbf{e}_r = \frac{\mathbf{r}}{r},$$

where $\mathbf{r}$ is the position vector with respect to the ball center, γ_{gr} is the gravitational constant, and d is the mass density.

Assume that the original ball is formed at time $t = t_0$ from the above-mentioned isotropic linearly viscoelastic aging material with zero onset time and, starting from some time $t = t_1 \geq t_0$, undergoes N successive stages of continuous growth by uniform influx of the same material to the surface. These intervals alternate with time intervals on which the material influx is absent and the ball surface is not loaded. Before the beginning and after the end of growth, the ball surface is not loaded either.

Neglecting inertial effects and assuming that the strains are small, we study the stress–strain state evolution in the ball before growth, at continuous growth stages and in the pauses between them, and after the end of growth.

B. Constitutive equations

We use the following constitutive equations to describe the mechanical behavior of the viscoelastic aging material of the growing solid [6]:

$$\begin{aligned}
&\mathbf{T}(\mathbf{r}, t) = G(t)(I + N_{\tau_0(\mathbf{r})})[2\mathbf{E}(\mathbf{r}, t) + (\kappa - 1)\mathbf{1}\text{tr}\mathbf{E}(\mathbf{r}, t)],\\
&I + N_s = (I - L_s)^{-1}, \quad \kappa = (1 - 2\nu)^{-1} = \text{const}.\\
&L_s f(t) = \int_s^t f(\tau)K(t, \tau)\, d\tau, \ N_s f(t) = \int_s^t f(\tau)R(t, \tau)\, d\tau,\\
&K(t, \tau) = G(\tau)\frac{\partial \Delta(t, \tau)}{\partial \tau},\\
&\Delta(t, \tau) = G(\tau)^{-1} + \omega(t, \tau) = (I - L_\tau)G(t)^{-1}.
\end{aligned}$$

Here t is the time variable, $\mathbf{r}$ is the position vector, $\mathbf{T}$ is the stress tensor, $\mathbf{E}$ is the linear strain tensor, $\mathbf{1}$ is the unit tensor, $\tau_0(\mathbf{r})$ is the instant of stress origination at a point of the body,

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L_s and N_s are Volterra integral operators with a parameter s, I is the identity operator, $K(t,\tau)$ is the creep kernel, $R(t,\tau)$ is the relaxation kernel, $\omega(t,\tau)$ is the shear creep function ($t \geq \tau \geq 0$), $G(t)$ is the elastic shear modulus, and ν is the constant Poisson ratio.

We approximate the mechanical characteristics of the body by the formulas

$$\omega(t,\tau) = \Omega(\tau)[1 - e^{-\gamma(t-\tau)}],$$
$$G(t) = G_\infty - \Delta G e^{-\alpha t}, \quad \Omega(t) = \Omega_\infty + \Delta\Omega e^{-\beta\tau},$$

where $\gamma > 0$ is the time scale factor, $\Omega(\tau) = \omega(+\infty,\tau)$ is the aging function, $\alpha/\gamma = 2$, $\Delta G/G_\infty = 0.5$, $\beta/\gamma = 31/60$, $G_\infty\omega_\infty = 0.5522$, $G_\infty\Delta\Omega = 4$, and $\nu = 0.1$. These data correspond to the experimental creep curves of concrete [6].

C. Boundary value problem for the pre-accretion period

Using the mathematical theory of growing solids [4], one can state initial–boundary value problems describing the deformation process for the objects in question in any time period.

Before the beginning of accretion, the process is described by the classical boundary value problem of linear viscoelasticity [6],

$$\nabla \cdot \mathbf{T} = \mathbf{e}_r c r, \quad 0 < r < a_0, \quad t_0 \leq t \leq t_1,$$
$$\mathbf{T} = G(I + N_{t_0})[2\mathbf{E} + (\kappa-1)\mathbf{1}\mathrm{tr}\mathbf{E}], \; \mathbf{E} = \frac{\nabla\mathbf{u} + \nabla\mathbf{u}^T}{2},$$
$$\mathbf{e}_r \cdot \mathbf{T} = \mathbf{0}, \quad r = a_0; \quad \mathbf{u} \to \mathbf{0}, \quad \mathbf{r} \to \mathbf{0}.$$

Here $\mathbf{u}(\mathbf{r},t) = \mathbf{e}_r u(r,t)$ is the displacement field for the original ball (core). Note that the distribution $\tau_0(\mathbf{r})$ of stress origination instants is uniform at the points of the original body and equals to the time t_0 at which the core was loaded.

We construct a closed-form analytic solution of this problem by using the well-known correspondence principle in linear viscoelasticity [6] and the solution of the corresponding boundary value problem of linear elasticity (e.g., see [7], [8]),

$$u = -\frac{cr}{10(\kappa+1)}\left(\frac{5\kappa+1}{3\kappa-1}a_0^2 - r^2\right)\Delta(t,t_0),$$
$$\sigma_r = -\frac{c(5\kappa+1)(a_0^2 - r^2)}{10(\kappa+1)},$$
$$\sigma_\vartheta = -\frac{c[(5\kappa+1)a_0^2 - (5\kappa-3)r^2]}{10(\kappa+1)}.$$

Here σ_r and σ_θ are the radial stress and the circumferential stress in any direction tangent to the central sphere, respectively. The other stress tensor components in the spherical coordinate system are zero.

Note that the stresses in the original body are independent of time until accretion begins. This is consistent with general theorems in linear viscoelasticity [6].

D. Boundary value problem for the accretion period

In the framework of the general mathematical theory of aging viscoelastic growing bodies [4], we can prove that the stress–strain state evolution in the ball under self-gravity after the beginning of accretion (i.e., at each stage of continuous material influx), in the pauses between the stages, and during an arbitrary long period after the material influx terminates is described by the nonclassical boundary value problem

$$\nabla \cdot \mathbf{S} = \mathbf{e}_r c r \frac{\partial\omega(t,\tau_0(r))}{\partial t}, \quad 0 < r < a(t), \quad t > t_1,$$
$$\mathbf{S} = 2\mathbf{D} + (\kappa-1)\mathbf{1}\mathrm{tr}\mathbf{D}, \quad \mathbf{D} = \frac{\nabla\mathbf{v} + \nabla\mathbf{v}^T}{2},$$
$$\mathbf{e}_r \cdot \mathbf{S} = -\mathbf{e}_r q(t), \quad r = a(t); \quad \mathbf{v} \to \mathbf{0}, \quad r \to 0.$$

Here $\mathbf{v}(\mathbf{r},t) = \mathbf{e}_r v(r,t)$ is the displacement rate field in the time-varying spatial domain occupied at time t by the entire growing ball of radius $a(t) > a_0$,

$$q(t) = \frac{c a(t) a'(t)}{G(t)},$$

$\mathbf{D}$ is the strain rate tensor, and $\mathbf{S}$ is the operator stress rate tensor, $\mathbf{S} = \partial\mathbf{T}^\circ/\partial t$, where

$$\mathbf{T}^\circ(\mathbf{r},t) = (I - L_{\tau_0(\mathbf{r})})\left[\frac{\mathbf{T}(\mathbf{r},t)}{G(t)}\right]$$

is the operator stress tensor.

The above-mentioned proof is too long to be presented here.

We obtain the following closed-form analytic solution of the problem in terms of characteristics rates of the stress–strain state:

$$v = -r\left[\Psi(r,t) + \Phi(r,t) + \frac{4\Psi(a(t),t) + q(t)}{3\kappa-1}\right],$$
$$S_{r,\vartheta} = m_{r,\vartheta}\Psi(r,t) - (3\kappa-1)\Phi(r,t) - 4\Psi(a(t),t) - q(t),$$
$$m_r = 4, \quad m_\vartheta = -2.$$

Here we have used the functions

$$\Phi(r,t) = \frac{c}{3(\kappa+1)}\int_r^{a(t)} \xi \frac{\partial\omega(t,\tau_0(\xi))}{\partial t} d\xi,$$
$$\Psi(r,t) = \frac{c}{3(\kappa+1)}\int_0^{r} \xi \frac{\partial\omega(t,\tau_0(\xi))}{\partial t}\left(\frac{\xi}{r}\right)^3 d\xi,$$

which depend on the entire accretion history of the body as well as on the gravitational, creeping, and aging properties and the Poisson ratio of the material.

E. Relations for the true stresses

Having solved the boundary value problems describing the process before and after accretion begins, we need to reconstruct the evolution of true stresses at every point $\mathbf{r}$ of the body starting from the beginning of the evolution, that is, from the time $\tau_0(\mathbf{r})$, and until arbitrarily large time.

First, we can reconstruct the evolution of the tensor field $\mathbf{T}^\circ(\mathbf{r},t)$ by integrating the rate values obtained for this tensor over time,

$$\mathbf{T}^\circ(\mathbf{r},t) = \mathbf{T}^\circ(\mathbf{r},\tau_1(\mathbf{r})) + \int_{\tau_1(\mathbf{r})}^{t} \mathbf{S}(\mathbf{r},\tau)\, d\tau,$$

$t \geq \tau_1(\mathbf{r})$, where we have introduced the auxiliary time instant distribution

$$\tau_1(\mathbf{r}) = \begin{cases} t_1, & 0 < r < a_0, \\ \tau_0(\mathbf{r}), & r > a_0. \end{cases}$$

For the original solid ($0 < r < a_0$), we have

$$\mathbf{T}^\circ(\mathbf{r},\tau_1(\mathbf{r})) \equiv \mathbf{T}^\circ(\mathbf{r},t_1) = \mathbf{T}_0(\mathbf{r})\Delta(t_1,t_0),$$

where $\mathbf{T}_0(\mathbf{r})$ is the stationary stress tensor field in the original solid (core) in the pre-accretion period (see subsection *C*). For the additional part of the solid formed during the accretion process ($r > a_0$), we have

$$\mathbf{T}^\circ(\mathbf{r}, \tau_1(\mathbf{r})) \equiv \mathbf{T}^\circ(\mathbf{r}, \tau_0(\mathbf{r})) \equiv \mathbf{0},$$

because the accretion process considered here has the property that the material particles entering the growing body owing to the substance influx to the surface are stress-free.

After we have reconstructed the entire evolution of the operator stress tensor $\mathbf{T}^\circ$ at any point $\mathbf{r}$ of the growing body, we can reconstruct the evolution of the true stress tensor $\mathbf{T}$ at this point by solving the corresponding tensor Volterra integral equation of the second kind

$$\frac{\mathbf{T}(\mathbf{r}, t)}{G(t)} - \int_{\tau_0(\mathbf{r})}^{t} \frac{\mathbf{T}(\mathbf{r}, \tau)}{G(\tau)} K(t, \tau)\, d\tau = \mathbf{T}^\circ(\mathbf{r}, t).$$

One can write out the solution of this integral equation in the resolvent form

$$\frac{\mathbf{T}(\mathbf{r}, t)}{G(t)} = \mathbf{T}^\circ(\mathbf{r}, t) + \int_{\tau_0(\mathbf{r})}^{t} \mathbf{T}^\circ(\mathbf{r}, \tau) R(t, \tau)\, d\tau.$$

But it is much more efficient to solve it numerically, for example, by using the trapezoid rule [9].

III. Conclusions

We carry out numerical computations of the stress–strain state evolution in a growing body at all stages of the deformation process for various growth modes. We estimate how the parameters of the growth process affect the stress–strain state in the course of evolution and the final stress–strain state. The final state of the accreted body is compared with the state obtained by solving the classical problem of mechanics of solids for an object similar in geometry and properties without taking into account the force and kinematic peculiarities of the growth process.

We reveal new mechanical effects specific to growing solids. Let us indicate some of the results obtained.

A. Some comments to the plots

Figures 2–4 show the normalized (by the gravitational factor c) dimensionless radial and circumferential stresses and the tangential stress intensity $\bar{\sigma}_r$, $\bar{\sigma}_\vartheta$, and $\bar{T}$, against the dimensionless radial coordinate $\bar{r}$. We use the following notation for dimensionless variables:

$$\overline{\text{stress}} = \frac{\text{stress}}{ca_0^2}, \quad \overline{\text{length}} = \frac{\text{length}}{a_0}, \quad \overline{\text{time}} = \gamma \cdot \text{time}.$$

For the numerical illustration of the results, we assume that the original ball (core) is loaded at time $\bar{t}_0 = 0.1$, and the material influx to its surface begins at time $\bar{t}_1 = 0.6$. Recall that the material itself is assumed to be formed at time $t = 0$.

The dashed lines in the figures show the radial distributions of the above-mentioned stress state characteristics in an instantly formed self-gravitating ball, both in the original core from which the considered growth process starts and in the fictitious nonaccreted ball of the final size. These distributions are found by solving the corresponding classical linear problem of viscoelasticity.

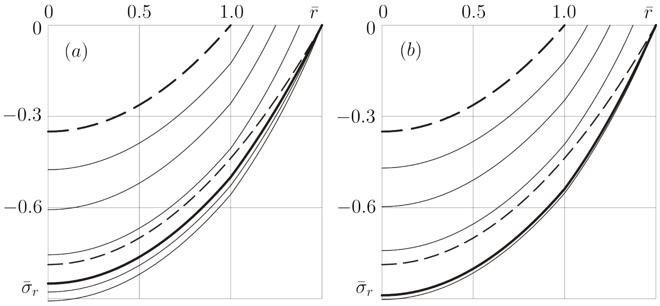

Fig. 2. Radial stress distribution

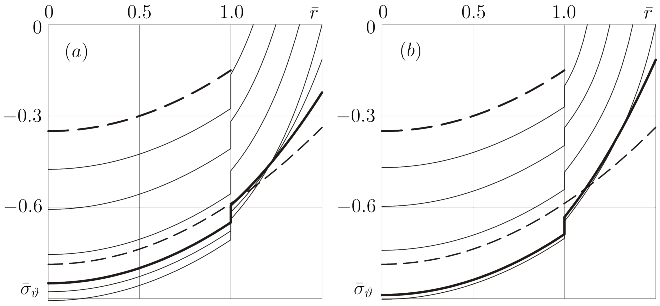

Fig. 3. Circumferential stress distribution

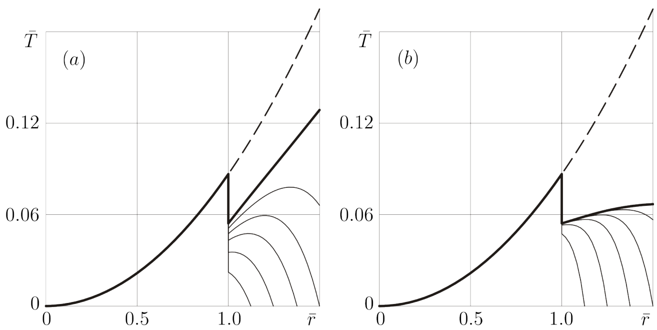

Fig. 4. Shear stress intensity distribution

The solid lines in the figures correspond to the single-stage growth process ($N = 1$, see Subsection 3*A*) with the constant rate of increase in the radius $a(t)$ due to material influx. In case (*a*), the process is sufficiently rapid; the accretion terminates at the time instant $\bar{t}_2 = 1.2$. In case (*b*), the process is similar to that in case (*a*) except that it is four times slower; the accretion terminates at time $\bar{t}_2 = 3.0$.

Bold solid lines depict the final distributions (at infinite time).

B. New detected effects

Let us discuss the main features of the numerical results.

There is a typical jump in the circumferential stress distribution diagrams at the interface between the original part of the ball and the part formed due to growth. It is also clear that the radial stress distribution diagrams are discontinuous at this interface.

Both the radial and the circumferential stresses increase in absolute value at all points of the body until the final termination of accretion, after which the absolute value of the radial stress decreases at each point to a steady-state value, and the circumferential stress is redistributed from the initial material to the added one and tends to a certain final distribution as well.

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The final radial and circumferential stress distributions depend on the material influx rate. The more rapid the accretion process, the higher the extent to which the radial stresses have time to decrease and the circumferential stresses have time to be redistributed, and the more uniform the final load of the added portion of the material is. The limit circumferential stress on the surface of the finally formed body can be higher (if the accretion process is sufficiently rapid) or lower (if the accretion process is relatively slow) than the initial value on the surface of the original body. However, the absolute value of the circumferential stress on the surface of the accreted body is always less than the circumferential stress on the surface of a similar solid that is not accreted.

The shear stress intensity is discontinuous on the common interface of two components of the ball as well. The tangential stress intensity diagram is independent of time in the original part of the body and always coincides with the intensity distribution in the intersection of this part with a nongrowing ball of arbitrary radius. This is a consequence of the following fact theoretically established in our investigation: the evolution of the stress deviator at any point of a self-gravitating ball is determined only by the deformation history of the latter until the time at which the point in question was incorporated in the solid.

It also follows from the above-mentioned theoretical fact that, in the case of a purely elastic material (which is a special case of the viscoelastic material considered in the present paper), the entire additional part of the ball formed by growth is loaded as a perfect fluid; i.e., it has a spherical stress state.

In the general viscoelastic case, the tangential stress intensity in the additional part of the ball is nonzero and develops into some limit dependence whose shape substantially depends on the growth rate. Its maximum value in the additional part of the ball can be larger (for sufficiently rapid growth processes) or smaller (for relatively slow accretion) than the maximum in the original part (kernel). However, the global intensity maximum in the growing ball is always less than in an instantly manufactured ball of the same size, in which the maximum is always attained on the surface.

One can also find the recent developments in the theory of growing solids with applications in [10]–[22].

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