Transcomplex Topology and Elementary Functions

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***Abstract*—The set of transcomplex numbers, introduced elsewhere, is a superset of the complex numbers that allows division by zero. Here we introduce a topology for the transcomplex numbers and extended the elementary functions from the complex domain to the transcomplex domain. We give a geometrical construction of non-finite angles and discuss the totalisation of computer subroutines to provide transcomplex functions.**

***Index Terms*—transcomplex topology, transcomplex exponential, transcomplex logarithm, transcomplex elementary functions, non-finite angles.**

I. Introduction

In [6] we introduced the set of transcomplex numbers, $\mathbb{C}^T$, and proved that transcomplex arithmetic is consistent. The set of transcomplex numbers contains the ordinary set of complex numbers and the set of transreal numbers as proper subsets. Transreal numbers were introduced in [5] and were proved consistent in [5] [10]. Transcomplex numbers are a new system of numbers which is total with respect to the four elementary arithmetical operations of addition, subtraction, multiplication and division. In particular, division by zero is allowed. This means that infinitely many exceptional states are removed from mathematics and from computer programs. This is of practical importance because it makes it possible to guarantee that if a program compiles then it does not terminate due to a logical exception. This is of very wide utility. In particular, meta-programs, such as genetic algorithms, can combine subprograms arbitrarily in the search for optimal solutions. The application of transarithmetic in computer hardware and software was discussed in, among other places, [1] [2].

The set $\mathbb{C}^T$ is given by

$$\mathbb{C}^T = \left\{ \frac{x}{y};\ x, y \in \mathbb{C} \right\}.$$

In [6] we defined fractions so that they allow a denominator of zero and we proved that transcomplex arithmetic is consistent. We named two special transcomplex numbers: *infinity*, $\infty := \frac{1}{0}$, and *nullity*, $\Phi := \frac{0}{0}$. Every transcomplex number can be written as a fraction, $\frac{x}{y}$, where x is an ordinary complex number and y is either one or zero. It is worth saying that $\frac{x_1}{y_1} \times \frac{x_2}{y_2} = \frac{x_1 x_2}{y_1 y_2}$ for all $x_1, x_2 \in \mathbb{C}$ and $y_1, y_2 \in \{0,1\}$. In particular, $\Phi \times z = \Phi$ for all $z \in \mathbb{C}^T$. Furthermore, $\frac{x_1}{y} = \frac{x_2}{y}$ if and only if there is a positive $\alpha \in \mathbb{R}$ such that $x_1 = \alpha x_2$, for all $x_1, x_2 \in \mathbb{C}$ and $y \in \{0,1\}$. Hence $\mathbb{C}^T = \mathbb{C} \cup \left\{ \frac{x}{0};\ x \in \mathbb{C},\ |x| = 1 \right\} \cup \left\{ \frac{0}{0} \right\}$.

Now let $\frac{x}{y} \in \mathbb{C}^T$. When $y \neq 0$ we have $\frac{x}{y} \in \mathbb{C}$, whence $\frac{x}{y} = re^{i\theta}$ for some $r \in [0, \infty)$ and $\theta \in (-\pi, \pi]$. When $y = 0$, $\frac{x}{y} = \frac{x}{0}$. If $x = 0$ then $\frac{x}{y} = \frac{0}{0} = \Phi = \Phi \times e^{i\theta}$ for any $\theta \in (-\pi, \pi]$. If $x \neq 0$ then $\frac{x}{y} = \frac{x}{0} = \frac{x/|x|}{0} = \frac{1}{0} \times \frac{x/|x|}{1} = \infty \times \frac{x}{|x|} = \infty \times e^{i\theta}$ for some $\theta \in (-\pi, \pi]$. From now on we write $\Phi e^{i\theta} := \frac{0}{0}$, where $\theta \in (-\pi, \pi]$ is arbitrary, and $\infty e^{i\theta} := \frac{x}{0}$ where $\theta = \text{Arg}(x)$. Hence every transcomplex number can be written as $re^{i\theta}$ where $r \in [0, \infty] \cup \{\Phi\}$ and $\theta \in (-\pi, \pi]$. Thus $\mathbb{C}^T \subset \left\{ re^{i\theta};\ \ r \in [0, \infty] \cup \{\Phi\},\ \theta \in (-\pi, \pi] \right\}$. Furthermore, clearly, $\left\{ re^{i\theta};\ \ r \in [0, \infty] \cup \{\Phi\},\ \theta \in (-\pi, \pi] \right\} \subset \mathbb{C}^T$. Therefore

$$\mathbb{C}^T = \left\{ re^{i\theta};\ \ r \in [0, \infty] \cup \{\Phi\},\ \theta \in (-\pi, \pi] \right\}.$$

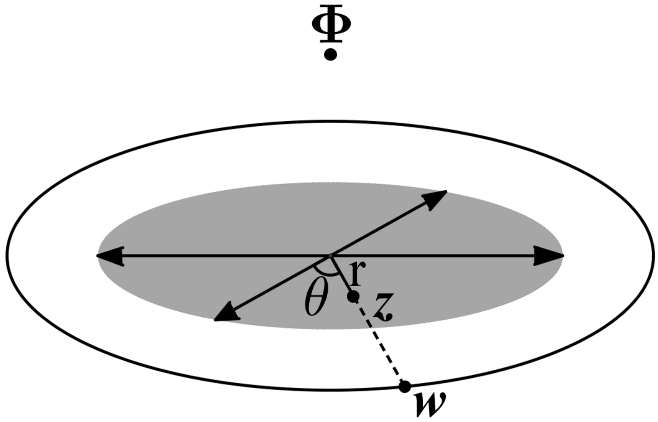

Fig. 1. Transcomplex plane showing $z = re^{i\theta}$ with r finite and $w = \infty e^{i\theta}$

The transcomplex plane is shown in Figure 1. The usual complex plane, $\mathbb{C}$, is shown as a grey disk. It has no real bound but, after a gap, it is surrounded by a circle at infinity, $\left\{ \infty e^{i\theta};\ \theta \in (-\pi, \pi] \right\}$. The point at nullity, $\Phi e^{i\theta}$, lies off the plane containing the complex plane and the circle at infinity.

In the present paper we propose a topology for the set of transcomplex numbers and we extend the elementary functions to the transcomplex domain. In [3] [7] [8] we introduced transreal calculus with transreal topology. ([7] won the best paper award of International Conference on Computer Science and Applications in the World Congress on Engineering and Computer Science 2014, San Francisco, USA.) That topology extends the ordinary, real topology. Here, in the same way, we extend the ordinary, complex topology to the transcomplex plane. Furthermore we establish some results about limits and continuity of transcomplex functions, analogous to complex functions. In [9] we extended every elementary function to the transreal domain. Here we extend every elementary function to the transcomplex domain. Remember that a complex, elementary function is defined in the following way. Every polynomial, root, exponential, logarithm, trigonometric and inverse trigonometric function is an elementary function; any finite composition of elementary functions is an elementary function; and any

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finite combination, using the four, elementary arithmetical operations, between elementary functions is an elementary function.

II. Topology, Limits and Continuity

Let $D := \{z \in \mathbb{C};\ |z| < 1\}$, $\overline{D} := \{z \in \mathbb{C};\ |z| \leq 1\}$ and

$$\begin{array}{rcl} \varphi : \mathbb{C}^T \setminus \{\Phi\} & \to & \overline{D} \subset \mathbb{C}^T \\ re^{i\theta} & \mapsto & \frac{1}{1+\frac{1}{r}} e^{i\theta} \end{array} . \tag{1}$$

Note that $\varphi_{|\mathbb{C}}$ is an homeomorphism between $\mathbb{C}$ and D with respect to the usual topology on $\mathbb{C}$.

Proposition 1: Define $d : \mathbb{C}^T \times \mathbb{C}^T \to \mathbb{R}$ where

$$d(z,w) = \begin{cases} 0, & \text{if } z = w = \Phi \\ 2, & \text{if } z = \Phi \text{ or else } w = \Phi. \\ |\varphi(z) - \varphi(w)|, & \text{otherwise} \end{cases}$$

We have that d is a metric on $\mathbb{C}^T$ and, therefore, $\mathbb{C}^T$ is a metric space.

Proof: Clearly, for all $z, w \in \mathbb{C}^T$, $d(z,w) = 0$ if and only if $z = w$, $d(z,w) = d(w,z)$ and $d(z,w) \geq 0$. If $z, w, u \in \mathbb{C}^T \setminus \{\Phi\}$ then $d(z,u) = |\varphi(z) - \varphi(u)| = |\varphi(z) - \varphi(w) + \varphi(w) - \varphi(u)| \leq |\varphi(z) - \varphi(w)| + |\varphi(w) - \varphi(u)| = d(z,w) + d(w,u)$. The reader can verify that the triangular inequality is also true when $z, w, u \in \mathbb{C}^T \setminus \{\Phi\}$ does not hold. ∎

Proposition 2: The topology on $\mathbb{C}$, induced by the topology of $\mathbb{C}^T$, is the usual topology of $\mathbb{C}$. That is, if $U \subset \mathbb{C}^T$ is open on $\mathbb{C}^T$ then $U \cap \mathbb{C}$ is open (in the usual sense) on $\mathbb{C}$ and if $U \subset \mathbb{C}$ is open (in the usual sense) on $\mathbb{C}$ then U is open on $\mathbb{C}^T$.

Proof: Let us denote the ball of centre z and radius ρ on $\mathbb{C}^T$ as $B_{\mathbb{C}^T}(z,\rho)$, that is, $B_{\mathbb{C}^T}(z,\rho) = \{w \in \mathbb{C}^T;\ |\varphi(z) - \varphi(w)| < \rho\}$, and denote the ball of centre z and radius ρ on $\mathbb{C}$ as $B_{\mathbb{C}}(z,\rho)$, that is, $B_{\mathbb{C}}(z,\rho) = \{w \in \mathbb{C};\ |z - w| < \rho\}$.

Let $U \subset \mathbb{C}^T$ be open on $\mathbb{C}^T$ and let $z \in U \cap \mathbb{C}$. As U is open on $\mathbb{C}^T$, there is a positive $\varepsilon \in \mathbb{R}$ such that $B_{\mathbb{C}^T}(z,\varepsilon) \subset U$. As φ is continuous, there is a positive $\delta \in \mathbb{R}$ such that if $w \in \mathbb{C}^T \setminus \{\Phi\}$ and $|z - w| < \delta$ then $|\varphi(z) - \varphi(w)| < \varepsilon$. Thus $B_{\mathbb{C}}(z,\delta) \subset B_{\mathbb{C}^T}(z,\varepsilon) \cap \mathbb{C} \subset U \cap \mathbb{C}$, whence $U \cap \mathbb{C}$ is open (in the usual sense) on $\mathbb{C}$.

Now, let $U \subset \mathbb{C}$ be open (in the usual sense) on $\mathbb{C}$ and let $z \in U$. Notice that $z = re^{i\theta}$ for some $r \in [0,\infty)$ and some $\theta \in (-\pi, \pi]$. As U is open (in the usual sense) on $\mathbb{C}$, there is a positive $\varepsilon \in \mathbb{R}$ such that $B_{\mathbb{C}}(z,\varepsilon) \subset U$. As φ^{-1} is continuous, there is a positive $\delta \in \mathbb{R}$ such that $\delta < |\varphi(z) - e^{i\theta}|$ and if $\varphi(w) \in \overline{D}$ and $|\varphi(z) - \varphi(w)| < \delta$ then $|z - w| < \varepsilon$. Thus $B_{\mathbb{C}^T}(z,\delta) \subset B_{\mathbb{C}}(z,\varepsilon) \subset U$, whence U is open on $\mathbb{C}^T$. ∎

Corollary 3: If $A \subset \mathbb{C}^T$ is closed on $\mathbb{C}^T$ then $A \cap \mathbb{C}$ is closed (in the usual sense) on $\mathbb{C}$.

Remark 4: Note that, by the definition of the metric of $\mathbb{C}^T$, obviously φ is an homeomorphism.

Remark 5: Because of Proposition 2:

i) Let $(x_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ and let $L \in \mathbb{C}$, it follows that $\lim_{n\to\infty} x_n = L$ on $\mathbb{C}^T$ if and only if $\lim_{n\to\infty} x_n = L$, in the usual, sense on $\mathbb{C}$.
ii) Let $A \subset \mathbb{C}$, $f : A \to \mathbb{C}$, $x \in A'$ and $L \in \mathbb{C}$, it follows that $\lim_{x\to x} f(z) = L$ on $\mathbb{C}^T$ if and only if $\lim_{x\to x} f(z) = L$, in the usual sense, on $\mathbb{C}$.
iii) Given $x \in A$, it follows that f is continuous in x on $\mathbb{C}^T$ if and only if f is continuous in x, in the usual sense, on $\mathbb{C}$.

Proposition 6: $\mathbb{C}^T$ is disconnected.

Proof: $\mathbb{C}^T = \{re^{i\theta};\ \ r \in [0,\infty],\ \theta \in (-\pi,\pi]\} \cup \{\Phi\}$ and the sets $\{re^{i\theta};\ \ r \in [0,\infty],\ \theta \in (-\pi,\pi]\}$ and $\{\Phi\}$ are open. ∎

Notice that Φ is the unique isolated point of $\mathbb{C}^T$.

Remark 7:

i) Let $(x_n)_{n\in\mathbb{N}} \subset \mathbb{C}^T$. Notice that $\lim_{n\to\infty} x_n = \Phi$ if and only if there is $k \in \mathbb{N}$ such that $x_n = \Phi$ for all $n \geq k$.
ii) Let $A \subset \mathbb{C}$, $f : A \to \mathbb{C}^T$ and $x \in A'$, it follows that $\lim_{x\to x} f(z) = \Phi$ if and only if there is a neighbourhood U of x such that $f(z) = \Phi$ for all $x \in U \setminus \{x\}$.
iii) If $\Phi \in A$ then f is continuous in Φ.

Proposition 8: $\mathbb{C}^T$ is a separable space.

Proof: $(\mathbb{Q} + \mathbb{Q}i) \cup \{\Phi\}$ is countable and dense in $\mathbb{C}^T$. ∎

Proposition 9: Every sequence of transcomplex numbers has a convergent subsequence.

Proof: Let $(x_n)_{n\in\mathbb{N}} \subset \mathbb{C}^T$. If $\{n;\ \ x_n \neq \Phi\}$ is a finite set then clearly $\lim_{n\to\infty} x_n = \Phi$. If $\{n;\ \ x_n \neq \Phi\}$ is an infinite set then denote, by $(y_k)_{k\in\mathbb{N}}$, the subsequence of $(x_n)_{n\in\mathbb{N}}$ of all elements of $(x_n)_{n\in\mathbb{N}}$ that are distinct from Φ. Note that $(\varphi(y_k))_{k\in\mathbb{N}}$ (φ defined in (1)) is a bounded sequence of complex numbers, whence it has a convergent subsequence, denoted $(\varphi(y_{k_m}))_{m\in\mathbb{N}}$. As φ is an homeomorphism, $(y_{k_m})_{m\in\mathbb{N}}$ is convergent. ∎

Proposition 10: $\mathbb{C}^T$ is compact.

Proof: As $\mathbb{C}^T$ is a metric space and every sequence from $\mathbb{C}^T$ has a convergent subsequence, $\mathbb{C}^T$ is compact. ∎

Corollary 11: Let $A \subset \mathbb{C}^T$. It follows that A is compact if and only if A is closed.

Proposition 12: $\mathbb{C}^T$ is complete.

Proof: Every compact, metric space is complete and $\mathbb{C}^T$ is compact and metric. ∎

III. Elementary Functions

A. Polynomial Functions

A function, f, is a complex, polynomial function if and only if there is $n \in \mathbb{N}$ and $a, \ldots, a_n \in \mathbb{C}$ such that $f(z) = a_n x^n + \cdots + a_1 x + a$ for all $x \in \mathbb{C}$. As every arithmetical operation is well-defined in transcomplex numbers, we extend the function f to $\mathbb{C}^T$ naturally. In the complex

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domain, $0 \times x^k = 0$ for all complex x but $0 \times x^k = 0$ does not hold for all transcomplex x. In order to avoid this problem we adopt the following definition.

Definition 13: A function, f, is a transcomplex, polynomial function if and only if there is $n, k \in \mathbb{N}$; $n_1, \ldots, n_k \in \{1, \ldots, n-1\}$ and $a, a_{n_1} \ldots, a_{n_k}, a_n \in \mathbb{C}$ such that $a_{n_1} \ldots, a_{n_k}, a_n$ are different from zero and

$$\begin{array}{rcl} f : \mathbb{C}^T & \longrightarrow & \mathbb{C}^T \\ z & \longmapsto & a_n x^n + a_{n_k} x^{n_k} + \cdots + a_{n_1} x^{n_1} + a \end{array}.$$

Remark 14: For every non-constant, transcomplex, polynomial function, f, we have that $f(\Phi) = \Phi$.

B. Exponential Functions

In [9] we defined the transreal, exponential function. We have that $e^{-\infty} = 0$, $e^{\infty} = \infty$ and $e^{\Phi} = \Phi$.

For every ordinary, complex number, $z = re^{i\theta}$, we have

$$\begin{aligned} \exp(z) &= \exp\left(re^{i\theta}\right) \\ &= \exp\left(r\cos(\theta) + ir\sin(\theta)\right) \\ &= e^{r\cos(\theta)}(\cos(r\sin(\theta)) + i\sin(r\sin(\theta))). \end{aligned}$$

In particular, when $\theta \in \{0, \pi\}$, we have that $\sin(\theta) = 0$ whence

$$\begin{aligned} \exp\left(re^{i\theta}\right) &= e^{r\cos(\theta)}(\cos(r\sin(\theta)) + i\sin(r\sin(\theta))) \\ &= e^{r\cos(\theta)}(\cos(0) + i\sin(0)) \\ &= e^{r\cos(\theta)}. \end{aligned}$$

Motivated by this, we extend the exponential function to the transcomplex domain in the following way.

Definition 15: A function, f, is a transcomplex, natural, exponential function if and only if

$$\begin{array}{rcl} f : \mathbb{C}^T & \longrightarrow & \mathbb{C}^T \\ re^{i\theta} & \longmapsto & \exp\left(re^{i\theta}\right) \end{array}$$

where $\exp\left(re^{i\theta}\right) = e^{r\cos(\theta)}$ if $\theta \in \{0, \pi\}$ and $\exp\left(re^{i\theta}\right) = e^{r\cos(\theta)}(\cos(\infty\sin(\theta)) + i\sin(\infty\sin(\theta)))$ if $\theta \notin \{0, \pi\}$.

Notice that $\exp(\infty) = \exp\left(\infty e^{i0}\right) = e^{\infty\cos(0)} = e^{\infty}$ and $\exp(-\infty) = \exp\left(\infty e^{i\pi}\right) = e^{\infty\cos(\pi)} = e^{-\infty}$. Furthermore, if $\theta \in (-\pi, \pi] \setminus \{0, \pi\}$ then

$$\begin{aligned} \exp\left(\infty e^{i\theta}\right) &= e^{\infty\cos(\theta)}(\cos(\infty\sin(\theta)) + i\sin(\infty\sin(\theta))) \\ &= e^{\infty\cos(\theta)}(\Phi + i\Phi) \\ &= e^{\infty\cos(\theta)}\Phi \\ &= \Phi. \end{aligned}$$

Therefore:

i) $\exp(z) = e^z$ for every $z \in \mathbb{C}$.
ii) $\exp(-\infty) = 0$.
iii) $\exp(\infty) = \infty$.
iv) $\exp\left(\infty e^{i\theta}\right) = \Phi$ for all $\theta \in (-\pi, \pi] \setminus \{0, \pi\}$.
v) $\exp(\Phi) = \Phi$.

Remark 16: Notice that exp is discontinuous in all infinities.

Remark 17: Unfortunately, the property $\exp(z + w) = \exp(z)\exp(w)$ does not hold for all $z, w \in \mathbb{C}^T$. For example, let $z = \frac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i}{0}$ and $w = \frac{\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i}{0}$. We have that $z + w = \frac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i}{0} + \frac{\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i}{0} = \frac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i}{0} = \frac{\sqrt{2}}{0} = \frac{1}{0} = \infty$ whence $\exp(z + w) = e^{\infty} = \infty$. But $z = \infty e^{\frac{\pi}{4}i}$ and $w = \infty e^{-\frac{\pi}{4}i}$ whence $\exp(z) = \exp(\infty e^{\frac{\pi}{4}i}) = \Phi$ and $\exp(w) = \exp(\infty e^{\frac{-\pi}{4}i}) = \Phi$. Thus $\exp(z)\exp(w) = \Phi \times \Phi = \Phi$. Therefore $\exp(z+w) \neq \exp(z)\exp(w)$. Another example is when $z = \infty$ and $w = i$. We have that $z + w = \infty + i = \infty$ whence $\exp(z+w) = \exp(\infty) = \infty$ but $\exp(z) = e^{\infty} = \infty$ and $\exp(w) = e^i$ whence $\exp(z)\exp(w) = \infty e^i$. Thus $\exp(z + w) \neq \exp(z)\exp(w)$.

C. Logarithmic Functions

In [9] we defined the transreal, logarithmic function. We have that $\ln(0) = -\infty$, $\ln(\infty) = \infty$ and $\ln(\Phi) = \Phi$.

A function, f, is the complex, logarithmic function if and only if $f(re^{i\theta}) = \ln(r) + i\theta$ for all $r \in (0, \infty)$ and $\theta \in (-\pi, \pi]$. Motivated by this, we extend the logarithmic function to the transcomplex domain in the following way.

Definition 18: A function, f, is a transcomplex, natural, logarithmic function if and only if

$$\begin{array}{rcl} f : \mathbb{C}^T & \longrightarrow & \mathbb{C}^T \\ re^{i\theta} & \longmapsto & \ln(r) + i\theta \end{array}.$$

Remark 19: Notice that, for every $\theta \in (-\pi, \pi]$, we have $\ln(\infty e^{i\theta}) = \ln(\infty) + i\theta = \infty + i\theta = \infty$. So $\ln(z) = \infty$ for every transcomplex infinity z.

Remark 20: The property $\ln(\exp(z)) = z$ does not hold for all $z \in \mathbb{C}^T$. If $\theta \in (-\pi, \pi] \backslash \{0, \pi\}$ then $\ln(\exp(\infty e^{i\theta})) = \ln(\Phi) = \Phi \neq \infty e^{i\theta}$. But $\ln(\exp(z)) = z$ holds in the other cases. Indeed:

- if $z = a + bi \in \mathbb{C}$, where $a, b \in \mathbb{R}$ and $b \in (-\pi, \pi]$, then we already know that $\ln(\exp(z)) = z$,
- if $z = \Phi$ then $\ln(\exp(z)) = \ln(\exp(\Phi)) = \ln(\Phi) = \Phi = z$,
- if $z = -\infty$ then $\ln(\exp(z)) = \ln(\exp(-\infty)) = \ln(0) = -\infty = z$, and
- if $z = \infty$ then $\ln(\exp(z)) = \ln(\exp(\infty)) = \ln(\infty) = \infty = z$.

In the same way $\exp(\ln(z)) = z$ does not hold for all $z \in \mathbb{C}^T$. If $\theta \in (-\pi, \pi] \setminus \{0\}$ then $\exp(\ln(\infty e^{i\theta})) = \exp(\infty) = \infty \neq \infty e^{i\theta}$. But $\exp(\ln(z)) = z$ holds in the other cases. Indeed:

- if $z \in \mathbb{C} \setminus \{0\}$ then we already know that $\exp(\ln(z)) = z$,
- if $z = 0$ then $\exp(\ln(z)) = \exp(\ln(0)) = \exp(-\infty) = 0 = z$,
- if $z = \Phi$ then $\exp(\ln(z)) = \exp(\ln(\Phi)) = \exp(\Phi) = \Phi = z$ and
- if $z = \infty$ then $\exp(\ln(z)) = \exp(\ln(\infty)) = \exp(\infty) = \infty = z$.

Remark 21: We know that, for all $z, w \in \mathbb{C} \setminus \{0\}$, $\ln(zw) = \ln(z) + \ln(w) + ki2\pi$ for some $k \in \mathbb{Z}$. Fortunately this property also holds in transcomplex domain. That is, for all $z, w \in \mathbb{C}^T$, $\ln(zw) = \ln(z) + \ln(w) + ki2\pi$ for some $k \in \mathbb{Z}$. In particular if the two conditions $z \in \mathbb{C} \setminus \{0\}$ and $w \in \mathbb{C} \setminus \{0\}$ do not hold simultaneously then

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$\ln(zw) = \ln(z) + \ln(w)$. The reader can prove this with simple calculations.

Remark 22: Definition 18 can give us powers of every transcomplex base so we define $z^w := \exp(w \ln(z))$ for all $z, w \in \mathbb{C}^T$.

D. Trigonometric Functions

In [9] we defined the transreal trigonometric functions. We have that $\sin(-\infty) = \cos(-\infty) = \tan(-\infty) = \csc(-\infty) = \sec(-\infty) = \cot(-\infty) = \sin(\infty) = \cos(\infty) = \tan(\infty) = \csc(\infty) = \sec(\infty) = \cot(\infty) = \sin(\Phi) = \cos(\Phi) = \tan(\Phi) = \csc(\Phi) = \sec(\Phi) = \cot(\Phi) = \Phi$.

A function, f, is the complex, sine function if and only $f(z) = \frac{\exp(iz)-\exp(-iz)}{2i}$ for all $z \in \mathbb{C}$ and f is the complex, cosine function if and only $f(z) = \frac{\exp(iz)+\exp(-iz)}{2}$ for all $z \in \mathbb{C}$. Furthermore, for all $k \in \mathbb{Z}$, it is the case that $\frac{\sin(z)}{\cos(z)}$, $\frac{1}{\sin(z)}$, $\frac{1}{\cos(z)}$ and $\frac{\cos(z)}{\sin(z)}$ are lexically well-defined at $\frac{\pi}{2} + k\pi$ and $k\pi$ in the transcomplex domain. Because of this we extend the trigonometric functions to $\mathbb{C}^T$ in the following way.

Definition 23: A function is a transcomplex, trigonometric function if and only if it is one of:

a) $\begin{array}{rcl} \sin : \mathbb{C}^T & \longrightarrow & \mathbb{C}^T \\ z & \longmapsto & \sin(z) = \frac{\exp(iz)-\exp(-iz)}{2i} \end{array}$,

b) $\begin{array}{rcl} \cos : \mathbb{C}^T & \longrightarrow & \mathbb{C}^T \\ z & \longmapsto & \cos(z) = \frac{\exp(iz)+\exp(-iz)}{2} \end{array}$,

c) $\begin{array}{rcl} \tan : \mathbb{C}^T & \longrightarrow & \mathbb{C}^T \\ z & \longmapsto & \tan(z) = \frac{\sin(z)}{\cos(z)} \end{array}$,

d) $\begin{array}{rcl} \csc : \mathbb{C}^T & \longrightarrow & \mathbb{C}^T \\ z & \longmapsto & \csc(z) = \frac{1}{\sin(z)} \end{array}$,

e) $\begin{array}{rcl} \sec : \mathbb{C}^T & \longrightarrow & \mathbb{C}^T \\ z & \longmapsto & \sec(z) = \frac{1}{\cos(z)} \end{array}$ and

f) $\begin{array}{rcl} \cot : \mathbb{C}^T & \longrightarrow & \mathbb{C}^T \\ z & \longmapsto & \cot(z) = \frac{\cos(z)}{\sin(z)} \end{array}$.

Remark 24: In [9] we show that $\sin^2(x) + \cos^2(x) = 1^x$ for all $x \in \mathbb{R}^T$. Unfortunately this property does not hold for all transcomplex numbers. We have that $\sin^2(z) + \cos^2(z) = 1^z$ if and only if $z \in \mathbb{C}^T \setminus \{-i\infty, i\infty\}$. Note that, by Remark 22, $1^z = \Phi$ if $z \in \mathbb{C}^T \setminus \mathbb{C}$.

IV. Totalisation

A. Recursive Totalisation

The work, above, develops the elementary, transcomplex functions as functions of transcomplex numbers, which numbers are expressible as tuples, $\langle r, \theta \rangle$, of a transreal radius, r, and a transreal angle, θ. This is adequate from a mathematical point of view but it is not sufficient for computer science where total functions are wanted whose domain can be recursively decomposed into the entire domain of transreal numbers so that, here, r and θ could be any transreals. As usual when a negative r occurs, we map r to its modulus and increment the angle by π so that all transreal radii are admitted. We observe that for all non-finite angles, $\theta \in \{-\infty, \infty, \Phi\}$, it is the case that $re^{i\theta} = r(\cos(\theta) + i\sin(\theta)) = r(\Phi + \Phi) = r\Phi = \Phi$ so that the exponential, logarithmic and trigonometric functions admit all transreal angles. The totalisation of the remaining elementary functions is immediate.

It is well known that the trigonometric functions can be defined, equivalently, by power series or by geometrical constructions. The totalisation of angle, just given, relies on power series. We now give a geometrical construction of the transreal angles.

B. Geometrical Construction of the Transreal Angles

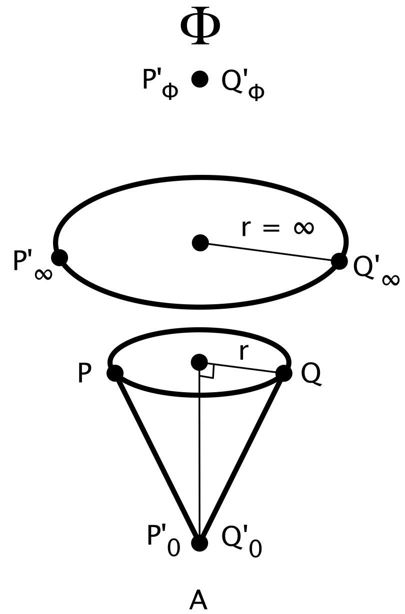

Fig. 2. Transreal cone

Let us explore both finite and non-finite angles in a geometrical construction before settling on a definition of transreal angle.

Consider a transreal cone with apex A, as shown in Figure 2. A right cross-section of the cone is a circle on which a radius, r, may be drawn. On the circle at unit radius, $r = 1$, mark off, not necessarily distinct, points P and Q. Project the lines AP and AQ, taking a point P' anywhere on AP, including the point P'_0 at A, the point P'_∞ on the circle at infinity and the point P'_Φ at the point at nullity, shown as Φ in the figure. Similarly take Q' on AQ.

At $r = 1$ the angle from P to Q is defined to be the arc length $\widehat{PQ}$ taken zero, positive or negative according to the usual sign convention. It is then shown that identical plane rotations arise for all non-negative, finite radii, $0 < r < \infty$, when the angle is given by $\widehat{PQ} = \widehat{P'Q'}/r$ when P' and Q' lie on the circle with radius r. We now consider the cases $r \in \{0, \infty, \Phi\}$. The reader is free to construct negative radii in a double cone.

At $r = \Phi$ we have $\widehat{P'_\Phi Q'_\Phi}/r_\Phi = \widehat{\Phi\Phi}/0 = 0/0 = \Phi$, which is to say that the angle nullity occurs at $r = \Phi$. Now $re^{i\Phi} = r(\cos(\Phi) + i\sin(\Phi)) = r(\Phi + \Phi) = r\Phi = \Phi$ for all transreal r. Thus the nullity rotation, by angle nullity, $\theta = \Phi$, maps the whole of its domain onto the point at nullity, Φ.

At $r = 0$ we have $\widehat{P'_0 Q'_0}/r_0 = \widehat{AA}/0 = 0/0 = \Phi$, which is to say that the angle nullity also occurs at $r = 0$. This is a redundancy which we shall presently resolve.

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At $r = \infty$ the zero angle, $\theta = 0$, arises when P'_∞ and Q'_∞ are co-punctal but when P'_∞ and Q'_∞ are distinct we have $\theta = \widehat{P'_\infty Q'_\infty}/r_\infty = \infty/\infty = \Phi$ so that only the angles zero and nullity can arise, from this geometrical construction, in the circle at infinity. This computed angle of nullity is degenerate in the sense that it hides the true value of any non-zero, finite angle in the circle at infinity. That is it hides all points $\infty e^{i\theta}$ with $\theta \neq 0$. Information hiding is discussed in [4]. We shall presently avoid this degeneracy.

We now construct the infinite angle via a winding on the ordinary, unit cone.

By definition P and Q lie in the unit circle, separated by an angle $\theta = \widehat{PQ}$. When P and Q are distinct we take an arc length $\alpha = \widehat{PQ}$ and when P and Q are co-punctal we take $\alpha = 2\pi$. We now take the arc at a smaller radius and wind it from P', once fully round the cone, and continue exactly to Q'. This winding marks off the angle $\theta_1 = \theta + 2\pi$. We continue in this way, recursively winding at smaller radii, to produce the family of angles $\theta_k = \theta + 2k\pi$. We suppose that the winding process is continuous to that at $r = 0$ we produce the winding $\theta_\infty = \theta + 2\infty\pi = \infty$. But this rotation is identically the rotation at $r = 0$ so the infinite angle is equivalent to the nullity angle. This agrees with the result obtained from power series.

Notice that all transreal angles are given uniquely when we define a zero angle at a fixed point, Z, on the base of the cone. Let us take $Z = Q$ in Figure 2. Now all angles, θ, in the principle range $-\pi < \theta \leq \pi$ are given uniquely by a point in the unit circle with radius $r = 1$. All finite angles, $-\infty < \theta < \infty$, are given uniquely by windings on the cone at all positive radii, $0 < r \leq 1$. And the equivalence class of all non-finite angles is given uniquely by the apex of the cone at $r = 0$. Hence the apex of the unit cone uniquely defines the non-finite angle and the unit cone, with the apex punctured and a zero point identified, defines each finite angle uniquely. Thus there is an injection from transreal angles to points on the unit cone, with the zero point identified. This being the case we accept the winding construction in the unit cone as our definition of transreal angle and simply note the extraneous behaviour in the circle at infinity and the point at nullity.

In the definition we have just adopted, we assume continuity of the winding process to obtain the infinite angle equal to the nullity angle, which is what we wanted to achieve. This commits us to continuity of winding everywhere, including all applications of winding in topology, in complex analysis and in mathematical physics.

V. Discussion

We have equipped the set of transcomplex numbers with a topology, given by a metric, that contains the usual topology of the complex numbers. This preserves many properties of complex numbers and leads to consistent generalisations of them. In particular it extends the usual geometrical constructions of the real trigonometric functions to their transreal counterparts, from which we obtain the transcomplex, trigonometric functions. Thus the topology provides a firm foundation for our work. As $\mathbb{C}^T$ is a metric space, all usual results of that space follow. For example: $\mathbb{C}^T$ is a Hausdorff space; the limit of a sequence, when it exists, is unique; when $A \subset \mathbb{C}^T$, $f : A \to \mathbb{C}^T$, $x \in A'$ and $L \in \mathbb{C}^T$, we have that $\lim_{x \to x} f(z) = L$ if and only if $\lim_{n\to\infty} f(x_n) = L$ for all $(x_n)_{n\in\mathbb{N}} \subset A \setminus \{x\}$ such that $\lim_{n\to\infty} x_n = x$; and when $A \subset \mathbb{C}^T$, $f : A \to \mathbb{C}^T$ and $x \in A$, we have that f is continuous in x if and only if $\lim_{n\to\infty} f(x_n) = f(z)$ for all $(x_n)_{n\in\mathbb{N}} \subset A$ and $\lim_{n\to\infty} x_n = x$.

In [8] we adopted the following procedure to extend an elementary function from the real to the transreal domain. If the usual expression of the function is lexically well-defined, at a transreal number, then we define the function by simply applying its expression at that transreal number. If the function, f, is not lexically well-defined at a transreal number, x, but there is a limit, $\lim_{x\to x} f(z)$, then we choose to define the function at x by $\lim_{x\to x} f(z)$. Otherwise we choose to define the function by way of its power series if it converges. And if, nevertheless, its power series does not converge, we keep the function undefined. But the transcomplex space is more complicated than transreal space. Transreal space has only two infinite numbers and there is only one path, one direction, to each one of these infinities but there are several (infinite) paths and directions to each infinite transcomplex number. Hence many limits do not exist at infinite transcomplexes.

Now let us address some remarks to why we did not adopt other ways to define the exponential function on the transcomplex plane.

i) We cannot define the transcomplex, exponential function by a lexical expression because the exponential is not defined by finitely many arithmetical operations. In particular we cannot take the usual algebraic definition that if $z = a + bi$ then $\exp(z) = e^a(\cos(b) + i\sin(b))$ because the infinite transcomplex numbers do not have any algebraic representation.
ii) We cannot define the transcomplex, exponential function by limits because, for every $r \in [0, \infty)$, $\exp\left(re^{i\theta}\right) = \exp\left(r\cos(\theta) + ir\sin(\theta)\right) = e^{r\cos(\theta)}(\cos(r\sin(\theta)) + i\sin(r\sin(\theta)))$, whence there is no $\lim_{r\to\infty} \exp\left(re^{i\theta}\right)$ for every $\theta \in (-\pi, \pi] \setminus \{0, \pi\}$.
iii) We cannot define the transcomplex, exponential function by power series because $1 + \sum_{n=1}^{\infty} \frac{\left(\infty e^{i\theta}\right)^n}{n!} = 1 + \sum_{n=1}^{\infty} \frac{\infty e^{in\theta}}{n!} = 1 + \sum_{n=1}^{\infty} \infty e^{in\theta}$ diverges for every $\theta \in (-\pi, \pi] \setminus \{0, \pi\}$.
iv) We could think about the homeomorphism φ. Notice that the circle at infinity, $\{\infty e^{i\theta};\ \theta \in (-\pi, \pi]\}$, is a homeomorphic copy, by the function φ, of the unitary circle, $\partial D := \{e^{i\theta};\ \theta \in (-\pi, \pi]\}$. So, in order to define exp at an infinite transcomplex number $\infty e^{i\theta}$, we could transform $\infty e^{i\theta}$ to $\varphi(\infty e^{i\theta}) = e^{i\theta}$, then we would take $\exp\left(e^{i\theta}\right) = \exp\left(\cos(\theta) + i\sin(\theta)\right) = e^{\cos(\theta)}(\cos(\sin(\theta)) + i\sin(\sin(\theta)))$, after that we would transform $e^{\cos(\theta)}(\cos(\sin(\theta)) + i\sin(\sin(\theta)))$ to $\frac{e^{\cos(\theta)}(\cos(\sin(\theta)) + i\sin(\sin(\theta)))}{|e^{\cos(\theta)}(\cos(\sin(\theta)) + i\sin(\sin(\theta)))|} = \cos(\sin(\theta)) + i\sin(\sin(\theta))$ and, finally, we would transform $\cos(\sin(\theta)) + i\sin(\sin(\theta))$ to $\varphi^{-1}\left(\cos(\sin(\theta)) + i\sin(\sin(\theta))\right)$. In this way, denoting the function $\mathbb{C} \setminus \{0\} \ni z \mapsto \frac{z}{|z|} \in \partial D$ by h, we would define $\exp\left(\infty e^{i\theta}\right) := \left(\varphi^{-1} \circ h \circ \exp \circ \varphi\right)\left(\infty e^{i\theta}\right)$. This would define the exponential of all transcomplex infinities but the transcomplex exponential, $\exp_{\mathbb{C}^T}$, would not be an extension of the transreal exponential,

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$\exp_{\mathbb{R}^T}$. In fact $\exp_{\mathbb{C}^T}(-\infty) = \exp_{\mathbb{C}^T}(\infty e^{i\pi}) = (\varphi^{-1} \circ h \circ \exp \circ \varphi)\left(\infty e^{i\pi}\right) = (\varphi^{-1} \circ h \circ \exp)\left(\varphi\left(\infty e^{i\pi}\right)\right) = (\varphi^{-1} \circ h \circ \exp)\left(e^{i\pi}\right) = (\varphi^{-1} \circ h \circ \exp)\left(-1\right) = (\varphi^{-1} \circ h)(\exp(-1)) = (\varphi^{-1} \circ h)(e^{-1}) = \varphi^{-1}(h(e^{-1})) = \varphi^{-1}\left(\frac{e^{-1}}{|e^{-1}|}\right) = \varphi^{-1}(1) = \infty \neq 0 = \exp_{\mathbb{R}^T}(-\infty)$.

In future we intend to extend the differential and integral calculi from the complex to the transcomplex domain, opening up the way to extend our generalisation of Newtonian Physics [4] to both relativistic and quantum physics.

VI. Conclusion

The transcomplex numbers, introduced elsewhere, contain the complex, transreal and real numbers and support division by zero, consistently, in all of their arithmetics.

Here we supply the set of transcomplex numbers with a topology that contains the usual topology of both the complex and real numbers. It is easy to see that our transcomplex topology also contains transreal topology. Thus we maintain all of these topologies within a single number system.

We extend the exponential from the complex to the transcomplex domain so that it contains the complex, transreal and real exponentials. Hence we obtain the transcomplex logarithm and the transcomplex, trigonometric functions and all transcomplex, elementary functions, such that they contain their complex, transreal and real counterparts.

We give a geometrical construction of transreal angle. Thus the equivalence of geometrical and power series definitions of the trigonometric functions is maintained.

All of the transarithmetics are total. This removes infinitely many exceptions from mathematics and from computer programs. Thus our corpus of work continues to offer both theoretical and practical advantages.

Our geometrical construction, of angle, assumes continuity of the winding process so we are committed to this continuity wherever winding occurs, for example in topology, in complex analysis and in mathematical physics.

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