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Legendre Wavelets Direct Method for the Numerical Solution of Fredholm Integral Equation of the First Kind

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***Abstract*—In this paper, an efficient direct method based on Legendre wavelets is introduced to approximate the solution of Fredholm integral equations of the first kind. These basic functions are orthonormal and have compact support. The properties of the Legendre wavelets are utilized to convert the integral equations into a system of linear algebraic equations. The main characteristic of the method is low cost of setting up the equations without using any projection method. Furthermore an estimation of error bound for the present method is proved. Finally, some numerical examples are provided to demonstrate the applicability and accuracy of the proposed technique.**

***Index Terms*—Fredholm integral equation of the first kind, Legendre wavelets, Direct method, Error bound.**

I. Introduction

INTEGRAL equation has been one of the essential tools for various areas of applied mathematics. Integral equations are widely involved in many scientific and engineering problems [1], [2], [3]. Fredholm integral equation of the first kind is one of the inverse problem that arise in many physical models and engineering fields, such as radiography, spectroscopy, cosmic radiation and image processing.

In this paper, we consider Fredholm integral equation of the first kind of the following form

$$\int_0^1 k(t,s)f(s)ds = g(t), \qquad 0 \le t \le 1, \tag{1}$$

where $g(t)$ and $k(t,s)$ are given continuous functions on $[0,1]$ and $[0,1]\times[0,1]$, respectively, and $f(t)$ is the solution to be determined.

Fredholm integral equation of the first kind is one of the ill-posed problems, and its numerical treatment is more difficult than second kind one, which has been widely studied [4], [5], [6], [7], [8], so, it is difficult to employ an appropriate numerical method. Several numerical methods have been used to approximate the solution of Eq. (1). Babolian and Delves [9] introduced an augmented Galerkin method for Fredholm integral equations of the first kind. In [10], computational projection methods were presented for the numerical solution of Fredholm integral equations. Shang and Han [11] obtained an approximate solution of these integral equations by Legendre multi-wavelets. Also, Adibi and Assari in [12] solved Fredholm integral equation of the first kind based on Chebyshev wavelet method.

Within recent years, wavelets lead to a huge number of applications in numerical approximations. A survey of some of their usages in various sciences can be found in [13]. The main characteristic of wavelets is their ability to convert the given differential and integral equations to a system of linear or nonlinear algebraic equations, which are then solved by existing numerical methods. Legendre wavelet has been used to solve different types of integral equations and integro-differential equations because of its good accuracy in approximations. An excellent survey on applications of Legendre wavelets for solving different problems can be found in [14], [15], [16], [17], [18].

In the present paper, we propose a direct computational method based on Legendre wavelets to determine the solution of Fredholm integral equations of the first kind. The properties of Legendre wavelets are applied to evaluate the unknown coefficients and find an approximate solution for Eq. (1).

The paper is organized as follows: In Section 2, we review the basic properties of wavelets and Legendre wavelets required for our subsequent development. Section 3 is devoted to the derivation of a computational method to solve Eq. (1) by using Legendre wavelets. In Section 4, we discuss the convergence analysis and error estimation. Several examples are given in section 5 to indicate the applicability and the accuracy of the numerical technique.

II. Wavelets and their properties

Wavelets constitute a family of functions constructed from dilation and translation of a single function called mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}}\psi\Big(\frac{t-b}{a}\Big), \qquad a,b \in R, a \neq 0.$$

If we consider the parameters a and b as discrete values $a = a_0^{-k}$, $b = nb_0a_0^{-k}$, where $a_0 > 0$, $b_0 > 0$ and n and k are positive integers, then we have the following discrete wavelets:

$$\psi_{n,k}(t) = |a_0|^{\frac{k}{2}}\psi(a_0^k t - nb_0),$$

where $\psi_{n,k}(t)$ form a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$ and $b_0 = 1$ then $\psi_{n,k}(t)$ form an orthonormal basis [13].

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A. Legendre wavelet

For any positive integer k, the Legendre wavelets are defined on the interval $[0,1)$ as follows:

$$\psi_{nm}(t)=\begin{cases}\sqrt{m+\frac{1}{2}}2^{\frac{k}{2}}L_m(2^kt-2n+1), & t\in[\frac{2n-2}{2^k},\frac{2n}{2^k}),\\ 0, & otherwise,\end{cases}$$

where $n=1,2,\cdots,2^{k-1}$ and $m=0,1,\cdots,M-1$. Here, $L_m(t)$ are the Legendre polynomials of order m and can be determined by the recurrence formulae as following:

$$L_0(t)=1,\quad L_1(t)=t,$$

$$L_{m+1}(t)=\frac{2m+1}{m+1}tL_m(t)-\frac{m}{m+1}L_{m-1}(t),$$

$$m=1,2,3,\ldots.$$

B. Function approximation

A function $f(t)\in L^2[0,1)$ may be expanded in terms of Legendre wavelets as follows:

$$f(t)\simeq\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{nm}\psi_{nm}(t)=\mathbf{C}^T\Psi(t),\tag{2}$$

where $\mathbf{C}$ and $\Psi(t)$ are $2^{k-1}M\times1$ vectors given by

$$\mathbf{C}=[c_{10},c_{11},\ldots,c_{1(M-1)},\ldots,c_{2^{k-1}0},\ldots,c_{2^{k-1}(M-1)}]^T,$$

$$\Psi(t)=[\psi_{10}(t),\ldots,\psi_{1(M-1)}(t),\psi_{20}(t),\ldots,\psi_{2(M-1)}(t),$$
$$\ldots,\psi_{2^{k-1}0}(t),\ldots,\psi_{2^{k-1}(M-1)}(t)]^T.$$

In Eq. (2), the wavelet coefficients are determined by $c_{nm}=\big(f(t),\psi_{nm}(t)\big)$, for $n=1,2,\cdots,2^{k-1}$, $m=0,1,\cdots,M-1$, and, $(.,.)$ denotes the inner product.
Similarly, the function $k(t,s)\in L^2([0,1)\times[0,1))$ may be estimated as:

$$k(t,s)\simeq\Psi^T(t)\mathbf{K}\Psi(s),\tag{3}$$

where $\mathbf{K}$ is a $2^{k-1}M\times2^{k-1}M$ matrix that

$$k_{ij}=\Big(\psi_i(t),\big(k(t,s),\psi_j(s)\big)\Big),\quad i,j=1,2,\ldots,2^{k-1}M.\tag{4}$$

III. The proposed numerical method

In this section, we introduce a numerical method to solve Eq. (1) by means of Legendre wavelets. For this purpose, assume that

$$f(t)\simeq\mathbf{C}^T\Psi(t),\tag{5}$$

$$g(t)\simeq\mathbf{G}^T\Psi(t),\tag{6}$$

$$k(t,s)\simeq\Psi^T(t)\mathbf{K}\Psi(s),\tag{7}$$

where $\mathbf{K}$ is a known $2^{k-1}M\times2^{k-1}M$-dimensional matrix and $\mathbf{G}$ is a known $2^{k-1}M$-vector. In Eq. (5), $\mathbf{C}$ is an unknown $2^{k-1}M$-vector.
By substituting Eqs. (5) $-$(7) into Eq. (1), we get

$$\int_0^1\Psi^T(t)\mathbf{K}\Psi(s)\Psi^T(s)\mathbf{C}ds=\Psi^T(t)\mathbf{G},\tag{8}$$

or

$$\Psi^T(t)\mathbf{K}\left(\int_0^1\Psi(s)\Psi^T(s)ds\right)\mathbf{C}=\Psi^T(t)\mathbf{G}.\tag{9}$$

Furthermore, the integration of the cross-product of two Legendre wavelets vectors is

$$\int_0^1\Psi(t)\Psi^T(t)dt=\mathbf{I},\tag{10}$$

where $\mathbf{I}$ is the $2^{k-1}M$ identity matrix.
Using Eq. (10), Eq. (9) can be replaced by

$$\Psi^T(t)\mathbf{KIC}=\Psi^T(t)\mathbf{G}.\tag{11}$$

Therefore, we have the following linear system of equations:

$$\mathbf{KC}=\mathbf{G},\tag{12}$$

by solving the linear system (12), we can find the unknown vector $\mathbf{C}$.

IV. Convergence analysis

In this section, we are concerned with the error bound and convergence of the proposed method by the following theorem.
Theorem. Suppose that $f\in C^{(M)}[0,1)$ is a real-valued function such that $f=\sum_{n=1}^{2^{k-1}}f_n$, and let $Y_n=span\{\psi_{n0}(t),\psi_{n1}(t),\ldots,\psi_{n(M-1)}(t)\}$, $n=1,2,\ldots,2^{k-1}$. If $\mathbf{C}_n^T\Psi_n(t)$ is the best approximation to f_n from Y_n, where

$$\mathbf{C}_n=[c_{n0},c_{n1},c_{n2},\ldots,c_{n(M-1)}]^T,$$

$$\Psi_n(t)=[\psi_{n0}(t),\psi_{n1}(t),\psi_{n2}(t),\ldots,\psi_{n(M-1)}(t)]^T,$$

then $f_{N,M}(t)=\mathbf{C}^T\Psi(t)$ approximates $f(t)$ with the following error bound

$$\|f(t)-f_{N,M}(t)\|_2\leq\frac{\gamma}{2^{M(k-1)}M!\sqrt{2M+1}},\tag{13}$$

where $\gamma=\max_{t\in[0,1)}|f^{(M)}(t)|$.

Proof. The Taylor expansion for the function $f_n(t)$ is

$$\tilde{f}_n(t)=f_n\Big(\frac{2n-2}{2^k}\Big)+f_n'\Big(\frac{2n-2}{2^k}\Big)\Big(t-\frac{2n-2}{2^k}\Big)+\cdots$$

$$f_n^{(M-1)}\Big(\frac{2n-2}{2^k}\Big)\frac{(t-\frac{2n-2}{2^k})^{M-1}}{(M-1)!},\quad\frac{2n-2}{2^k}\leq t<\frac{2n}{2^k},$$

we know that

$$|f_n(t)-\tilde{f}_n(t)|\leq|f^{(M)}(\eta)|\frac{(t-\frac{2n-2}{2^k})^M}{M!},$$

$$\eta\in[\frac{2n-2}{2^k},\frac{2n}{2^k}),\quad n=1,2,\cdots,2^{k-1}.\tag{14}$$

Since $\mathbf{C}_n^T\Psi_n(t)$ is the best approximation of $f_n(t)$ and $\tilde{f}_n\in Y_n$, using (14), we have

$$\Big\|f_n(t)-\mathbf{C}_n^T\Psi_n(t)\Big\|_2^2\leq\Big\|f_n(t)-\tilde{f}_n(t)\Big\|_2^2$$

$$=\int_{\frac{2n-2}{2^k}}^{\frac{2n}{2^k}}|f_n(t)-\tilde{f}_n(t)|^2dt$$

$$\leq\int_{\frac{2n-2}{2^k}}^{\frac{2n}{2^k}}\Big[\frac{f^{(M)}(\eta)(t-\frac{2n-2}{2^k})^M}{M!}\Big]^2dt$$

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TABLE I
ABSOLUTE ERRORS FOR EXAMPLE 1

t	Present method with k=2, $M=2$	Present method with k=2, $M=3$	Method in [12] with k=2, $m=2$
0.1	1.93×10^{-7}	5.37×10^{-11}	5.32×10^{-5}
0.2	9.22×10^{-8}	1.81×10^{-11}	1.37×10^{-4}
0.3	8.48×10^{-9}	3.61×10^{-11}	2.21×10^{-4}
0.4	1.09×10^{-7}	1.57×10^{-13}	3.04×10^{-4}
0.5	5.23×10^{-8}	1.70×10^{-11}	5.98×10^{-4}
0.6	3.85×10^{-8}	8.14×10^{-12}	3.80×10^{-4}
0.7	2.47×10^{-8}	2.16×10^{-12}	5.51×10^{-4}
0.8	1.09×10^{-8}	8.99×10^{-13}	7.21×10^{-4}
0.9	2.82×10^{-9}	1.06×10^{-12}	8.92×10^{-4}

$$\leq \left[\frac{\gamma}{M!}\right]^2 \int_{\frac{2n-2}{2^k}}^{\frac{2n}{2^k}} \left(t-\frac{2n-2}{2^k}\right)^{2M} dt$$

$$= \left[\frac{\gamma}{M!}\right]^2 \frac{1}{2^{(k-1)(2M+1)}(2M+1)}.$$

Now,

$$\left\|f(t)-\mathbf{C}^T\Psi(t)\right\|_2^2 \leq \sum_{n=1}^{2^{k-1}} \left\|f_n(t)-\mathbf{C}_n^T\Psi_n(t)\right\|_2^2$$

$$\leq \frac{\gamma^2}{2^{(k-1)(2M)}(M!)^2(2M+1)}.$$

By taking the square roots, we get the error estimate of approximate $f(t)$ with $\mathbf{C}^T\Psi(t)$. Obviously, by considering assumptions of this theorem, we infer that $\mathbf{C}^T\Psi(t)\to f(t)$ as M,k are sufficiently large. ■

V. NUMERICAL EXAMPLES

In this section, we present several examples to approximate the solution of Fredholm integral equations of the first kind using numerical method described in the previous sections. In order to demonstrate the performance of the method and clarify the accuracy and the efficiency of the proposed method, we compare the results of our method with the results from some other methods. The numerical experiments are implemented in the software Mathematica 9.

A. Example 1

Consider the following Fredholm integral equation of the first kind:

$$\int_0^1 \sin(ts)f(s)ds = \frac{\sin t - t\cos t}{t^2}, \quad 0\leq t\leq 1,$$

with the exact solution $f(t)=t$. The computational results are displayed in Table I and Fig. 1. In Table I the absolute errors of the proposed method are compared with the results obtained by the Chebyshev wavelet method [12]. Fig. 1 shows the exact and approximate solution for k = 2 and M = 3.

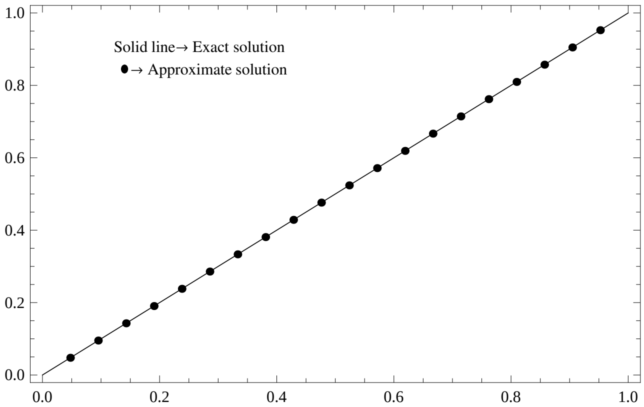

Fig. 1. The Exact Solution and Approximate Solution for Example 1 .

TABLE II
ABSOLUTE ERRORS FOR EXAMPLE 2

t	Present method with k=2, $M=2$	Present method with k=2, $M=3$	Method in [12] with k=2, $m=3$
0.1	2.99×10^{-3}	2.04×10^{-4}	9.37×10^{-4}
0.2	1.42×10^{-2}	2.35×10^{-4}	2.65×10^{-5}
0.3	1.66×10^{-2}	4.66×10^{-4}	1.08×10^{-3}
0.4	5.47×10^{-3}	7.97×10^{-4}	1.10×10^{-3}
0.5	6.36×10^{-2}	6.20×10^{-3}	1.25×10^{-4}
0.6	1.68×10^{-2}	6.00×10^{-4}	2.11×10^{-3}
0.7	1.16×10^{-2}	9.01×10^{-4}	2.26×10^{-3}
0.8	2.00×10^{-2}	2.16×10^{-4}	7.28×10^{-4}
0.9	6.13×10^{-3}	5.37×10^{-4}	3.83×10^{-4}

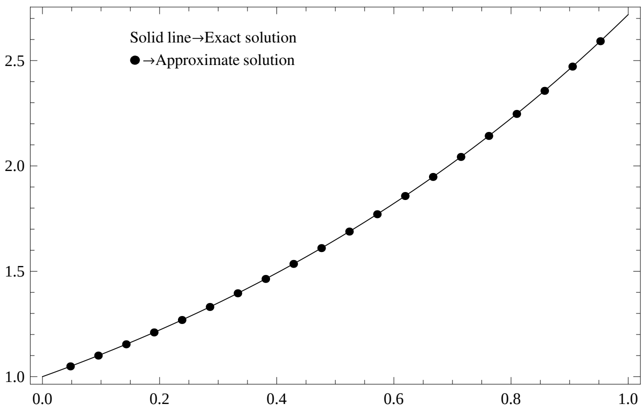

Fig. 2. The Exact Solution and Approximate Solution for Example 2.

B. Example 2

As the second example, the following Fredholm integral equation of the first kind is considered

$$\int_0^1 e^{ts}f(s)ds = \frac{e^{t+1}-1}{t+1}, \quad 0\leq t\leq 1.$$

The exact solution of this problem is $f(t)=e^t$. Table II and Fig. 2 show the numerical results for this example. A comparison between the absolute errors of our method together with Chebyshev wavelet method [12] for $t\in[0,1)$ is shown in Table II. The approximate solution (for $k=2$ and $M=3$) together with the exact solution are depicted in Fig. 2.

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TABLE III
ABSOLUTE ERRORS FOR EXAMPLE 3

t	Present method with k=2, $M=2$	Present method with k=2, $M=3$	method in [10] with j=3
0.1	6.29×10^{-3}	1.94×10^{-13}	5.53×10^{-2}
0.2	1.63×10^{-2}	2.59×10^{-14}	5.09×10^{-2}
0.3	6.37×10^{-3}	6.19×10^{-13}	1.59×10^{-1}
0.4	2.35×10^{-2}	1.74×10^{-12}	3.94×10^{-1}
0.5	1.74×10^{-2}	1.25×10^{-11}	6.24×10^{-1}
0.6	1.56×10^{-2}	2.74×10^{-12}	6.14×10^{-1}
0.7	2.88×10^{-2}	2.70×10^{-12}	6.27×10^{-1}
0.8	4.80×10^{-2}	3.81×10^{-12}	3.44×10^{-1}
0.9	4.90×10^{-3}	5.60×10^{-13}	3.37×10^{-2}

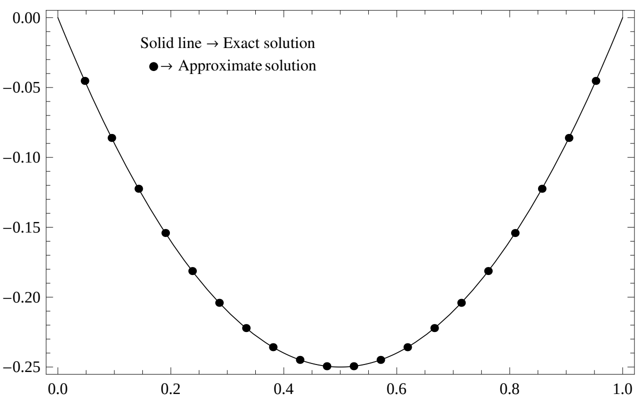

Fig. 3. The Exact Solution and Approximate Solution for Example 3 .

C. Example 3

Consider the following Fredholm integral equation of the first kind:

$$\int_0^1 \sqrt{s^2+t^2} f(s)ds = g(t), \quad 0 \le t \le 1,$$

where

$$g(t) = \frac{1}{48}\Big(16(t^2)^{\frac{3}{2}} + 3t^4\big(\log[t^2]$$
$$-2\log[1+\sqrt{1+t^2}]\big) - 2\sqrt{1+t^2}(2+5t^2)\Big),$$

and with the exact solution $f(t) = t(t-1)$. Table III and Fig. 3 show the numerical results for this example. The comparison results between the proposed method and Computational projection methods [10] are tabulated in Table III. Moreover, Fig. 3 displays the exact solution and approximate solution with $k = 2$ and $M = 3$.

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