Numerical Solution of the Arbitrary Order Weakly Singular Integral Equation Using Block Pulse Functions

Hao Song and Mingxu Yi

***Abstract*— In this paper, an approximate formulation of the arbitrary order weakly singular integral is acquired by using Block-Pulse functions. The formulation contributes a numerical scheme for solving the higher order linear and nonlinear weakly singular Volterra integral equation of the second kind. By implementing the Block-Pulse functions and the approximation, the considered equations will be reduced to a system of algebraic equations. Also, the error analyses of the suggested numerical method are provided. Some examples are considered to demonstrate the efficiency and accuracy of proposed numerical approach.**

***Index Terms*— *weakly* singular integral, weakly singular integral equation, Block Pulse functions, operational matrix, error analysis, numerical solution.**

I. Introduction

INTEGRAL equations with higher order have recently proved to be valuable tools to the modeling of many physical phenomena and it starts to attract much more attention of Physicists and Mathematicians [1-3]. These equations are represented by linear and nonlinear integral equations and solving such higher order integral equations is very important [4-10]. So it is very important to find efficient methods for solving higher order integral equations. The weakly singular Volterra integral equations are also found in a lot of physical, chemical, and biological problems, such as reaction-diffusion problems, crystal growth and so on [11-13]. Most of the higher order integral equations do not have exact analytical solutions; hence considerable need has been focused on approximate and numerical solutions of these equations.

Recently, various researchers have introduced new methods in the literature. These methods include operational matrix method [5], Adomian decomposition method (ADM) [7], differential transform method (DTM) [8], Laplace decomposition method (LDM) [10], homotopy analysis method (HAM) [14] and homotopy perturbation method (HPM) [15]. In this article, the integration operational matrix of the Block Pulse functions is got by using the operational matrix of Legendre wavelet. The operational matrix will be used to solve the higher order linear and nonlinear weakly singular Volterra integral equation.

In this paper, we describe application of Block Pulse functions basis in solving the higher order linear and nonlinear Volterra integral equation with a weakly singular kernel. Consider the higher order linear Volterra integral equation with weakly singular kernel

$$\sum_{i=0}^{n} a_i(t)y^{(i)}(t)+\lambda\int_0^t (t-s)^{-\alpha} y(s)ds = f(t). \tag{1}$$

where $a_i(t)$, $f(t)$ are known continuous functions on $[0,1]$ and $y^{(i)}(t)$ stands for the i th-order derivative of $y(t)$. λ is a real constants.

The higher order nonlinear Volterra integral equation with weakly singular kernel is as following

$$\sum_{i=0}^{n} a_i(t)y^{(i)}(t)+\lambda\int_0^t (t-s)^{-\alpha} [y(s)]^p ds = f(t). \tag{2}$$

II. The Quadrature Formulation of the Arbitrary Order Weakly Singular Integral

Block Pulse functions have been studied by many authors and also have been applied for solving different problems. Here, we present a brief review of Block Pulse functions and its properties [15]. The m -set of Block Pulse functions are defined as

$$b_i(x)=\begin{cases}1, & \dfrac{iT}{m}\le x<\dfrac{(i+1)T}{m};\\ 0, & otherwise.\end{cases} \tag{3}$$

where $i=0,1,2,\cdots m-1$,
with a positive integer value for m. In this paper, we set $T=1$, and

$$b_i(x)b_j(x)=\begin{cases}b_i(x), & i=j;\\ 0 & i\ne j.\end{cases} \tag{4}$$

The set of Block Pulse functions are orthogonal with each other, that is

$$\int_0^1 b_i(x)b_j(x)dx=\begin{cases}1/m, & i=j;\\ 0, & i\ne j.\end{cases} \tag{5}$$

As m tends to infinity, the m -set of Block Pulse functions become a complete basis for any $L^2[0,1)$, so that an arbitrary real bounded function $f(t)$, which is square integrable in the interval $[0,T)$, can be expanded into

$$\int_0^1 f^2(x)dx=\sum_{i=0}^{\infty} f_i^2\left\|b_i(x)\right\|^2. \tag{6}$$

where $f_i=m\int_0^1 b_i(x)f(x)dx.$

Arbitrary order weakly singular integral is given by as following

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$$I(t)=\int_0^t \frac{g(s)}{(t-s)^\alpha}ds,\ \ 0\le t\le 1, 0<\alpha<1. \tag{7}$$

where $g(s)\in L^2([0,1])$.

Using the orthogonality property of the Block-Pulse functions, the function $g(s)$ can be written as

$$g(s)\cong\sum_{i=0}^{m-1}c_i b_i(s)=c^T B_m(s). \tag{8}$$

where $c=(c_0,c_1,\cdots,c_{m-1})^T$, $B_m(x)=(b_0(s),b_1(s),\cdots,b_{m-1}(s))^T$.

Substituting the Equation (8) into Equation (7), we have:

$$I(t)=\int_0^t \frac{g(s)}{(t-s)^\alpha}ds=c^T\int_0^t \frac{B_m(s)}{(t-s)^\alpha}ds=c^T D(t). \tag{9}$$

where

$$D(t)=\int_0^t \frac{B_m(s)}{(t-s)^\alpha}ds. \tag{10}$$

Combining Equation (3) and Equation (10), we can obtain

$$\begin{aligned}D(t)&=\int_0^t \frac{B_m(s)}{(t-s)^\alpha}ds\\ &=\int_0^{1/m}\frac{B_m(s)}{(t-s)^\alpha}ds+\int_{1/m}^{2/m}\frac{B_m(s)}{(t-s)^\alpha}ds+\cdots+\int_{i/m}^{t}\frac{B_m(s)}{(t-s)^\alpha}ds\\ &=\Big(-\frac{(t-\frac{1}{m})^{1-\alpha}-t^{1-\alpha}}{1-\alpha}-\frac{(t-\frac{2}{m})^{1-\alpha}-(t-\frac{1}{m})^{1-\alpha}}{1-\alpha}\\ &\quad\cdots\frac{(t-\frac{i}{m})^{1-\alpha}}{1-\alpha}\ \ 0\cdots\ 0\Big)^T\end{aligned} \tag{11}$$

where $D(0)=(0,0,\cdots,0)^T$.

Let $t=k/m,\quad k\in\{1,2,\cdots,m-1\}$, we have

$$\begin{aligned}D(\frac{k}{m})=&\Big(-\frac{(\frac{k}{m}-\frac{1}{m})^{1-\alpha}-(\frac{k}{m})^{1-\alpha}}{1-\alpha},\\ &-\frac{(\frac{k}{m}-\frac{2}{m})^{1-\alpha}-(\frac{k}{m}-\frac{1}{m})^{1-\alpha}}{1-\alpha},\cdots,\frac{(\frac{k}{m}-\frac{i}{m})^{1-\alpha}}{1-\alpha},0,\cdots,0\Big)^T.\end{aligned}$$

At this time, $i=k-1$.

Using Equation (9) and Equation (11), we can obtain the approximation of Equation (7).

III. Method of Solution

In this section, a collocation method based on Block Pulse functions is presented for solving the following linear weakly singular Volterra integral equation:

$$\sum_{i=0}^{n}a_i(t)y^{(i)}(t)+\lambda\int_0^t (t-s)^{-\alpha}y(s)ds=f(t). \tag{12}$$

with condition

$$\begin{aligned}&y^{(n-1)}(0)=y_{n-1},\ y^{(n-2)}(0)=y_{n-2},\ \cdots,y(0)=y_0,\\ &\quad 0\le t\le 1,\ 0<\alpha<1.\end{aligned} \tag{13}$$

where $a_i(t), f(t)$ are known continuous functions on $[0,1]$ and $y^{(i)}(t)\in L^2([0,1])$ stands for the i th-order derivative of $y(t)$. λ and y_k $(k=0,1,2,\cdots,n-1)$ are real constants.

Before solving Equation (12), the integration operational matrix of Block-Pulse functions can be got by using the operational matrix of Legendre wavelet.

Legendre wavelet in the interval $[0,1)$ can be defined as [16-17]:

$$\psi_{nm}^{(k)}(x)=\begin{cases}\sqrt{2m+1}2^{\frac{k}{2}}P_m(2^{k+1}x-2n+1), & x\in[\frac{n-1}{2^k},\frac{n}{2^k});\\ 0, & otherwise.\end{cases} \tag{14}$$

P_m is said Legendre polynomial.

Set P is the Legendre wavelet operational matrix of integration,

where $P=\frac{1}{2^k}\begin{bmatrix}L & F & F & \cdots & F\\ O & L & F & \cdots & F\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ O & O & O & L & F\\ O & O & O & O & L\end{bmatrix}_{m\times m}$,

$$F=\begin{bmatrix}2 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{bmatrix}_{M\times M}$$

and

$$L=\begin{bmatrix}1 & \frac{1}{\sqrt{3}} & 0 & 0 & \cdots & 0 & 0 & 0\\ -\frac{1}{\sqrt{3}} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} & 0 & \cdots & 0 & 0 & 0\\ 0 & -\frac{\sqrt{3}}{3\sqrt{5}} & 0 & \frac{\sqrt{5}}{5\sqrt{7}} & \cdots & 0 & 0 & 0\\ 0 & 0 & -\frac{\sqrt{7}}{5\sqrt{5}} & 0 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & -\frac{\sqrt{2M-3}}{(2M-3)\sqrt{2M-5}} & 0 & \frac{\sqrt{2M-3}}{(2M-3)\sqrt{2M-1}}\\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{\sqrt{2M-1}}{(2M-3)\sqrt{2M-5}} & 0\end{bmatrix}_{M\times M},$$

where $m=2^{k-1}M$.

Let $t_i=\frac{i-\frac{1}{2}}{2^{k-1}M}$, $i=1,2,\cdots,2^{k-1}M$, the Legendre wavelet matrix [15] can be acquired

$$\Phi_{m\times m}=[\Psi(t_1),\Psi(t_2),\cdots,\Psi(t_{2^{k-1}M})]. \tag{15}$$

where $\begin{aligned}\Psi(t)=[&\psi_{1,0}^{(k)}(t),\cdots,\psi_{1,M-1}^{(k)}(t),\psi_{2,0}^{(k)}(t),\cdots,\\ &\psi_{2,M-1}^{(k)}(t),\cdots,\psi_{2^{k-1},0}^{(k)}(t),\cdots,\psi_{2^{k-1},M-1}^{(k)}(t)]^T.\end{aligned}$

The relation between the Block-Pulse functions and Legendre wavelet is given by

$$\Psi(t)=\Phi_{m\times m}B_m(t). \tag{16}$$

Then we have

$$\int_0^t B_m(s)ds=\int_0^t\Phi_{m\times m}^{-1}\Psi(s)ds=\Phi_{m\times m}^{-1}P\Phi_{m\times m}B_m(t). \tag{17}$$

Let $Q=\Phi_{m\times m}^{-1}P\Phi_{m\times m}$, Q is called the Block-Pulse operational matrix of integration

$$\int_0^t B_m(s)ds=QB_m(t). \tag{18}$$

Suppose $y^{(n)}(t)\in L^2([0,1])$, then we get

$$y^{(n)}(t) \cong \sum_{i=0}^{m-1} d_i b_i(t) = d^T B_m(t). \tag{19}$$

$$y^{(n-1)}(t) = \int_0^t y^{(n)}(s)ds + y^{(n-1)}(0) = [d^T Q + A^T y^{(n-1)}(0)]B_m(t). \tag{20}$$

$$y^{(n-2)}(t) = [d^T Q^2 + A^T y^{(n-1)}(0)Q + A^T y^{(n-1)}(0)]B_m(t). \tag{21}$$

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$$\begin{aligned} y(t) = [&d^T Q^n + A^T y^{(n-1)}(0)Q^{n-1} \\ &+ \cdots + A^T y'(0)Q + A^T y(0)]B_m(t). \end{aligned} \tag{22}$$

$\forall i \in \{0,1,2\cdots,n\}$,

$$y^{(i)}(t) = [d^T Q^{n-i} + A^T y^{(n-1)}(0)Q^{n-i-1} + \cdots + A^T y^{(i)}(0)]B_m(t). \tag{23}$$

where $A = m\int_0^1 B_m(t)dt$.

Substituting the Equation (23), Equation (22) and Equation (9) into Equation (12), we can obtain.

$$\begin{aligned} &d^T(\sum_{i=0}^{n} a_i(t)Q^{n-i}B_m(t) + \lambda Q^n D(t)) \\ &= f(t) - \sum_{i=0}^{n} a_i(t)(A^T y^{(n-1)}(0)Q^{n-i-1} + \cdots + A^T y^{(i)}(0))B_m(t) \\ &\quad - \lambda(A^T y^{(n-1)}(0)Q^{n-1} + \cdots + A^T y(0))D(t). \end{aligned} \tag{24}$$

Discreting the Equation (24) by taking step $\Delta = \frac{1}{m}$ of t, a linear system of algebraic equations can be easily got. Then d^T can be got by solving Equation (24).

Consider the following nonlinear weakly singular Volterra integral equation:

$$\sum_{i=0}^{n} a_i(t)y^{(i)}(t) + \lambda\int_0^t (t-s)^{-\alpha}[y(s)]^p ds = f(t). \tag{25}$$

with the initial conditions Equation (13).

Let $\beta^T = d^T Q^n + A^T y^{(n-1)}(0)Q^{n-1} + \cdots + A^T y'(0)Q + A^T y(0)$, namely $\beta = (\beta_0, \beta_1, \cdots, \beta_{m-1})^T$, Equation (22) can be rewritten as

$$f(t) = \beta^T B_m(t). \tag{26}$$

Using the properties of the Block-Pulse functions, Equation (27) can be got.

$$[f(t)]^p = [\beta^p]^T B_m(t). \tag{27}$$

where $\beta^p = (\beta_0^p, \beta_1^p, \cdots, \beta_{m-1}^p)^T$.

Substituting the Equation (23), Equation (27) and Equation (9) into Equation (25), we have:

$$\begin{aligned} &\beta^T \sum_{i=0}^{n} a_i(t)Q^{-i}B_m(t) + \lambda[\beta^p]^T D(t) \\ &= f(t) + \sum_{i=0}^{n} a_i(t)(A^T y^{(i-1)}Q^{-1} + \cdots + A^T y(0)Q^{-i})B_m(t). \end{aligned} \tag{28}$$

when $p=1$, Equation (28) is Equation (24).

Discreting the Equation (28) by taking step $\Delta = \frac{1}{m}$ of t, a nonlinear system of algebraic equations can be easily got. Then β and $y(t)$ can be also obtained.

IV. Error Analysis

In this section, we analyze the error when a differentiable function $y(x)$ is represented in a series of block pulse functions over the interval $I = [0,1)$. We need the following theorem.

Theorem 4.1 Suppose $y(x)$ is continuous in I, is differentiable in $(0,1)$, and there is a number M such that $|y'(x)| \le M$, for every $x \in I$. Then

$$|y(b) - y(a)| \le M|b-a|,$$

for all $a,b \in I$.

Now, we assume that $y(x)$ is a differentiable function on I such that $|y'(x)| \le M$. We define the error between $y(x)$ and its block pulse functions expansion over every subinterval I_i as follows:

$$e_i(x) = c_i - y(x), \quad x \in I_i. \tag{29}$$

where $I_i = \left[\frac{i}{n}, \frac{i+1}{n}\right)$.

It can be shown that

$$\|e_i\|^2 = \int_{i/n}^{(i+1)/n} e_i^2(x)dx = \frac{1}{n}(c_i - y(\eta))^2, \quad \eta \in I_i. \tag{30}$$

where we used mean value theorem for integral. Using Equation (6) and the mean value theorem, we have

$$c_i = n\int_{i/n}^{(i+1)/n} y(x)dx = n\frac{1}{n}y(\varsigma) = y(\varsigma), \ \varsigma \in I_i. \tag{31}$$

Substituting Equation (31) into Equation (30) and using Theorem 4.1, we have

$$\|e_i\|^2 = \frac{1}{n}(y(\varsigma) - y(\eta))^2 \le \frac{M^2}{n}|\varsigma - \eta|^2 \le \frac{M^2}{n^3}. \tag{32}$$

This leads to

$$\begin{aligned} \|e(x)\|^2 &= \int_0^1 e^2(x)dx = \int_0^1 \left(\sum_{i=0}^{n-1} e_i(x)\right)^2 dx \\ &= \int_0^1 \left(\sum_{i=0}^{n-1} e_i^2(x)\right)dx + 2\sum_{i\le j}\int_0^1 e_i(x)e_j(x)dx. \end{aligned} \tag{33}$$

Since for $i \ne j$, $I_i \cap I_j = \varnothing$, then

$$\|e(x)\|^2 = \sum_{i=0}^{n-1}\left(\int_0^1 e_i^2(x)dx\right) = \sum_{i=0}^{n-1}\|e_i\|^2. \tag{34}$$

Substituting Equation (33) into Equation (34), we get

$$\|e(x)\|^2 \le \frac{M^2}{n^2}. \tag{35}$$

hence, $\|e(x)\| = O\left(\frac{1}{n}\right)$,

where $e(x) = y_n(x) - y(x)$ and $y_n(x) = \sum_{i=0}^{n-1} c_i b_i(x)$.

V. Numerical Examples

Example 1. Consider the weakly singular integral [18]

$$I_1(t) = \int_0^t \frac{s^n}{\sqrt{t-s}}ds. \tag{36}$$

The exact solution is $\dfrac{\sqrt{\pi}t^{(\frac{1}{2}+n)}\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}$. Taking $m = 64, m = 128,$ and making use of MATLAB2011a, **Fig. 1** and **Fig. 2** are comparison of the numerical solutions with the exact.

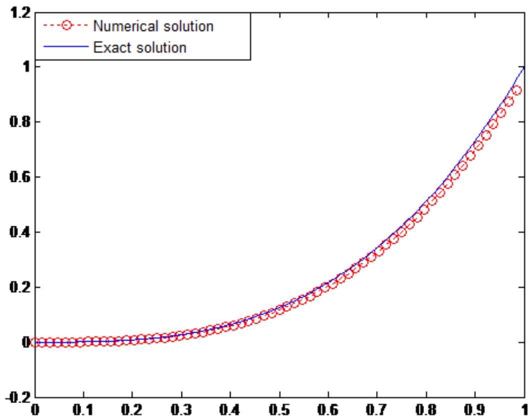

Fig. 1. $m = 64, n = 4$.

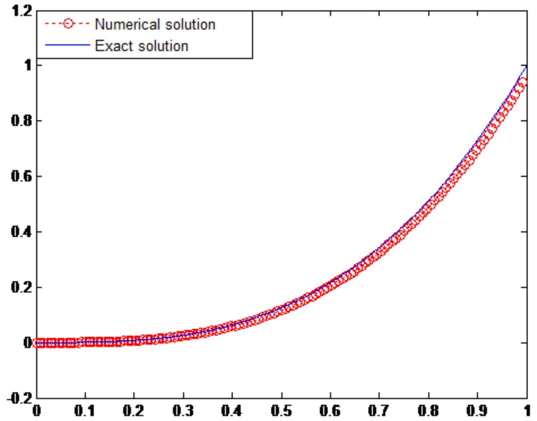

Fig. 2. $m = 128, n = 4$.

Example 2. Consider the weakly singular Volterra integral equation [20]:

$$y(t) + \int_0^t y(s)(t-s)^{-1/2} ds = \frac{1}{2}\pi t + \sqrt{t}, \quad 0 \le t < 1. \tag{37}$$

The exact solution is $\sqrt{t}$. The absolute errors for different m are shown in **Fig. 3**.

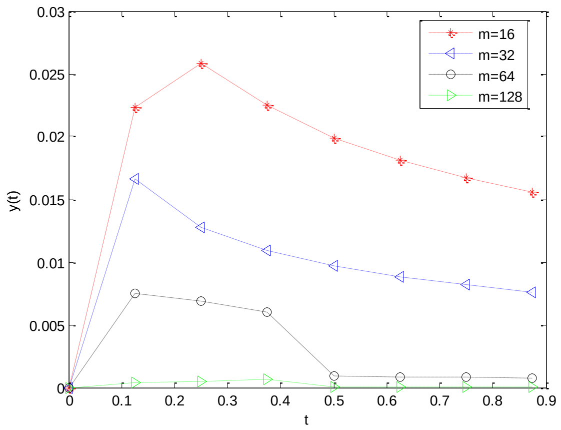

Fig. 3. The absolute errors for different m .

VI. Conclusion

In this work, the Block-Pulse functions and their good properties has been successfully applied to construct approximate solutions for higher order linear and nonlinear weakly singular Volterra integral equation of the second kind. The Block-Pulse functions method provides the solution in terms of convergent series with easily computable components. The approximation of the arbitrary order weakly singular integral and integration operational matrix are obtained. The initial equations can be transformed into a system of algebraic equations. The Block-Pulse functions method is effective and simple to solve the higher order linear and nonlinear weakly singular Volterra integral equation.

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