Optimal Control of a Hyperbolic Distributed Parameter System subject to Actuators

M. A. Bokhari, *Member, IAENG*, and Ismail Kucuk

***Abstract*— A problem of optimal control of vibrations in a hyperbolic distributed system is considered by applying pointwise actuators. Modal space technique simplifies this problem to an optimal control of a linear time-invariant lumped-parameter system. A direct computational method is then used to evaluate the modal optimal control and trajectory. The method is based, in general, on various orthogonal expansions that approximate the modal state variables. The formulation is straightforward and convenient for digital computation.**

***Index Terms*— Pointwise actuators, Distributed parameter systems, Lumped parameter system, Hyperbolic partial differential equation, Optimal control, Orthogonal polynomials**

I. Introduction

Optimal control of hyperbolic distributed parameter systems (DPS) has a wide range of applications in the modern engineering structures. Control of undesired vibrations in mechanical systems has been a challenging problem in many engineering applications. Active vibration suppression is one of the many approaches to control structural vibrations by incorporating actuators [1]-[2]. The number of actuators and their locations optimally are important factors in studying vibration control of mechanical systems [3].

Some orthogonal functions are convenient and sharp tools for obtaining approximate solutions of dynamic systems [4]-[11]. These techniques are linear, non-iterative, non-differential, non-integral, and appropriate for computation. Here, we solve the optimal problem by applying state and/or control parametrization by orthogonal expansion that results to finite-term series whose coefficients are determined optimally [12]. In fact, the problem reduces to an algebraic system of equations, thus avoiding the difficult integral equations created from variational methods and the maximum principle [13]. It is an approach different from the standard variational method [14], which offers an attractive computational scheme.

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M. A. Bokhari is with the Department of Mathematics & Statistics, King Fahd University of Petroleum & Minerals, Dhahran, Saudi Arabia (phone: +966-50-898-3765; fax: +966-13-860-2340; e-mail: mbokhari@kfupm.edu.sa).

Ismail Kucuk is with Department of Mathematical Engineering, Yildiz Technical University, Istanbul, Turkey (e-mail: ikucuk@fbe.yildiz.edu.tr).

This paper studies the active control of vibrations for hyperbolic distributed systems using pointwise actuators that suppress unwanted vibrations in the system. The equation of motion is given by a partial differential equation that includes Dirac functions due to the pointwise actuators. The objective function is specified as a weighted quadratic functional of the dynamic responses of the hyperbolic system, which is to be minimized at a specified terminal time using continuous piezoelectric patches (voltages). The expenditure of the control force is included in the objective function as a penalty term. In this paper, we employ the suggested technique to a wave equation subject to the initial and boundary conditions and then minimize the cost functional in a given time period. A detailed formulation of the method is given in sections 3, 4 and 5.

II. Control Problem

We consider a control system based on the wave equation

$$u_{tt}(x,t) - c^2 u_{xx}(x,t) = \sum_{n=1}^{L} h_n(t)\delta(x - x_n),$$
$$0 \le x, \le l;\ 0 < x_n < l;\ 0 \le t \le t_f,$$

where δ is the Dirac delta function; t_f is the terminal time; $u(x,t) \in L^2\left([0,l]\times[0,t_f]\right)$ denotes the deflection of the mechanical system and for each *n*, x_n is a location of actuator with $h_n(t)$, the corresponding applied control voltage, to be determined. After changing the right side of the above equation to vector form, we have

$$u_{tt}(x,t) - c^2 u_{xx}(x,t) = D^T(x)H(t), \tag{1}$$

where $D(t) = \left[\delta(x - x_n)\right]_{n=1}^{L}$ and $H(t) = \left[h_n(t)\right]_{n=1}^{L}$. We consider (1) subject to the homogeneous boundary conditions

$$u(0,t) = u(l,t) = 0, \tag{2}$$

along with the initial and terminal conditions

$$u(x,0) = g(x);\ u_t(x,0) = 0\,. \tag{3}$$

The problem is to determine the solution from the admissible set

$$U_{ad} = \left\{ H = [h_n]_{n=1}^{L} : h_n \in L^2[0,t_f] \text{ and the pair } \{H,u\} \text{ satisfies (1)-(3)} \right\} \tag{4}$$

which solves the minimization problem

$$\min_{H \in U_{ad}} J(H) \tag{5}$$

where

$$J(H) = \mu_1 \left\| u(.,t_f) \right\|_{L^2[0,l]}^2 + \mu_2 \left\| u_t(.,t_f) \right\|_{L^2[0,l]}^2 + \mu_3 \left\| H^T H \right\|_{L^2[0,t_f]}^2 \tag{6}$$

Here, $\mu_1,\ \mu_2,\ \mu_3$ are nonnegative weighting constants satisfying the conditions $\mu_1 + \mu_2 > 0,\ \mu_3 > 0.$

III. MODAL SPACE OPTIMIZATION PROBLEM

We transform the DPS optimization problem (5) into a modal lumped parameter (MLP) problem by means of the eigenfunction $\phi_j(x) = \sin \lambda_j x,\quad j = 1,2,3,\ldots$ where $\lambda_j = j\pi / l$ are the corresponding eigenvalues. To do this, we substitute

$$u(x,t) = \sum_{n=1}^{N} z_n(t) \phi_n(x) \tag{7}$$

in (1), and then use the orthogonal properties of $\phi_j(x)$ over interval [0, l]. This results to

$$\ddot{z}_m(t) - c^2 \lambda_m^2 z_m(t) = \boldsymbol{\phi}_{m,L}^T H(t), \quad m = 1,2,3,\ldots,N \tag{8}$$

where $\boldsymbol{\phi}_{m,L} = \left[2\phi_m / l \right]_{m=1}^{L}$, $H(t) = \left[h_m(t) \right]_{m=1}^{L}$ and the two over-dots represent the 2[nd] derivative with respect to time. In a similar manner, the initial and terminal conditions (3) are replaced by

$$\left. \begin{aligned} z_m(0) &= \left\langle g, \phi_m \right\rangle_{L^2[0,l]} \\ \dot{z}_m(0) &= 0 \end{aligned} \right\}, \ m = 1,2,3,\ldots \tag{9}$$

We set $Z(t) = \left[z_m(t) \right]_{m=1}^{M}, Z_0 = [z_m(0)]_{m=1}^{M}, \ddot{Z}(t) = \left[\ddot{z}_n(t) \right]_{m=1}^{M}$ and $\Lambda = \text{diag}\left[\lambda_m^2 \right]_{m=1}^{M}$ in order to have (8)-(9) in the vector form:

$$\left. \begin{aligned} &\ddot{Z}(t) - c^2 \Lambda Z(t) = \Phi_{M,L} H(t) \\ &Z_0 = \left[z_m(0) \right]_{m=1}^{M} \\ &\dot{Z}_0 = \mathbf{0} \end{aligned} \right\} \tag{10}$$

Here, $\Phi_{M,L} = \left[\boldsymbol{\phi}_{m,L}^T(t) \right]_{m=1}^{M}$ is an $M \times L$ matrix. The optimal problem (4)-(6) thus reduces to an equivalent MLP problem:

$$\left. \begin{aligned} &\min_{h \in U_{ad,M}} J_M(H) \\ &J_M(H) = \mu_1 Z^T(t_f) Z(t_f) + \mu_2 \dot{Z}^T(t_f) \dot{Z}(t_f) + \mu_3 \left\| H^T H \right\|_{L^2[0,t_f]}^2 \\ &U_{ad,M} = \left\{ H = [h_n]_{n=1}^{L} : h_n \in L^2[0,t_f], \text{ and the pair } \{H,Z\} \text{ satisfies (10)} \right\} \end{aligned} \right\} \tag{11}$$

IV. STATE PARAMETRIZATION

This section deals with developing a direct method [12] for solving the model control problem (11) by means of orthogonal functions $p_n(t)$ over time interval [0, t_f]. We parametrize $z_m(t)$ by setting

$$z_m(t) = z_m(0) + \sum_{k=1}^{K} a_{mk} p_k(t), \ m = 1,2,\ldots,M \tag{12}$$

where $a_{mk}, k = 1,2,3,\ldots,K$, are unknown coefficients. Next, we set $\boldsymbol{P}(t) = \text{diag}\left[\boldsymbol{p}^T(t) \right]_{MK \times MK}$ and $\boldsymbol{A} = \left[\boldsymbol{a}_m \right]_{m=1}^{M}$ where $\boldsymbol{p}(t) = \left[p_k(t) \right]_{k=1}^{K}$ and $\boldsymbol{a}_m = \left[a_{mk} \right]_{k=1}^{K}$. This allows us to express the vectors $Z(t)$ and $\ddot{Z}$ as

$$\left. \begin{aligned} Z(t) &= Z_0 + \boldsymbol{P}(t)\boldsymbol{A} \\ \ddot{Z}(t) &= \ddot{\boldsymbol{P}}(t)\boldsymbol{A} \end{aligned} \right\} \tag{13}$$

The minimization problem from (11) thus takes the form

$$\left. \begin{aligned} &\min_{h \in U_{ad,M,K}} J_{M,K}(H) \\ &J_{M,K}(H) = \mu_1 \left(Z_0 + \boldsymbol{P}(t_f)\boldsymbol{A} \right)^T \left(Z_0 + \boldsymbol{P}(t_f)\boldsymbol{A} \right) + \mu_2 \left(\dot{\boldsymbol{P}}(t_f)\boldsymbol{A} \right)^T \left(\dot{\boldsymbol{P}}(t_f)\boldsymbol{A} \right) + \mu_3 \left\| H^T H \right\|_{L^2[0,t_f]} \\ &U_{ad,M,K} = \left\{ H = [h_n]_{n=1}^{L} : h_n \in L^2[0,t_f] \text{ and the pair } \{H,\boldsymbol{A}\} \text{ satisfies (12)} \right\} \end{aligned} \right\} \tag{14}$$

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V. SOLUTION OF MAIN PROBLEM

The second order differential equation in (10) after using (13) is written as

$$\Phi_{M,L} H(t) = \left(\ddot{\boldsymbol{P}}(t) - c^2 \Lambda \boldsymbol{P}(t)\right)\boldsymbol{A} - c^2 \Lambda Z_0 \tag{15}$$

Thus, the control vector $H(t)$ in (14), after slight algebraic manipulations, turns out as an explicit function of $\boldsymbol{A}$, i. e.,

$$H(t) = \Psi_{M,M} \Phi_{M,L}^T \left[\left(\ddot{\boldsymbol{P}}(t) - c^2 \Lambda \boldsymbol{P}(t)\right)\boldsymbol{A} - c^2 \Lambda Z_0\right]. \tag{16}$$

Here, $\Psi_{M,M} = \left(\Phi_{M,L}^T \Phi_{M,L}\right)^{-1}$ is an $M \times M$ matrix. With this, we arrive at a simple alternate form of problem (14) stated below

$$\left.\begin{aligned}
&\min_A J_{M,K}(\boldsymbol{A}) \\
&J_{M,K}(\boldsymbol{A}) = \mu_1 \left(Z_0 + \boldsymbol{P}(t_f)\boldsymbol{A}\right)^T \left(Z_0 + \boldsymbol{P}(t_f)\boldsymbol{A}\right) \\
&\qquad + \mu_2 \left(\dot{\boldsymbol{P}}(t_f)\boldsymbol{A}\right)^T \left(\dot{\boldsymbol{P}}(t_f)\boldsymbol{A}\right) + \mu_3 \int_0^{t_f} \Theta(A)\, dt. \\
&\Theta(A) = \left[\Psi_{M,M}\Phi_{M,L}^T \left\{\left(\ddot{\boldsymbol{P}}(t) - c^2 \Lambda \boldsymbol{P}(t)\right)\boldsymbol{A} + c^2 \Lambda Z_0\right\}\right]^T \\
&\qquad \times \left[\Psi_{M,M}\Phi_{M,L}^T \left\{\left(\ddot{\boldsymbol{P}}(t) - c^2 \Lambda \boldsymbol{P}(t)\right)\boldsymbol{A} + c^2 \Lambda Z_0\right\}\right].
\end{aligned}\right\} \tag{17}$$

Now the cost functional $J_{M,K}(\boldsymbol{A})$ in (17) can be differentiated easily with respect to $\boldsymbol{A}$. The necessary condition for optimality '$d(J_{M,K}(\boldsymbol{A}))/dt = 0$' thus leads us to an MK linear algebraic equations that determine the unknown vector $\boldsymbol{A}$. The optimal value of $J_{M,K}(\boldsymbol{A})$ brings to an approximate solution of the desired problem (4)-(7).

VI. A NOTE ON SIMULATIONS

Different kinds of orthogonal functions may be applied to parameterize the state variable in section 4. These include shifted Legendre and Radau polynomials. In addition, we may consider specific orthogonal interpolants $p_k(t)$ [15] that have additional properties like $p_k(0) = 0$ and/or $\dot{p}_k(0) = 0,\ \ k = 1,2,3,\ldots,K$. These functions prove useful in numerical computation of the optimal value.

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