Lyapunov Function and Global Stability for the Model of Influenza Dynamics

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***Abstract*—In this paper, we study the dynamics of Influenza transmission in population by using mathematical modeling in infectious. We present the construction of Lyapunov functions for the model. The analysis of the model for the transmission of Influenza is done in this study. From the analysis the model, we use the standard methods for analyzing the system of Ordinary Differential Equations. The stability of the model is determined by using Lyaponov function to show the global asymptotically stables in this study.**

***Index Terms*—Influenza, Lyaponov Function, Mathematical modeling**

I. Introduction

INFLUENZA is caused by a virus that attacks mainly the upper respiratory tract, nose, throat and bronchi and rarely also the lungs. Spreading from person to person through sneezing, coughing, or touching contaminated surfaces. There are three types of influenza viruses, labeled A, B, and C. Type A influenza viruses are further divided into subtypes according to the specific variety and combinations of two proteins that occur on the surface of the virus, the hemagglutinin or "H" protein and the neuraminidase or "N" protein. Currently, influenza A (H1N1) and A (H3N2) are the circulating seasonal influenza A virus subtypes. This A (H1N1) virus is the same virus that caused the 2009 influenza pandemic, as it is now circulating seasonally. In addition, there are two type B viruses that are also circulating as seasonal influenza viruses, which are named after the areas where they were first identified, Victoria lineage and Yamagata lineage. Type C influenza causes milder infections and is associated with sporadic cases and minor localized outbreaks. As influenza C poses much less of a disease burden than influenza A and B, only the latter two are included in seasonal influenza vaccines [1].

Mathematical modeling for infectious disease, the researchers in infectious disease are one of the foremost to realize the important role of mathematics and mathematical models in providing an explicit framework for understanding the disease transmission dynamics within and between hosts and parasites. Mathematical models have been widely used by epidemiologists as tools to predict the occurrence of epidemics of infectious diseases, and also as a tool for guiding research for eradication of the disease at the present time [2].

The method of Lyapunov functions is used to establish global stability results for biological models by general theory of Lyapunov functions and for applications in mathematical biology. The systematic use of Lyapunov function in studying stability problems is relatively recent. However, LaSalle-Lyapunov theory has been used in [3]-[4] to study the stability of classic Susceptible-Infectious-Removed-Susceptible (SIRS) models. Lyapunov function method has been a popular technique to study global stability of epidemiological models. A Volterra-type Lyapunov function has been used in [5]-[6] to prove global stability of the steady states of classic Susceptible-Infectious-Susceptible (SIS), Susceptible-Infectious-Removed (SIR) and Susceptible-Infectious-Removed-Susceptible (SIRS) epidemiological models with bilinear incidence rate and in 2004, Korobeinikov [7] construct a Lyapunov function to demonstrate the simplification of the result for the endemic equilibrium state for SIR and Susceptible-Exposed-Infectious-Recovered (SEIR) epidemiological models.

In this paper, the global dynamics of the fourth-dimensional model of Influenza transmission model which construct a system of nonlinear differential equations from our study [8] is resolved through the use of Lyapunov functions. We prove the global asymptotic stability of the equilibrium states using Lyapunov functions. Our discussion and conclusion are contained in the last section

II. Formulation of the Model

A. Concept of the Model

In the epidemiological model, the total population is divided into 4 classes, the following sub-classes that are the susceptible individuals S^h, infectious individuals I^h, the recovered individuals who are totally immune to that strain R^h and the recovered individuals who are partially immune to that strain classes C^h.

B. Parameter of the model

Let S^h, I^h, R^h and C^h represent the fraction of susceptible , infectious, recovered human who are totally immune to that strain and the recovered human who are partially immune to that strain groups. Note that N^h is the total of fraction for human population where $S^h + I^h + R^h + C^h = 1$. The birth rate of human, the

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natural death rate of human, the transmission rate which the susceptible human become to infectious human, the transmission rate which the infectious human become to the recovered human who are totally immune to that strain and the transmission rate which the recovered human who are totally immune to that strain become to the recovered human who are partially immune to that strain classes represented by B_n, μ, β_1, α and δ, respectively.

C. Equations of the Model

For the SIRC model in this study, the flow is from the S^h group to the I^h group, and then either directly to the R^h group. After that, the flow is from R^h group to the C^h group. The transfer diagram leads to the following system of ordinary differential equations:

$$\begin{aligned}\frac{dS^h}{dt} &= B_n - \beta_1 I^h S^h - \mu S^h,\\ \frac{dI^h}{dt} &= \beta_1 S^h I^h - (\mu+\alpha) I^h,\\ \frac{dR^h}{dt} &= \alpha I^h - (\mu+\delta) R^h,\\ \frac{dC^h}{dt} &= \delta R^h - \mu C^h.\end{aligned} \tag{1}$$

III. ANALYSIS

A. Analysis the Model

The differential equation of the total population of (1) is

$$\frac{d}{dt}(S^h + I^h + R^h + C^h) = B_n - \mu(S^h + I^h + R^h + C^h). \tag{2}$$

Thus the total population size may vary in time. In the absence of disease, the population size converges to the steady stable $\frac{B_n}{\mu}$. Then, the studying (1) in the following feasible region:

$$\Omega = \left\{(S^h, I^h, R^h, C^h) \in \mathbb{R}^4_+ ;\ S^h, I^h, R^h, C^h \ge 0,\ S^h + I^h + R^h + C^h \le \frac{B_n}{\mu}\right\} \tag{3}$$

which can be shown to be positively invariant with respect to (1). Direct calculations shows that system (1) has two possible steady states

$$E^0 = (S^0, 0, 0, 0,) \tag{4}$$

where $S^0 = \frac{B_n}{\mu}$ and a unique endemic steady state

$$E^* = (S^*, I^*, R^*, C^*,) \tag{5}$$

with

$$S^* = \frac{B_n}{\mu R_0}, \tag{6}$$

$$I^* = \frac{\mu}{\beta_1}(R_0 - 1), \tag{7}$$

$$R^* = \frac{\alpha\mu}{\beta_1(\mu+\delta)}(R_0 - 1), \tag{8}$$

$$C^* = \frac{\delta\alpha}{\beta_1(\mu+\delta)}(R_0 - 1) \tag{9}$$

which $R_0 = \frac{\beta_1 B_n}{\mu(\mu+\alpha)}$, R_0 is the basic reproductive number, is

$$R_0 = \frac{\beta_1 B_n}{\mu(\mu+\alpha)} = \frac{\beta_1 S^0}{\mu(\mu+\alpha)}. \tag{10}$$

B. Global Stability of the Disease Free Steady State

The global stability of the disease free steady state E^0 is proved by using common quadratic and linear Lyapunov functions and LaSalle's invariance principle.

Theorem 1. If $R_0 \le 1$ then the disease free steady state E^0 of (1) is globally asymptotically stable in Ω.

Proof

$$V : \{(S^h, I^h, R^h, C^h) \in \mathbb{R}^4_+ : S^h > 0\} \to \mathbb{R}$$

by

$$V(S^h, I^h, R^h, C^h) = \frac{(\mu+\delta)}{2S^0}(S^h - S^0)^2 + (\mu+\delta)I^h + (\mu+\delta)R^h + \delta C^h. \tag{11}$$

The derivative of (11) with respect to t along solution curves of (1) is given by

$$\begin{aligned}&V'(S^h, I^h, R^h, C^h)\\ &= \frac{(\mu+\delta)}{S^0}(S^h - S^0)\frac{dS^h}{dt} + (\mu+\delta)\frac{dI^h}{dt} + (\mu+\delta)\frac{dR^h}{dt} + \delta\frac{dC^h}{dt}\\ &= \frac{(\mu+\delta)}{S^0}(S^h - S^0)(B_n - \beta_1 I^h S^h - \mu S^h)\\ &\quad +(\mu+\delta)(\beta_1 S^h I^h - (\mu+\alpha) I^h)\\ &\quad +(\mu+\delta)(\alpha I^h - (\mu+\delta) R^h) + \delta(\delta R^h - \mu C^h)\\ &= -\frac{\mu(\mu+\delta)}{S^0}(S^h - S^0)^2 - \frac{\mu(\mu+\delta)}{S^0}(S^h - S^0)\beta_1 I^h S^h\\ &\quad +(\mu+\delta)(\beta_1 S^h I^h - (\mu+\delta+\alpha) I^h - (\mu+\delta) R^h)\\ &\quad +\delta(\delta R^h - \mu C^h).\end{aligned} \tag{12}$$

Using the expression

$$\frac{\beta_1 I^h S^h (S^h - S^0)}{S^0} = \frac{\beta_1 I^h (S^h - S^0)^2}{S^0} + \beta_1 I^h (S^h - S^0).$$

We obtain,

$$\begin{aligned}&V'(S^h, I^h, R^h, C^h)\\ &= -\frac{\mu(\mu+\delta)}{S^0}(S^h - S^0)^2 - \frac{(\mu+\delta)\beta_1 I^h}{S^0}(S^h - S^0)^2\\ &\quad -(\mu+\delta)\beta_1 I^h (S^h - S^0) + (\mu+\delta)(\beta_1 S^h I^h\\ &\quad -(\mu+\delta+\alpha) I^h - (\mu+\delta) R^h) + \delta(\delta R^h - \mu C^h)\\ &= -\frac{(\mu+\delta)(\mu+\beta_1 I^h)}{S^0}(S^h - S^0)^2 + (\mu+\delta)\beta_1 I^h S^0\end{aligned}$$

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$$-(\mu+\delta)(\mu+\delta+\alpha)I^h-(\mu+\delta)(\mu+\delta)R^h+\delta^2R^h-\delta\mu C^h$$
$$=-\frac{(\mu+\delta)(\mu+\beta_1 I^h)}{S^0}(S^h-S^0)^2+((\mu+\delta)\beta_1 I^hS^0$$
$$-(\mu+\delta)(\mu+\alpha)I^h)-\delta(\mu+\delta)I^h-\mu\delta R^h$$
$$=-\frac{(\mu+\delta)(\mu+\beta_1 I^h)}{S^0}(S^h-S^0)^2-\delta(\mu+\delta)I^h-2\mu\delta R^h$$
$$-\delta^2R^h+((\mu+\delta)(\mu+\alpha)I^h)(\frac{(\mu+\delta)\beta_1S^0}{(\mu+\delta)(\mu+\alpha)}-1)$$
$$=-\frac{(\mu+\delta)(\mu+\beta_1 I^h)(S^h-S^0)^2}{S^0}-\delta(\mu+\delta)I^h-2\mu\delta R^h$$
$$-\delta^2R^h+(\mu+\delta)(\mu+\alpha)I^h(\frac{\beta_1S^0}{(\mu+\alpha)}-1)$$
$$=-\frac{(\mu+\delta)(\mu+\beta_1 I^h)(S^h-S^0)^2}{S^0}-\delta(\mu+\delta)I^h-2\mu\delta R^h$$
$$-\delta^2R^h+(\mu+\delta)(\mu+\alpha)I^h(1-R_0)\,. \quad (13)$$

Therefore, $R_0\le 1$ ensure that $V'(S^h,I^h,R^h,C^h)\le 0$ for all the $S^h,I^h,R^h,C^h>0$ and that $V'(S^h,I^h,R^h,C^h)=0$ holds when $S^h=S^0$ and $I^h=R^h=C^h=0$. The steady E^0 is globally asymptotically stable.

C. Global Stability of the Endemic Steady State

The globally asymptotic stability of the endemic steady state is proved by constructing a globally Lyapunov function. We obtain the Lyapunov function of a suitable combination of common quadratic and Volterra type functions.

Theorem 2 If $R_0>1$, then the unique endemic steady state E^* of (1) is globally asymptotically stable in the interior of Ω.

Proof Define

$$L:\{(S^h,I^h,R^h,C^h)\in\Omega: S^h,I^h,R^h,C^h>0\}\to \mathbb{R}$$

by

$$L(S^h,I^h,R^h,C^h)$$
$$=\frac{(S^h-S^*)^2}{2S^*}+(I^h-I^*-I^*\ln\frac{I^h}{I^*})+\frac{\mu R^*}{\alpha I^*}(R^h-R^*-R^*\ln\frac{R^h}{R^*})$$
$$+\frac{\mu C^*}{\delta R^*}(C^h-C^*-C^*\ln\frac{C^h}{C^*})\,.$$

This function is defined, continuous and positive definite for all $S^h,I^h,R^h,C^h>0$. It can be verified that the function $L(S^h,I^h,R^h,C^h)$ takes the value $L(S^h,I^h,R^h,C^h)=0$ at the steady state E^*, and thus, the global minimum of $L(S^h,I^h,R^h,C^h)$ occurs at the endemic steady state E^*. Since (S^*,I^*,R^*,C^*) is an endemic steady state point of (1) we have

$$B_n=\beta_1 I^*S^*+\mu S^*, \quad (14)$$
$$(\mu+\alpha)=\beta_1S^*, \quad (15)$$
$$(\mu+\delta)=\frac{\alpha I^*}{R^*}, \quad (16)$$
$$\mu=\frac{\delta R^*}{C^*}\,. \quad (17)$$

Computing the derivative of $L(S^h,I^h,R^h,C^h)$ along the solutions of system (1), we obtain

$$L'(S^h,I^h,R^h,C^h)$$
$$=\frac{2(S^h-S^*)}{2S^*}\frac{dS^h}{dt}+\frac{(I^h-I^*)}{I^h}\frac{dI^h}{dt}+\frac{\mu R^*}{\alpha I^*}\frac{(R^h-R^*)}{R^h}\frac{dR^h}{dt}$$
$$+\frac{\mu C^*}{\delta R^*}(1-\frac{C^*}{C^h})\frac{dC^h}{dt}$$
$$=\frac{(S^h-S^*)}{S^*}(B_n-\beta_1 I^hS^h-\mu S^h)+\frac{(I^h-I^*)}{I^h}(\beta_1S^hI^h$$
$$-(\mu+\alpha)I^h)+\frac{\mu R^*}{\alpha I^*}\frac{(R^h-R^*)}{R^h}(\alpha I^h-(\mu+\delta)R^h)$$
$$+\frac{\mu C^*}{\delta R^*}(\frac{C^h-C^*}{C^h})(\delta R^h-\mu C^h)$$
$$=\frac{(S^h-S^*)}{S^*}(\beta_1 I^*S^*+\mu S^*-\beta_1 I^hS^h-\mu S^h)$$
$$+\frac{(I^h-I^*)}{I^h}(\beta_1S^hI^h-\beta_1S^*I^h)$$
$$+\frac{\mu R^*}{\alpha I^*}\frac{(R^h-R^*)}{R^h}(\alpha I^h-\frac{\alpha I^*}{R^*}R^h)$$
$$+\frac{\mu C^*}{\delta R^*}(\frac{C^h-C^*}{C^h})(\delta R^h-\frac{\delta R^*}{C^*}C^h)$$
$$=-\frac{(S^h-S^*)}{S^*}(\mu(S^h-S^*)+\beta_1(I^hS^h-I^*S^*))$$
$$+\beta_1(I^h-I^*)(S^h-S^*)+\frac{\mu R^*}{\alpha I^*}(1-\frac{R^*}{R^h})(\alpha I^h-\frac{\alpha I^*}{R^*}R^h)$$
$$+\frac{\mu C^*}{\delta R^*}(1-\frac{C^*}{C^h})(\delta R^h-\frac{\delta R^*}{C^*}C^h)\,. \quad (18)$$

Notice that

$$S^hI^h-S^*I^*=S^*(I^h-I^*)+I^h(S^h-S^*)\,. \quad (19)$$

Then, we have

$$L'(S^h,I^h,R^h,C^h)$$
$$=-\frac{(S^h-S^*)}{S^*}(\mu(S^h-S^*)+\beta_1S^*(I^h-I^*)+\beta_1I^h(S^h-S^*))$$
$$+\beta_1(I^h-I^*)(S^h-S^*)+\frac{\mu R^*}{\alpha I^*}(1-\frac{R^*}{R^h})(\alpha I^h-\frac{\alpha I^*}{R^*}R^h)$$
$$+\frac{\mu C^*}{\delta R^*}(1-\frac{C^*}{C^h})(\delta R^h-\frac{\delta R^*}{C^*}C^h)$$

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$$= -\frac{(S^h - S^*)^2}{S^*}(\mu + \beta_1 I^h) + \mu R^*(\frac{I^h}{I^*} - \frac{R^h}{R^*} - \frac{R^* I^h}{I^* R^h} + 1)$$
$$+\mu C^*(\frac{R^h}{R^*} - \frac{C^h}{C^*} - \frac{R^h C^*}{C^h R^*} + 1)$$
$$= -\frac{(S^h - S^*)^2}{S^*}(\mu + \beta_1 I^h) - \mu R^*(-\frac{I^h}{I^*} + \frac{R^h}{R^*} + \frac{R^* I^h}{I^* R^h} - 1)$$
$$-\mu C^*(-\frac{R^h}{R^*} + \frac{C^h}{C^*} + \frac{R^h C^*}{C^h R^*} - 1)$$
$$= -\frac{(S^h - S^*)^2}{S^*}(\mu + \beta_1 I^h) - \mu R^*(\frac{R^h}{R^*} + \frac{R^* I^h}{I^* R^h} - \frac{I^h}{I^*} - 1)$$
$$-\mu C^*(\frac{C^h}{C^*} + \frac{R^h C^*}{C^h R^*} - \frac{R^h}{R^*} - 1)$$
$$= -\frac{(S^h - S^*)^2}{S^*}(\mu + \beta_1 I^h)$$
$$-\mu(\frac{\alpha\mu}{\beta_1(\mu+\delta)}(R_0 - 1))(\frac{R^h}{R^*} + \frac{R^* I^h}{I^* R^h} - \frac{I^h}{I^*} - 1)$$
$$-\mu(\frac{\delta\alpha}{\beta_1(\mu+\delta)}(R_0 - 1))(\frac{C^h}{C^*} + \frac{R^h C^*}{C^h R^*} - \frac{R^h}{R^*} - 1)$$
$$= -\frac{(S^h - S^*)^2}{S^*}(\mu + \beta_1 I^h)$$
$$-\mu(\frac{\alpha\mu}{\beta_1(\mu+\delta)}(R_0 - 1))(\frac{R^h}{R^*} + \frac{R^* I^h}{I^* R^h})$$
$$-\mu(\frac{\delta\alpha}{\beta_1(\mu+\delta)}(R_0 - 1))(\frac{C^h}{C^*} + \frac{R^h C^*}{C^h R^*})$$
$$+\mu\frac{\alpha\mu}{\beta_1(\mu+\delta)}(\frac{\beta_1 B_n}{\mu(\mu+\alpha)} - 1)(\frac{I^h}{I^*} + 1)$$
$$+\mu\frac{\delta\alpha}{\beta_1(\mu+\delta)}(\frac{\beta_1 B_n}{\mu(\mu+\alpha)} - 1)(\frac{R^h}{R^*} + 1)$$
$$= -\frac{(S^h - S^*)^2}{S^*}(\mu + \beta_1 I^h)$$
$$-\mu(\frac{\alpha\mu}{\beta_1(\mu+\delta)}(R_0 - 1))(\frac{R^h}{R^*} + \frac{R^* I^h}{I^* R^h})$$
$$-\mu(\frac{\delta\alpha}{\beta_1(\mu+\delta)}(R_0 - 1))(\frac{C^h}{C^*} + \frac{R^h C^*}{C^h R^*})$$
$$+\frac{\alpha\mu}{\beta_1(\mu+\delta)(\mu+\alpha)} I^*(\beta_1 B_n - \mu(\mu+\alpha))(\frac{I^h + I^*}{I^{*2}})$$
$$+\frac{\delta\alpha}{\beta_1(\mu+\delta)(\mu+\alpha)} R^*(\beta_1 B_n - \mu(\mu+\alpha))(\frac{R^h + R^*}{R^{*2}}) \quad (20)$$

The fourth and fifth terms in (20) are always negative when $R_0 > 1$. Therefore, $L'(S^h, I^h, R^h, C^h) \le 0$,the endemic steady stable E^* is globally asymptotically stable in the interior of Ω .

IV. CONCLUSION

In this study, we prove the global stability of the mathematical model for Influenza Dynamics in the SIRC model. The quantity is called the basic reproductive number of the disease. It is the number of secondary cases generated from a single infective case introduced into a susceptible population. We prove that if the basic reproductive number R_0 is less than one or equal to unity, then the disease free steady state is globally asymptotically stable in Theorem 1. If R_0 is greater than unity, unique endemic steady state exists and is globally asymptotically stable in Theorem 2. We construct Lyapunov functions for each steady state. The constructions are obtained by suitable combinations of well known Lyapunov functions.

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