A Corrector Result for the Homogenization of a Class of Nonlinear PDE in Perforated Domains

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Abstract—**This paper is devoted to the corrector of the homogenization of a quasilinear elliptic problem with oscillating coefficients in a periodically perforated domain. A nonlinear Robin condition is prescribed on the boundary of the holes, which depends on a real parameter $\gamma \geq 1$. We suppose that the data satisfy some suitable assumptions to ensure the existence and uniqueness of a weak solution of the problem. The periodic unfolding method is used to prove the result.**

Index Terms—**corrections, homogenization, nonlinear, unfolding method, quasilinear.**

I. INTRODUCTION

IN this paper, we study the corrector of the homogenization of a quasilinear elliptic problem with oscillating coefficients posed in a periodically perforated domain. We assume that the holes are of the same size as the period and on the boundary of the holes, we prescribe a nonlinear Robin condition, which depend on a real parameter $\gamma \geq 1$. Some suitable growth conditions are also assumed on the nonlinear boundary term while a weaker than a Lipschitz condition is prescribed on the quasilinear term. The assumptions used here are the same as those considered in [9] and [16] (see also [4]) for the existence and uniqueness of the weak solution of the problem to hold.

The physical motivation is that, in several composites the thermal conductivity depends in a nonlinear way from the temperature itself like the case of a glass or wood, where the conductivity is nonlinearly increasing with the temperature, as well as ceramics, where it is decreasing, or aluminium and semi-conductors, where the dependence is not even monotone (see [1], [2] and [23] for details). On the other hand, nonlinear Robin conditions appear in several physical situations like in some chemical reactions (see for instance [17]) or climatization (see [24]).

To prove the main result in this work, we apply the Periodic Unfolding Method **(PUM)**, a method of homogenization recently formed for the periodic case. It was first introduced in [11] for fixed domains (see also [12] for a general setting and detailed proofs and [20] for more simple approach), extended to perforated domains in [14] (see also [15] for complete proofs and [16] for more applications), to more general situations and comprehensive presentation in [10] and to time-dependent functions in [22]. Some nice properties of this method are: it only deals with the classical notion of convergences in L^p-spaces; any function in the perforated domain is mapped to the unfolded function in a fixed domain; and one do not need any extensions operators anymore when dealing with nonhomogeneous boundary conditions, like the case in this work.

Manuscript received: March 12, 2016.
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The correctors for the homogenizations of a class of linear elliptic problem in a periodically perforated domain when the oscillating matrix field depends on a weakly converging sequence, a linear elliptic problem with Dirichlet condition in a fixed domain and a linear elliptic problem with Robin boundary conditions in a perforated domain have been done in [8], [12] and [15], respectively, via **PUM**. Some corrector results were also obtained by applying mainly some lemmas, for composites with imperfect interface in [21] (see also the references therein). One can also refer to [18] for a corrector result for H-converging parabolic problems with time-dependent coefficients via Tartar's oscillating test functions. The reader can also see [6] and the references therein, for the corrector of some wave equation with discontinuous coefficients in time. For the correctors for linear Dirichlet problems with simultaneously varying operators and domains obtained by using some special test function depending on the varying matrices and domains, see [19]. The reader is also referred to [3] and the references therein, for the correctors for the homogenizations of the wave and heat equations and to [7] for a corrector result for the wave equation with high oscillating periodic coefficients.

Let us also mention here, that the existence and uniqueness of a solution of the problem as well as the homogenization were already studied in [4] and [5], respectively.

This paper is organized as follows: Section 2 gives the geometric setting of the problem as well as the assumptions on the data to ensure existence and uniqueness of the weak solution of our problem; Section 3 contains a short discussion on **PUM** together with the operators and the corresponding properties that we need to prove the main result; homogenization results for the problem obtained in [5], for which the corrector result is based, are also recalled in Section 4; and the main results for this paper, which completes the study of the asymptotic bahavior in [5], are presented in Section 5.

II. SETTING OF THE PROBLEM

Let us recall the geometric framework for the perforated domain (see e.g. [14]).

Let $\mathbf{b} = (b_1, b_2, \ldots, b_N)$ be a basis of $\mathbb{R}^N$ (the set of reference periods) and Y a subset of $\mathbb{R}^N$ such that,

$$\mathbb{R}^N = \sum_{k\in\mathbb{Z}^N} \left(Y + \sum_{j=1}^{N} k_j b_j \right) = \sum_{\xi\in\mathbf{G}} (Y+\xi),$$

where

$$\mathbf{G} = \left\{ \xi \in \mathbb{R}^N \mid \xi = \sum_{i=1}^{N} k_i b_i, \ (k_1, \ldots, k_N) \in \mathbb{Z}^N \right\},$$

and

$$\Xi_\varepsilon = \{\xi \in \mathbf{G}, \ \varepsilon(\xi + Y) \subset \Omega\}.$$

For $z \in \mathbb{R}^N$, we let $[z]_Y = \sum_{j=1}^{N} k_j b_j$, be the unique integer combination of periods such that $z - [z]_Y$ is in Y, and

$$\{z\}_Y = z - [z]_Y.$$

That is,

$$z = \{z\}_Y + [z]_Y, \quad z \in \mathbb{R}^N.$$

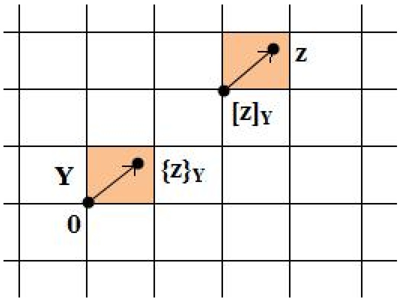

Fig. 1. The numbers $\{z\}_Y$ and $[z]_Y$.

The set $Y = (0,1)^N$ is called the reference cell, that is,

$$Y = \left\{ y \in \mathbb{R}^N | y = \sum_{i=1}^{N} y_i b_i, (y_1, \dots, y_N) \in Y \right\}.$$

Let $\{\varepsilon\}$ be a positive sequence converging to zero and for each positive ε, one can write

$$x = \varepsilon \left(\left\{ \frac{x}{\varepsilon} \right\}_Y + \left[\frac{x}{\varepsilon} \right]_Y \right),$$

for all $x \in \mathbb{R}^N$.

Denote a bounded open set on $\mathbb{R}^N$ by Ω. Let also S, a compact subset of Y, be the reference hole and suppose that S has a Lipschitz continuous boundary with a finite number of connected components. Let us also define $Y^* = Y \backslash S$ the perforated reference cell.

The perforated domain Ω_ε^* is then given by

$$\Omega_\varepsilon^* = \Omega \setminus S_\varepsilon, \quad \text{where} \quad S_\varepsilon = \bigcup_{\xi \in G} \varepsilon(\xi + S).$$

As introduced in [10] (see also [12]), we set

$$\widehat{\Omega}_\varepsilon = \text{interior} \left\{ \bigcup_{\xi \in \Xi_\varepsilon} \varepsilon(\xi + \bar{Y}) \right\} \quad \text{and} \quad \Lambda_\varepsilon = \Omega \backslash \widehat{\Omega}_\varepsilon,$$

that is, $\widehat{\Omega}_\varepsilon$ is the interior of the largest union of $\varepsilon(\xi + \bar{Y})$ cells fully contained in Ω, and Λ_ε contains the parts from the $\varepsilon(\xi + \bar{Y})$ cells that intersects the boundary $\partial\Omega$.

The corresponding perforated sets are then, given by,

$$\widehat{\Omega}_\varepsilon^* = \widehat{\Omega}_\varepsilon \setminus S_\varepsilon \quad \text{and} \quad \Lambda_\varepsilon^* = \Omega_\varepsilon^* \backslash \widehat{\Omega}_\varepsilon^*.$$

The boundary of the perforated domain Ω_ε^* is,

$$\partial\Omega_\varepsilon^* = \Gamma_0^\varepsilon \cup \Gamma_1^\varepsilon, \quad \text{where} \quad \Gamma_1^\varepsilon = \partial\widehat{\Omega}_\varepsilon^* \cap \partial S_\varepsilon$$

and

$$\Gamma_0^\varepsilon = \partial\Omega_\varepsilon^* \setminus \Gamma_1^\varepsilon.$$

Thus, Γ_1^ε is the boundary of the set of holes contained in $\widehat{\Omega}_\varepsilon$.

In the perforated domain in Figure 2 below, the dark perforated part is the set $\widehat{\Omega}_\varepsilon^*$ and the boundary of the holes contained is Γ_1^ε while the remaining part is the set Λ_ε^*, the boundary of the holes contained being the Γ_0^ε.

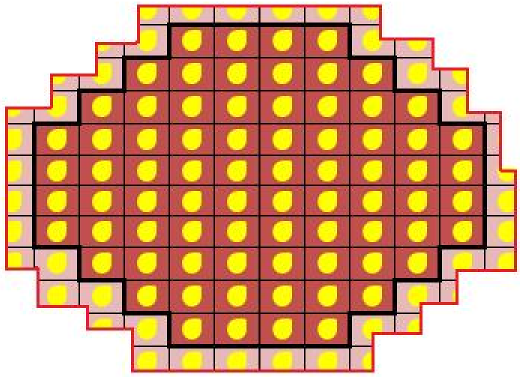

Fig. 2. The perforated domain Ω_ε^* and its boundary $\partial\Omega_\varepsilon^* = \Gamma_0^\varepsilon \cup \Gamma_1^\varepsilon$.

Lastly, let $\alpha, \beta \in \mathbb{R}$ with $0 < \alpha < \beta$, and denote by $M(\alpha, \beta, Y)$ the set of $N \times N$ matrix fields

$$A = (a_{ij})_{1 \le i,j \le N} \in (L^\infty(Y))^{N \times N},$$

satisfying

$$(A(y)\lambda, \lambda) \ge \alpha|\lambda|^2 \quad \text{and} \quad |A(y)\lambda| \le \beta|\lambda|,$$

for all $\lambda \in \mathbb{R}^N$ and a.e. in Y.

The goal of this paper is to provide a corrector result for the homogenization of the following quasilinear elliptic problem **(P)**:

$$\begin{cases} -\operatorname{div}(A^\varepsilon(x, u_\varepsilon)\nabla u_\varepsilon) = f & \text{in } \Omega_\varepsilon^*, \\ u_\varepsilon = 0 & \text{on } \Gamma_0^\varepsilon, \\ A^\varepsilon(x, u_\varepsilon)\nabla u_\varepsilon \cdot n + \varepsilon^\gamma \tau_\varepsilon(x) h(u_\varepsilon) = g_\varepsilon(x) & \text{on } \Gamma_1^\varepsilon, \end{cases}$$

where n is the unit exterior normal to Ω_ε^* and γ is a real parameter, with $\gamma \ge 1$.

Let us denote by $\mathcal{M}_\mathcal{O}$ the mean value of an integrable function on $\mathcal{O}$, given by

$$\mathcal{M}_\mathcal{O}(\Phi) = \frac{1}{|\mathcal{O}|} \int_\mathcal{O} \Phi(y) dy, \quad \forall \Phi \in L^1(\mathcal{O}).$$

We also set

$$\tau_\varepsilon(x) = \tau\left(\frac{x}{\varepsilon}\right),$$

$$g_\varepsilon(x) = \begin{cases} g\left(\frac{x}{\varepsilon}\right) & \text{if } \mathcal{M}_{\partial S}(g) = 0, \\ \varepsilon g\left(\frac{x}{\varepsilon}\right) & \text{if } \mathcal{M}_{\partial S}(g) \ne 0, \end{cases}$$

and

$$A^\varepsilon(x, t) = A\left(\frac{x}{\varepsilon}, t\right), \quad \text{for every } (x, t) \in \mathbb{R}^N \times \mathbb{R}.$$

Suppose that the data satisfy the following assumptions:

A1. f, g, τ are functions such that for every ε,
- (i) $f \in L^2(\Omega_\varepsilon^*)$,
- (ii) g is a Y-periodic function in $L^2(\Gamma_1^\varepsilon)$,
- (iii) τ is a positive Y-periodic function in $L^\infty(\partial S)$;

A2. h is a function from $\mathbb{R}$ to $\mathbb{R}$ such that
- (i) h is an increasing function in $C^1(\mathbb{R})$ such that $h(0) = 0$,
- (ii) there exists a constant $C > 0$ and an exponent q with $1 \le q \le \infty$ if $N = 2$ and $1 \le q \le \dfrac{N}{N-2}$ if $N > 2$ such that $\forall s \in \mathbb{R}$,

$$|h'(s)| \le C(1 + |s|^{q-1});$$

ISBN: 978-988-19253-0-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$A3$. $A : Y \times \mathbb{R} \longmapsto \mathbb{R}^{N^2}$, is a matrix field satisfying the following conditions:

(i) A is a Caratheodory function and $Y-$periodic for every t

(ii) for every $t \in \mathbb{R},\ A(\cdot, t) \in M(\alpha, \beta, Y)$,

(iii) there exists a function $\omega : \mathbb{R} \mapsto \mathbb{R}$ such that

a. ω is continuous, nondecreasing and $\omega(t) > 0$ for all $t > 0$,

b. $|A(y,t) - A(y,t_1)| \le \omega(|t-t_1|)$ a.e. $y \in\ Y$, for $t \neq t_1\ \in \mathbb{R}$,

c. for any $r > 0$, $\displaystyle\lim_{s\to 0^+}\int_s^r \frac{dt}{\omega(t)} = +\infty$.

Now, for $p \in [1, +\infty)$, we define

$$V_\varepsilon^p = \left\{\phi \in W^{1,p}(\Omega_\varepsilon^*) \mid \phi = 0 \text{ on } \Gamma_0^\varepsilon\right\}$$

and

$$V_\varepsilon = V_\varepsilon^2,$$

which is a Banach space for the norm

$$\|u\|_{V_\varepsilon^p} = \|\nabla u\|_{L^p(\Omega_\varepsilon^*)}\ \ \forall u \in V_\varepsilon^p.$$

The variational formulation **(VF)** of problem **P** is then given by

$$\begin{cases} \text{Find } u_\varepsilon \in\ V_\varepsilon \text{ such that} \\ \displaystyle\int_{\Omega_\varepsilon^*} A^\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon \nabla v\ dx + \varepsilon^\gamma \int_{\Gamma_1^\varepsilon} \tau_\varepsilon(x) h(u_\varepsilon) v\ d\sigma_x \\ \displaystyle = \int_{\Omega_\varepsilon^*} fv\ dx + \int_{\Gamma_1^\varepsilon} g_\varepsilon(x) v\ d\sigma_x, \qquad \forall\ v \in V_\varepsilon. \end{cases}$$

Under assumptions $A1 - A3$, the existence and uniqueness of a solution for problem **VF** has been proved in [4].

III. A SHORT REVIEW OF THE UNFOLDING METHOD

In this section, we briefly recall the main definitions and properties of the unfolding operators under **PUM**, that we need.

DEFINITION 1. For any Lebesgue-measurable function ϕ on Ω_ε^*, the unfolding operator $\mathcal{T}_\varepsilon^*$ is a function from $L^p(\Omega_\varepsilon^*)$ to $L^p(\Omega \times Y^*)$, and is defined by

$$\mathcal{T}_\varepsilon^*(\phi)(x,y) = \begin{cases} \phi\left(\varepsilon\left[\dfrac{x}{\varepsilon}\right]_Y + \varepsilon y\right), & \text{a.e. in } \widehat{\Omega}_\varepsilon \times Y^*, \\ 0, & \text{a.e. in } \Lambda_\varepsilon \times Y^*. \end{cases}$$

PROPOSITION 2. *[10], [12], [14], [15]*

Let $p \in [1, +\infty)$.

1) $\mathcal{T}_\varepsilon^*$ *is linear and continuous.*
2) $\mathcal{T}_\varepsilon^*(\phi\psi) = \mathcal{T}_\varepsilon^*(\phi)\mathcal{T}_\varepsilon^*(\psi)$ *for every* $\phi, \psi \in L^p(\Omega_\varepsilon^*)$.
3) *For* $w \in L^p(\Omega)$,
$$\mathcal{T}_\varepsilon^*(w) \to w \quad \textit{strongly in } L^p(\Omega \times Y^*).$$
4) *For all* $\phi \in L^1(\Omega_\varepsilon^*)$ *one has*
$$\int_{\widehat{\Omega}_\varepsilon^*} \phi(x)\ dx = \int_{\Omega_\varepsilon^*} \phi(x)\ dx - \int_{\Lambda_\varepsilon^*} \phi(x)\ dx$$
$$= \frac{1}{|Y|}\int_{\Omega\times Y^*} \mathcal{T}_\varepsilon^*(\phi)(x,y)\ dx\,dy.$$
5) $\nabla_y \mathcal{T}_\varepsilon^*(\phi)(x,y) = \varepsilon\mathcal{T}_\varepsilon^*(\nabla_x\phi)(x,y)$ *for every* (x,y) *in* $\mathbb{R}^N \times Y^*$.
6) *Let* $\phi_\varepsilon \in L^p(\Omega)$ *such that*
$$\phi_\varepsilon \to \phi \quad \textit{strongly in } L^p(\Omega).$$
Then
$$\mathcal{T}_\varepsilon^*(\phi_\varepsilon) \to \phi \quad \textit{strongly in } L^p(\Omega \times Y^*).$$

DEFINITION 3. For $p \in [1, +\infty]$, the averaging operator $\mathcal{U}_\varepsilon^* : L^p(\Omega \times Y^*) \longmapsto L^p(\Omega_\varepsilon^*)$ is defined as

$$\mathcal{U}_\varepsilon^*(\Phi)(x) = \frac{1}{|Y^*|}\int_{Y^*} \Phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon z, \left\{\frac{x}{\varepsilon}\right\}_Y\right)\ dz,$$

a.e. for $x \in \widehat{\Omega}_\varepsilon^*$ and,

$$\mathcal{U}_\varepsilon^*(\Phi)(x) = 0,$$

a.e. for $x \in \Lambda_\varepsilon^*$.

One has

$$\mathcal{U}_\varepsilon^*\left(\Phi|_{\Omega\times Y^*}\right) = \mathcal{U}_\varepsilon\left(\Phi\right)|_{\Omega_\varepsilon^*},$$

when Φ belongs to $L^p(\Omega \times Y^*)$.

Some of the properties of the averaging operator are given in the next two propositions.

PROPOSITION 4. *[10], [12], [14], [15]*

Let $p \in [1, +\infty[$.

1) $\mathcal{U}_\varepsilon^*$ *is linear and continuous.*
2) $\mathcal{U}_\varepsilon^*$ *is almost a left inverse of* $\mathcal{T}_\varepsilon^*$ *on* Ω_ε^*.
3) *For any* ϕ *in* $L^p(\Omega)$,
$$\|\phi - \mathcal{U}_\varepsilon^*(\phi)\|_{L^p(\Omega_\varepsilon^*)} \to 0.$$
4) *Let* w_ε *be in* $L^p(\Omega_\varepsilon^*)$. *Then the following are equivalent:*
   (i) $\mathcal{T}_\varepsilon^*(w_\varepsilon) \to \widehat{w}$ *strongly in* $L^p(\Omega \times Y^*)$ *and* $\displaystyle\int_{\Lambda_\varepsilon^*} |w_\varepsilon|^p\ dx \to 0$,
   (ii) $\|w_\varepsilon - \mathcal{U}_\varepsilon^*(\widehat{w})\|_{L^p(\Omega_\varepsilon^*)} \to 0$.

PROPOSITION 5. *[10] For* $p \in [1, +\infty)$, *suppose that* ρ *is in* $L^p(\Omega)$ *and* θ *in* $L^\infty(\Omega; L^p(Y))$. *Then the product* $\mathcal{U}_\varepsilon(\rho)\mathcal{U}_\varepsilon(\theta)$ *belongs to* $L^p(\Omega)$ *and*

$$\mathcal{U}_\varepsilon(\rho\,\theta) - \mathcal{U}_\varepsilon(\rho)\mathcal{U}_\varepsilon(\theta) \to 0 \quad \textit{strongly in } L^p(\Omega).$$

Let us now define the boundary unfolding operator. Here, we assume that $p \in]1, +\infty[$ and that ∂S has a finite number of connected components.

DEFINITION 6. For any Lebesgue-measurable function φ on $\partial\widehat{\Omega}_\varepsilon^* \cap \partial S_\varepsilon$, the boundary unfolding operator is defined by

$$\mathcal{T}_\varepsilon^b(\varphi)(x,y) = \begin{cases} \varphi\left(\varepsilon\left[\dfrac{x}{\varepsilon}\right]_Y + \varepsilon y\right), & \text{a.e. in } \widehat{\Omega}_\varepsilon \times \partial S, \\ 0 & \text{a.e. in } \Lambda_\varepsilon \times \partial S. \end{cases}$$

PROPOSITION 7. *[10], [15] Let* $p \in [1, \infty[$. *Then*

1) $\mathcal{T}_\varepsilon^b$ *is a linear operator.*
2) $\mathcal{T}_\varepsilon^b(\phi\psi) = \mathcal{T}_\varepsilon^b(\phi)\mathcal{T}_\varepsilon^b(\psi)$ *for every* $\phi, \psi \in L^p(\partial S_\varepsilon)$.
3) *Let* $\phi \in L^p(\partial S)$ *be a* Y*-periodic function. Set* $\phi_\varepsilon(x) = \phi\left(\frac{x}{\varepsilon}\right)$. *Then*
$$\mathcal{T}_\varepsilon^b(\phi_\varepsilon)(x,y) = \phi(y).$$
4) *For all* $\phi \in L^1(\partial S_\varepsilon)$, *the integration formula is given by*
$$\int_{\Gamma_1^\varepsilon} \phi(x)\ d\sigma_x = \frac{1}{\varepsilon|Y|}\int_{\Omega\times\partial S} \mathcal{T}_\varepsilon^b(\phi)(x,y)\ dx d\sigma_y.$$

ISBN: 978-988-19253-0-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

5) *Let* $\phi \in L^p(\partial S_\varepsilon)$. *Then*

$$\mathcal{T}^b_\varepsilon(\phi) \to \phi \quad \textit{strongly in } L^p(\Omega \times \partial S).$$

IV. Some known results

In this section, we recall the homogenization results as obtained in [5], on which the corrector result will be based upon.

Theorem 8. *Under assumptions* $A1 - A3$, *let* u_ε *be the unique solution of* ***VF*** *and* $\gamma \geq 1$. *Then, there exists* $(u_0, \widehat{u})$ *in* $H^1_0(\Omega) \times L^2(\Omega; H^1_{per}(Y^*))$ *with* $\mathcal{M}_{Y^*}(\hat{u}) = 0$, *such that:*

$$\begin{cases} (i)\ \mathcal{T}^*_\varepsilon(u_\varepsilon) \to u_0 \ \textit{ strongly in } L^2(\Omega; H^1(Y^*)), \\ (ii)\ \mathcal{T}^*_\varepsilon(\nabla u_\varepsilon) \rightharpoonup \nabla u_0 + \nabla_y \widehat{u} \ \textit{ weakly in } L^2(\Omega \times Y^*), \\ (iii) \mathcal{T}^b_\varepsilon(h(u_\varepsilon)) \rightharpoonup h(u_0) \ \textit{weakly in } L^t(\Omega; W^{1-\frac{1}{t},t}(\partial S)). \end{cases}$$

Case 1. If $\gamma = 1$, *the couple* $(u_0, \widehat{u})$ *is the unique solution in the space* $H^1_0(\Omega) \times L^2(\Omega; H^1_{per}(Y^*))$ *with* $\mathcal{M}_{Y^*}(\hat{u}) = 0$, *of the limit equation*

$$\int_{\Omega \times Y^*} A(y, u_0)(\nabla u_0 + \nabla_y \widehat{u})(\nabla \phi(x) + \nabla_y \Psi(x, y))\ dx\, dy$$
$$+ |\partial S| \mathcal{M}_{\partial S}(\tau) \int_\Omega h(u_0)\phi\ dx$$
$$= |Y^*| \int_\Omega f\phi\ dx + |\partial S| \mathcal{M}_{\partial S}(g) \int_\Omega \phi\ dx$$

for all $\phi \in H^1_0(\Omega)$ *and* $\Psi \in L^2(\Omega; H^1_{per}(Y^*))$.
Case 2. If $\gamma > 1$, *the couple* $(u_0, \widehat{u})$ *is the unique solution in the space* $H^1_0(\Omega) \times L^2(\Omega; H^1_{per}(Y^*))$ *with* $\mathcal{M}_{Y^*}(\hat{u}) = 0$, *of the limit equation*

$$\int_{\Omega \times Y^*} A(y, u_0)(\nabla u_0 + \nabla_y \widehat{u})(\nabla \phi(x) + \nabla_y \Psi(x, y))\ dx\, dy$$
$$= |Y^*| \int_\Omega f\phi\ dx + |\partial S| \mathcal{M}_{\partial S}(g) \int_\Omega \phi\ dx$$

for all $\phi \in H^1_0(\Omega)$ *and* $\Psi \in L^2(\Omega; H^1_{per}(Y^*))$.

Corollary 9. *Under assumptions* $A1 - A3$, *let* u_ε *be the unique solution of* ***VF***. *Then, if* $\gamma \geq 1$,

$$\widetilde{u_\varepsilon} \rightharpoonup \frac{|Y^*|}{|Y|} u_0 \quad \textit{weakly in } L^2(\Omega),$$

where $\tilde{}$ *denotes the extension by 0 to* Ω.

If $\gamma = 1$, *the function* u_0 *is the unique solution of the limit problem*

$$\begin{cases} -\ div(A^0(u)\nabla u_0) + \dfrac{|\partial S|}{|Y|} \mathcal{M}_{\partial S}(\tau) h(u_0) \\ \quad = \dfrac{|Y^*|}{|Y|} f + \dfrac{|\partial S|}{|Y|} \mathcal{M}_{\partial S}(g) & \textit{in } \Omega \\ u_0 = 0 & \textit{on } \partial\Omega. \end{cases}$$

If $\gamma > 1$, *the function* u_0 *is the unique solution of the problem*

$$\begin{cases} -\ div(A^0(u)\nabla u_0) = \dfrac{|Y^*|}{|Y|} f + \dfrac{|\partial S|}{|Y|} \mathcal{M}_{\partial S}(g) & \textit{in } \ \Omega \\ u_0 = 0 & \textit{on } \partial\Omega. \end{cases}$$

The homogenized matrix field $A^0(t)$ *is given by*

$$A^0(t)\lambda = \frac{1}{|Y|} \int_{Y^*} A(y, t) \nabla w_\lambda(t, y) dy, \quad \forall \lambda \in \mathbb{R}^N,$$

where $w_\lambda(y, t) = -\chi_\lambda(y, t) + \lambda \cdot y$ *a.e. in* Y^* *and* $\chi_\lambda(\cdot, t)$ *is, for every* t, *the solution of the cell problem*

$$\begin{cases} -\ div(A(\cdot, t)\nabla \chi_\lambda(\cdot, t) = -\ div(A(\cdot, t)\lambda & \textit{in } Y^*, \\ A(\cdot, t)\nabla \chi_\lambda(\cdot, t) \cdot n = 0 & \textit{on } \partial S, \\ \chi_\lambda(\cdot, t) \quad Y - \textit{periodic}, \\ \mathcal{M}_{Y^*}(\chi_\lambda(\cdot, t)) = 0. \end{cases}$$

Remark 10. One has (see e.g. [13] for the details of the computation),

$$\widehat{u}(x, y) = -\sum_{i=1}^N \chi_{e_i}(y, u_0(x)) \frac{\partial u_0}{\partial x_i}(x), \tag{1}$$

where u_0 is the one given in Theorem 8.

V. Main Result

First we recall a classical result that we need to prove the main result given in Theorem 14.

Lemma 11. *[10] Let* $\{D_\varepsilon\}$ *be a sequence of* $n \times n$ *matrix fields in* $M(\alpha, \beta, \mathcal{O})$ *for some open set* $\mathcal{O}$, *such that*

$$D_\varepsilon \to D \textit{ a.e. on } \mathcal{O},$$

(or more generally, in measure in $\mathcal{O}$*). If the sequence* $\{\zeta_\varepsilon\}$ *converges weakly to* ζ *in* $[L^2(\mathcal{O})]^N$, *then*

$$\liminf_{\varepsilon \to 0} \int_{\mathcal{O}} D_\varepsilon \zeta_\varepsilon \zeta_\varepsilon\, dx \geq \int_{\mathcal{O}} D\zeta\zeta\, dx.$$

Furthermore, if

$$\limsup_{\varepsilon \to 0} \int_{\mathcal{O}} D_\varepsilon \zeta_\varepsilon \zeta_\varepsilon\, dx \leq \int_{\mathcal{O}} D\zeta\zeta\, dx,$$

then

$$\int_{\mathcal{O}} D\zeta\zeta\, dx = \lim_{\varepsilon \to 0} \int_{\mathcal{O}} D_\varepsilon \zeta_\varepsilon \zeta_\varepsilon\, dx,$$

and

$$\zeta_\varepsilon \to \zeta \quad \textit{strongly in } [L^2(\mathcal{O})]^N.$$

Next, we present a proposition which is also essential in proving the corrector result.

Proposition 12. *Let* $\gamma \geq 1$ *and suppose* $\mathcal{M}_{\partial S}(g) \neq 0$. *Under the assumptions in Theorem 8, one has*

$$\mathcal{T}^*_\varepsilon(\nabla u_\varepsilon) \to \nabla u_0 + \nabla_y \widehat{u} \textit{ strongly in } L^2(\Omega \times Y^*), \tag{2}$$

and

$$\lim_{\varepsilon \to 0} \int_{\Lambda_\varepsilon} |\nabla u_\varepsilon|^2\ dx = 0. \tag{3}$$

Proof:

We prove the case $\gamma = 1$. When $\gamma > 1$, the proof is similar but the the nonlinear term in the boundary approaches 0 at the limit.

By applying **PUM**,

$$\int_{\Omega^*_\varepsilon} f u_\varepsilon\, dx = \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}^*_\varepsilon(f) T^*_\varepsilon(u_\varepsilon)\ dx\, dy,$$

so that from Proposition 2-3) and Theorem 8 convergence (i),

$$\lim_{\varepsilon \to 0} \int_{\Omega^*_\varepsilon} f u_\varepsilon\, dx = \frac{1}{|Y|} \lim_{\varepsilon \to 0} \int_{\Omega \times Y^*} \mathcal{T}^*_\varepsilon(f) T^*_\varepsilon(u_\varepsilon)\ dx\, dy$$
$$= \frac{1}{|Y|} \int_{\Omega \times Y^*} f u_0\ dx\, dy.$$

ISBN: 978-988-19253-0-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Since f and u_0 are just functions of x, one gets

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^*} f u_\varepsilon\, dx = \frac{|Y^*|}{|Y|}\int_\Omega f u_0\ dx. \tag{4}$$

Also, from Proposition 3.5 of [10],

$$\int_{\Gamma_1^\varepsilon} g_\varepsilon(x) u_\varepsilon\ d\sigma_x = \frac{|\partial S|}{|Y|}\mathcal{M}_{\partial S}(g)\int_\Omega u_0\ dx. \tag{5}$$

Moreover,

$$\varepsilon\int_{\Gamma_1^\varepsilon}\tau_\varepsilon(x)h(u_\varepsilon)u_\varepsilon\ d\sigma_x$$

$$= \frac{1}{|Y|}\int_{\Omega\times\partial S}\tau(y)\mathcal{T}_\varepsilon^b(h(u_\varepsilon))\mathcal{T}_\varepsilon^*(u_\varepsilon)\ dx\, d\sigma_y,$$

from Proposition 7. Thus, one obtains from Theorem 8 (i) and (iii) that,

$$\frac{1}{|Y|}\lim_{\varepsilon\to 0}\left(\int_{\Omega\times\partial S}\tau(y)\mathcal{T}_\varepsilon^b(h(u_\varepsilon))\mathcal{T}_\varepsilon^*(u_\varepsilon)\ dx\, d\sigma_y\right)$$

$$= \frac{|\partial S|}{|Y|}\mathcal{M}_{\partial S}(\tau)\int_\Omega h(u_0)u_0\ dx,$$

which yields,

$$\lim_{\varepsilon\to 0}\left(\varepsilon\int_{\Gamma_1^\varepsilon}\tau_\varepsilon(x)h(u_\varepsilon)u_\varepsilon\ d\sigma_x\right)$$
$$= \frac{|\partial S|}{|Y|}\mathcal{M}_{\partial S}(\tau)\int_\Omega h(u_0)u_0\ dx. \tag{6}$$

Now, using Lemma 11 with

$$D_\varepsilon = \mathcal{T}_\varepsilon^*(A^\varepsilon(x,u_\varepsilon)) \quad\text{and}\quad \zeta_\varepsilon = \mathcal{T}_\varepsilon^*(\nabla u_\varepsilon),$$

together with (4)-(6), one obtains

$$\frac{1}{|Y|}\int_{\Omega\times Y^*} A(y,u_0)(\nabla u_0 + \nabla_y\widehat{u})(\nabla u_0+\nabla_y\widehat{u})\ dx\, dy$$

$$\leq \liminf_{\varepsilon\to 0}\left(\frac{1}{|Y|}\int_{\Omega\times Y^*}\mathcal{T}_\varepsilon^*(A^\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon\nabla u_\varepsilon)\ dx\, dy\right)$$

$$\leq \limsup_{\varepsilon\to 0}\left(\frac{1}{|Y|}\int_{\Omega\times Y^*}\mathcal{T}_\varepsilon^*(A^\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon\nabla u_\varepsilon)\ dx\, dy\right)$$

$$\leq \limsup_{\varepsilon\to 0}\int_{\Omega_\varepsilon^*} A^\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon\nabla u_\varepsilon\ dx$$

$$= \limsup_{\varepsilon\to 0}\left(\int_{\Omega_\varepsilon^*} f v\ dx + \int_{\Gamma_1^\varepsilon} g_\varepsilon(x) v\ d\sigma_x - \varepsilon^\gamma\int_{\Gamma_1^\varepsilon}\tau_\varepsilon(x)h(u_\varepsilon)v\ d\sigma_x\right)$$

$$= \frac{|Y^*|}{|Y|}\int_\Omega f u_0\ dx + \frac{|\partial S|}{|Y|}\mathcal{M}_{\partial S}(g)\int_\Omega u_0\ dx - \frac{|\partial S|}{|Y|}\mathcal{M}_{\partial S}(\tau)\int_\Omega h(u_0)u_0\ dx$$

$$= \frac{1}{|Y|}\int_{\Omega\times Y^*} A(y,u_0)(\nabla u_0+\nabla_y\widehat{u})(\nabla u_0+\nabla_y\widehat{u})\ dx\, dy,$$

by choosing $\phi = u_0$ and $\Psi = \widehat{u}$ in the limit equation for Case 1 in Theorem 8. Thus,

$$\lim_{\varepsilon\to 0}\left(\frac{1}{|Y|}\int_{\Omega\times Y^*}\mathcal{T}_\varepsilon^*(A^\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon\nabla u_\varepsilon)\ dx\, dy\right)$$
$$= \frac{1}{|Y|}\int_{\Omega\times Y^*} A(y,u_0)(\nabla u_0+\nabla_y\widehat{u})(\nabla u_0+\nabla_y\widehat{u})\ dx\, dy \tag{7}$$

which implies, using Lemma 11, the convergence given in (2).

On the other hand, from Proposition 2-4) and the ellipticity of A^ε,

$$\alpha\int_{\Lambda_\varepsilon}|\nabla u_\varepsilon|^2\ dx \leq \int_{\Lambda_\varepsilon} A^\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon\nabla u_\varepsilon\ dx$$

$$= \int_{\Omega_\varepsilon^*} A^\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon\nabla u_\varepsilon\ dx$$

$$-\frac{1}{|Y|}\int_{\Omega\times Y^*}\mathcal{T}_\varepsilon^*(A^\varepsilon(x,u_\varepsilon))\mathcal{T}_\varepsilon^*(\nabla u_\varepsilon)\mathcal{T}_\varepsilon^*(\nabla u_\varepsilon)\ dx\, dy.$$

This, together with (7) gives the second convergence in (3). ■

REMARK 13. From the computations above, we have the following convergence of the energy:

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^*} A^\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon\nabla u_\varepsilon\ dx$$

$$= \frac{1}{|Y|}\int_{\Omega\times Y^*} A(y,u_0)(\nabla u_0+\nabla_y\widehat{u})(\nabla u_0+\nabla_y\widehat{u})\ dx\, dy.$$

Let us now have the main result in this paper, the corrector result.

THEOREM 14. *Let $\gamma \geq 1$ and suppose $\mathcal{M}_{\partial S}(g) \neq 0$. Under the assumptions in Theorem 8, we have*

$$\left\|\nabla u_\varepsilon - \nabla u_0 + \sum_{i=1}^N \mathcal{U}_\varepsilon^*(\nabla_y\chi_{e_i}(y,u_0(x)))\mathcal{U}_\varepsilon^*\left(\frac{\partial u_0}{\partial x_i}\right)\right\|_{L^2(\Omega_\varepsilon^*)}$$

converges to 0.

Proof:

From Proposition 12, and Proposition 4-4), one has

$$\|\nabla u_\varepsilon - \mathcal{U}_\varepsilon^*(\nabla u_0+\nabla_y\widehat{u})\|_{L^2(\Omega_\varepsilon^*)} \to 0.$$

This together with Proposition 4-1),3) gives

$$\|\nabla u_\varepsilon - \nabla u_0 - \mathcal{U}_\varepsilon^*(\nabla u_0) + \mathcal{U}_\varepsilon^*(\nabla u_0) - \mathcal{U}_\varepsilon^*(\nabla_y\widehat{u})\|_{L^2(\Omega_\varepsilon^*)}$$
$$\leq \|\nabla u_\varepsilon - \mathcal{U}_\varepsilon^*(\nabla u_0) - \mathcal{U}_\varepsilon^*(\nabla_y\widehat{u})\|_{L^2(\Omega_\varepsilon^*)}$$
$$+ \|\nabla u_0 - \mathcal{U}_\varepsilon^*(\nabla u_0)\|_{L^2(\Omega_\varepsilon^*)}$$
$$\to 0.$$

Thus, from (1), Proposition 5 and the computations above, one obtains

$$\left\|\nabla u_\varepsilon - \nabla u_0 - \mathcal{U}_\varepsilon^*\left(-\sum_{i=1}^N\nabla_y\chi_{e_i}(y,u_0(x))\frac{\partial u_0}{\partial x_i}\right)\right.$$
$$+\mathcal{U}_\varepsilon^*\left(-\sum_{i=1}^N\nabla_y\chi_{e_i}(y,u_0(x))\frac{\partial u_0}{\partial x_i}\right)$$

$$- \left(- \sum_{i=1}^{N} \mathcal{U}_\varepsilon^*(\nabla_y \chi_{e_i}(y, u_0(x))) \mathcal{U}_\varepsilon^* \left(\frac{\partial u_0}{\partial x_i} \right) \right) \Bigg\|_{L^2(\Omega_\varepsilon^*)}$$

$$\leq \left\| \mathcal{U}_\varepsilon^* \left(- \sum_{i=1}^{N} \nabla_y \chi_{e_i}(y, u_0(x)) \frac{\partial u_0}{\partial x_i} \right) - \right.$$

$$\left. \left(- \sum_{i=1}^{N} \mathcal{U}_\varepsilon^*(\nabla_y \chi_{e_i}(y, u_0(x))) \mathcal{U}_\varepsilon^* \left(\frac{\partial u_0}{\partial x_i} \right) \right) \right\|_{L^2(\Omega_\varepsilon^*)}$$

$$+ \left\| \nabla u_\varepsilon - \nabla u_0 - \mathcal{U}_\varepsilon^* \left(- \sum_{i=1}^{N} \nabla_y \chi_{e_i}(y, u_0(x)) \frac{\partial u_0}{\partial x_i} \right) \right\|_{L^2(\Omega_\varepsilon^*)}$$

$$\to 0,$$

which yields the desired result. ■

Remark: The case $\mathcal{M}_{\partial S}(g) = 0$ has been recently done in [8].

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