Solution of an f(R) Theory of Gravitation in Cylindrical Symmetric Godel Space-time

S. N. Pandey, A. M. Mishra

***Abstract*—Einstein theory of gravity has been modified in different ways in which an attractive modification based on non-conformal invariant property of gravitational waves which are inevitable consequence of Einstein theory that describes our universe fairly well. We have solved this theory for cylindrically symmetric space-time, particularly Godel type space-time and obtained the condition under which Godel space-time will be a solution of $f(R)$ theory of gravity. We have also compared the results of solutions of higher order field equations and Einstein field equation with cosmological constant showing that solution of Einstein field equation is a particular case of solution of higher order field equations without cosmological constant.**

***Index Terms*—Godel metric, Hilbert-Lagrangian, Riccci curvature tensor.**

I. Introduction

AFTER introduction of GR in 1915, questions related to limitations become more and more pertinent. If only took four years that Weyl in 1919 started considering a modification of the theory, and late in 1922 Eddington by introducing higher order invariant in its action. There efforts were mainly by scientific curiosity to understand this newly proposed theory. However in 1960, there appears indication that complicating the gravitational action might indeed have its merits. GR is not renormalizable and therefore cannot be conveniently quantized [1]. Utiyama and De witt [2] showed that renormlization in one loop demands that Einstein-Hilbert action be supplemented by higher order curvature terms. Such considerations stimulated the interest in higher-order theories of gravity, that is, modification of Einstein Hilbert action in order to include higher-order curvature invariants with respect to Ricci scalar. Einstein's theory of gravitation describes the universe as observed fairly well. It predicts the presence of angularity which is usually interpreted to mean the limit of applicability of the theory, GR. In a classical regime, there exists a possibility of avoiding the initial singularities in higher order gravity theories [3]. It appears natural that when gravity is very strong and the curvature is large, the linearity in scalar curvature R in the Lagrangian would be restrictive. Thus higher curvature gravity seems to work in both classical and quantum theories [4]. From the cosmological point of view, it may give a better description of the initial stages of the universe then Einstein theory and is expected to give a similar description late as Einstein's theory because the curvature effects would then be small [5].

Gravitational waves which are one of two important prediction of Einstein theory, demand low energy conditions unlike black holes [6]. These are non-conformal unlike electromagnetic waves and hence, there is a need to modify Einstein theory as par with Electromagnetic waves [7]. So we have chosen $f(R)$ as a polynomial in R of a finite number of terms without associating it with any other field except gravitation [8]. Taking Lagrangian in the form:

$$\mathfrak{L} = R + \sum_{n=2}^{N} c_n \frac{(l^2R)^n}{6l^2}$$

Or in other way:

$$\mathfrak{L} = R + \sum_{n=2}^{N} a_n R^n \tag{1}$$

where l is the characteristic length and C_n are the dimensionless arbitrary coefficients corresponding the values of n. This Lagrangian is used to modify Einstein field equation so that the gravitational waves can be reduced to a conformally invariant form by a proper choice of arbitrary coefficients.

An application of variational principle to this action [9] yields the following field equations:

$$G_{ij} - \sum_{n=2}^{N} \frac{nC_n}{6}(l^2R)^{n-1}[R_{ij} - \frac{1}{2n}g_{ij}R - \frac{n-1}{R}R_{;i;j} - g_{ij}\Box R)$$
$$-\frac{(n-1)(n-2)}{R^2}(R_{;i}R_{;j} - g_{ij}R_{;\alpha}R^{;\alpha})] = \kappa T_{ij} \tag{2}$$

Here square box ($\Box$) represents Laplacian operator in 4-dimension space-time. κ is a constant and

$$T_{ij} = \sqrt{-g}\frac{\partial L_S}{\partial g^{ij}} \tag{3}$$

stands for energy momentum tensor which is responsible for production of gravitational potential. We can observe that equation of continuity holds for these field equations in a similar way as it in case of Einstein general relativity [10].

On the other hand, when we talk about the anisotropy and inhomogeneity of the universe, we cannot ignore cylindrical symmetry [11]. One can study a general cylindrically symmetric space time [12] but we are interested in an special one, Godel type, because of its useful possibility to explore time travel by studying closed time like curves [13]. The Godel solution of Einstein field equation is given by space-time [14]

$$ds^2 = (dt + X(r)d\phi)^2 - Y^2(r)d\phi^2 - dr^2 - dz^2 \tag{4}$$

Here, $X(r)$ and $Y(r)$ are arbitrary parameter which need to be determined. If we put $X(r) = e^r$ and Y(r)=e$^r/\sqrt{2}$ we get the space-time that had considered originally by Godel [16]. The geometric properties of the space-time are

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as follows:
The scalar curvature is

$$R = \frac{X'^2 - 4YY''}{2Y^2} \tag{5}$$

Similarly, components of nonzero Ricci curvature tensor of this space-time are:

$$R_{11} = \frac{X'^2 - 2YY''}{2Y^2} \tag{6}$$

$$R_{22} = \frac{1}{2}\left(\left(1 - \frac{X^2}{Y^2}\right)X'^2 - \frac{2XX'Y'}{Y} + 2XX'' - 2YY''\right) \tag{7}$$

$$R_{42} = \frac{AA'^2 - BA'B' + B^2A''}{2B^2} \tag{8}$$

$$R_{44} = \frac{A'^2}{2B^2} \tag{9}$$

The paper is mainly divided into three parts: Field equations, its solutions and interpretation of the solutions. In previous section, we have discussed field equations of higher order $f(R)$ theory motivated by gravitational waves. In next section we split field equations corresponding to $n = 2$ and 3. The solutions of field equations are to be discussed in section 3 and a condition in terms of Ricci scalar curvature be obtained. In section 4, some space-times are to be discussed which are solutions of higher order theory of gravity under some specific conditions. In section 5, we conclude the results and compare it with the solution of Einstein field equation with cosmological constant.

II. Field Equation in Vacuum

We consider the field equation (2) in vacuum

$$G_{ij} - \sum_{n=2}^{N} \frac{nC_n}{6}(l^2R)^{n-1}[R_{ij} - \frac{1}{2n}g_{ij}R - \frac{n-1}{R}R_{;i;j} - g_{ij}\Box R)$$

$$-\frac{(n-1)(n-2)}{R^2}(R_{;i}R_{;j} - g_{ij}R_{;\alpha}R^{;\alpha})] = 0 \tag{10}$$

Equation (10) converts into Einstein field equation when we take $n = 0$ and $n = 1$, so we must take n from 2 onwards for higher order gravity.
For $n = 2$;

$$R_{ij} - \frac{1}{2}g_{ij}R + 2a_2R\left[R_{ij} - \frac{1}{4}g_{ij}R - \frac{2}{R}(R_{;i;j} - g_{ij}\Box R)\right] = 0 \tag{11}$$

and similarly for $n = 3$:

$$R_{ij} - \frac{1}{2}g_{ij}R + 2a_2R\left[R_{ij} - \frac{1}{4}g_{ij}R - \frac{2}{R}(R_{;i;j} - g_{ij}\Box R)\right]$$

$$+3a_3R^2[R_{ij} - \frac{1}{6}g_{ij}R - \frac{6}{R}(R_{;i;j} - g_{ij}\Box R)$$

$$-\frac{2}{R^2}(R_{;i}R_{;j} - g_{ij}R_{;\alpha}R^{;\alpha})] = 0 \tag{12}$$

Now we will find the solutions of these field equations.

III. Solution of Field Equations

In order to solve equation (10) we need geometric parameter of the metric given by equation (4). Using these parameters which have calculated in equation(5 -9)we get:

$$-\frac{1}{8Y^4}[3a_2X'^2(X'^2 - 64Y'^2) + 16a_2Y(16X'Y'X'' + 3X'2Y''$$

$$+16Y'^2Y'') - 2Y^2(X'^2 + 32a_2X'X''') + 8a_2(4X''^2 + 7Y''^2$$

$$+16Y'Y''') + 12a_2Y^3Y^{iv}] = 0 \tag{13}$$

$$\frac{-1}{8Y^4}(Y^2 - X^2)(4YY'' - X'^2)(2Y^2 - a_2X'^2 + 4a_2YY'')$$

$$+\frac{1}{2}\left[\left(1 + \frac{X^2}{Y^2}\right)X'^2 - \frac{2XX'Y'}{B} + 2XX'' - 2YY''\right]$$

$$+\left[1 + \frac{a_2(4YY'' - X'^2)}{Y^2}\right] + \frac{1}{Y^4}[4a_2(X^2 - Y^2)$$

$$[X'^2(YY'' - 3Y'^2) + YX'(4Y'X'' - YX''') + Y(4Y'^2Y''$$

$$-Y(X''^2 + 2Y''^2 + 4Y'Y''') + 2Y^2Y^{iv})]] = 0 \tag{14}$$

$$\frac{1}{8Y^4}[a_2X'^2(X'^2 + 96Y'^2) - 8a_2Y(16X'Y'X'' + 5X'^2Y''$$

$$+16Y'^2Y'') + Y^2(-2X'^2 + 32a_2X'X''') + 16a_2(2X''^2 + 5Y''^2$$

$$+8Y'Y''') + 8Y^3(Y'' - 8a_2Y^{iv})] = 0 \tag{15}$$

$$\frac{1}{8Y^4}[4Y(YX'' - X'Y')(Y^2 - a_2X'^2 + 4a_2YY'')$$

$$+X(-a_2X'^2(5X'^2 + 96Y'^2) + 8a_2Y(16X'Y'X'' + 7X'^2Y''$$

$$+16Y'^2Y'') + 2Y^2(3X'^2 - 16a_2X'X''' - 8a_2(2X''^2 + 5Y''^2$$

$$+8Y'Y''')) - 8Y^3(Y'' - 8a_2Y^{iv}))] = 0 \tag{16}$$

$$\frac{1}{8Y^4}[-a_2X'^2(5X'^2 + 96Y'^2) + 8a_2Y16X'Y'X'' + 7X'^2Y''$$

$$+2Y^2[3X'^2 - 16a_2X'X''' - 8a_2(2X''^2 + 5Y''^2 + 8Y'Y''')]$$

$$-8Y^3(Y'' - 8a_2Y^{iv})] = 0 \tag{17}$$

From equations (16-17), we get:

$$4Y(YX'' - X'Y')[Y^2 + a_2(4YY'' - X'^2)] = 0 \tag{18}$$

Also, adding equations (15-17), we get:

$$4X'^2[Y^2 + a_2(4YY'' - X'^2)] = 0 \tag{19}$$

Similarly if we solve equations (13-14):

$$[Y^2 + a_2(4YY'' - X'^2)] = 0 \tag{20}$$

which is equivalent to

$$1 + a_2R = 0 \tag{21}$$

Here it should be noted that scalar curvature R of space time (4) becomes constant as it depends only upon the constant coefficient in this case. Also, we are avoiding trivial

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case, that is, $'a_n'$ coefficients in the Lagrangian (1) can not be zero.

If we consider $n = 3$, the field equation (12) in view of the values for the space-time (4) yields:

$$Y^2 + a_2(4YY'' - X'^2) + \frac{3a_3}{4Y^2}(4YY'' - X'^2)^2 = 0 \quad (22)$$

This implies

$$1 + 2a_2R + 3a_3R^2 = 0 \quad (23)$$

likewise we obtained for $n = 4$,

$$1 + 2a_2R + 3a_3R^2 + 4a_4R^3 = 0 \quad (24)$$

This result can be generalized for arbitrary natural number by mathematical induction. Hence

$$1 + 2a_2R + 3a_3R^2 + 4a_4R^3 + 5a_5R^4 + ... + na_nR^{n-1} = 0 \quad (25)$$

Here, we emphasise the fact that equation (25)gives a relation between $X(r)$ and $Y(r)$ by equation (5). The relation depends upon arbitrary constants. These arbitrary constants are introduced in the Hilbert-Lagrangian (1) in order to nullify the additional gravitational potential in gravitational wave equation [15].

IV. Some Examples

Space-time (5) is a solution of higher order field equations (10) under the condition (25). There are two parameter $X(r)$ and $Y(r)$ involved in space-time and we have only one condition. So we suppose that $X(r)$ and $Y(r)$ are dependent in a way that $X(r) = cY(r)$ where c is a constant. Under this condition, space-time would be:

$$ds^2 = dt^2 - dr^2 - (1 - c^2)Y^2d\phi^2 - dz^2 + 2cYdtd\phi \quad (26)$$

c is a constant which takes values between -1 to 1. Nature of parameter Y depends upon the roots of equation (25). The number of roots of equation (25) must be $n - 1$. We assume ω as a real root of equation (25) to avoid physical absurdness in space-time. If ω is negative then

$$Y(r) = c_1[Sin(\sqrt{\alpha\beta}r + c_2)]^{1/\omega}$$

and for positive values:

$$Y(r) = [c_1e^{\sqrt{\alpha\beta}r} + c_2e^{-\sqrt{\alpha\beta}r}]^{1/\omega}$$

where $\alpha = \omega/4$ and $\beta = (4 - c^2)/4$.

For a special value of $c = 1/\sqrt{2}$,$c_1 = \sqrt{2}$ and $c_2 = 0$ we get:

$$ds^2 = dt^2 - dr^2 + \frac{e^{2r}}{2}d\phi^2 - dz^2 + 2e^rdtd\phi \quad (27)$$

under the condition

$$1 + 4a_2 + 12a_3 + ... + na_n2^{n-1} = 0 \quad (28)$$

Equation (25) is the metric used by K. Godel in [16].

V. Discussion and Conclusion

We have obtained the solution of field equations of a higher order theory of gravity which is obtained by modifying Einstein theory incorporation based on conformal non-invariance in Godel space-time. Field equation (10) have been solved recursively for some natural numbers in cylindrically symmetric space-time. We observed that each successive term of equation (25) corresponds to successive higher order terms in equation (10) respectively. In order to predict precise nature of space-time we need one more constraint in the functions $X(r)$ and $Y(r)$. Using arbitrary and natural relation between $X(r)$ and $Y(r)$, we produced some interesting examples, one of them is equation (27).
In vacuum Einstein field equation implies that Ricci curvature tensor vanishes and so scalar curvature becomes null. It is very trivial situation for this space-time. But if we solve Einstein field equation considering cosmological constant in vacuum, we get cylindrical symmetric solution which may be given by equation (27). Interestingly we have found that solution of higher order and $f(R)$ theory of gravity (off-course without cosmological constant) in vacuum yields the same result in particular way. This conclusion strengthen the fact that coefficients of higher order term in this $f(R)$ gravity plays cumulative role same as cosmological constant in Einstein theory of gravity. This fact has also been established [17] for spherically symmetric space-time and [18] for Peres space-time.

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