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Contact Problem for a Foundation with a Rough Coating

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***Abstract*—We study the contact interaction between a rigid punch and a viscoelastic foundation with a thin rough coating. The case in which the punch and the coating surfaces are conformal (mutually repeating) is under consideration. Such problems can arise, for example, when the punch immerses into a solidifying coating before its complete solidification, e.g. into some sort of glue or new concrete. The shape of the coating roughness as well as the shape of the punch surface may be described by fast oscillating functions. We obtain basic integral equation and construct its solution by using a projection method. We also discuss qualitative behavior of main contact characteristics.**

***Index Terms*—conformal contact, foundation, coating, roughness, projection method**

INTRODUCTION

THE shape of real contacting surfaces is always definitely rough. Such a roughness can be efficiently described only by rapidly oscillating functions. We develop a projection method for solving multidimensional integral equations with rapidly oscillating function in initial data. This method allows us to solve the governing integral equation with high accuracy in the case when the classical method of separation of variables gives up to 100% mistake. A model contact problem for a coating with experimental profilogram for the shape of its surface is solved. Plots for the distribution of contact stress are presented.

I. STATEMENT OF THE PROBLEM

We assume that a viscoelastic layer with a coating lies on a rigid basis. At time τ_0, the force $P(t)$ with eccentricity $e(t)$ starts to indent a smooth rigid punch of width $2a$ (Fig. 1) into the surface of such a foundation. A specific characteristic of this contact interaction is the fact that the coating shape (the shape of the surface of the layer packet) coincides with the punch base shape. Such a contact interaction will be called conformal. The coating is assumed to be thin compared with the contact area, i.e., its thickness satisfies the condition $h(x) \ll 2a$. Both the thin coating and the lower layer of an arbitrary thickness H are made of viscoelastic materials. We denote the moments of their production by τ_1 and τ_2, respectively. We assume that the coating rigidity is less than the rigidity of the lower layer or they are of the same order of magnitude [1–12]. We consider the case of plane strain.

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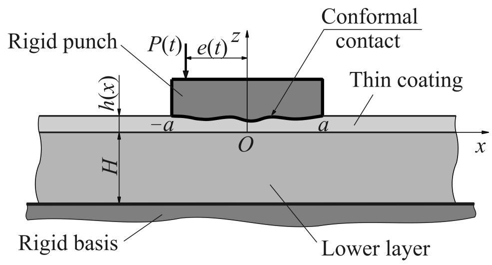

Fig. 1. Basic scheme of contact interaction

We note that the simple case of conformal contact is the contact of a punch with a plane base and a plane part of a solid (including basements with coating of constant thickness).

To obtain the integral equation of the problem, we replace the punch by some normally distributed load $p(x,t) = -q(x,t)$ acting upon the same region ($-a \le x \le a$) and equal to zero outside this region. Then the vertical displacement of the upper face of the foundation described above under the action of the normal force $q(x,t)$ can be written in the form [1, 4, 5]:

$$u_z(x,t) = (\mathbf{I} - \mathbf{V}_1)\frac{\theta q(x,t)h(x)}{E_1(t-\tau_1)} + (\mathbf{I} - \mathbf{V}_2)\mathbf{F}\frac{2(1-\nu_2^2)q(x,t)}{\pi E_2(t-\tau_2)}, \quad (1)$$

$$\mathbf{F}f(x,t) = \int_{-a}^{a} k_{\mathrm{pl}}\Big(\frac{x-\xi}{H}\Big) f(\xi,t)\,d\xi,$$

$$\mathbf{V}_k f(x,t) = \int_{\tau_0}^{t} K^{(k)}(t-\tau_k, \tau-\tau_k) f(x,\tau)\,d\tau,$$

$$K^{(k)}(t,\tau) = E_k(\tau)\frac{\partial}{\partial\tau}\Big[\frac{1}{E_k(\tau)} + C^{(k)}(t,\tau)\Big], \quad k = 1,\, 2,$$

where $E_k(t)$ are the Young moduli of the coating ($k = 1$) and the lower layer ($k = 2$) and ν_2 is Poisson's ratio of the lower layer; $\mathbf{I}$ is the identity operator; $\mathbf{V}_k$ are the Volterra integral operators with tensile creep kernels $K^{(k)}(t,\tau)$ ($k = 1,\, 2$); $C^{(k)}(t,\tau)$ ($k = 1,\, 2$) are the tensile creep functions; θ is a dimensionless coefficient depending on the contact conditions between coating and lower layer; in the case of a smooth coating-layer contact, we have

$$\theta = 1 - \nu_1^2,$$

and in the case of an perfect contact,

$$\theta = \frac{1 - \nu_1 - 2\nu_1^2}{1 - \nu_1},$$

where ν_1 is Poisson's ratio of the coating; $\mathbf{F}$ is the integral operator with the known kernel of the plane contact problem $k_{\rm pl}[(x-\xi)/H]$, which has the form [1]

$$k_{\rm pl}(s) = \int_0^\infty \frac{L(u)}{u}\cos(su)\,du,$$

and, in the case of a smooth contact between the lower layer and the rigid base,

$$L(u) = \frac{\cosh 2u - 1}{\sinh 2u + 2u},$$

and in the case of a perfect contact,

$$L(u) = \frac{2\varkappa \sinh 2u - 4u}{2\varkappa\cosh 2u + 4u^2 + 1 + \varkappa^2}, \quad \varkappa = 3 - 4\nu_2.$$

By equating the vertical displacements of the upper face of the coating with the displacement of the rigid punch and taking into account (1) and the fact that the contact interaction is conformal, we obtain the integral equation of our problem in the form

$$(\mathbf{I}-\mathbf{V}_1)\frac{\theta q(x,t)h(x)}{E_1(t-\tau_1)} + (\mathbf{I}-\mathbf{V}_2)\mathbf{F}\frac{2(1-\nu_2^2)q(x,t)}{\pi E_2(t-\tau_2)} = \delta(t) + \alpha(t)x \quad (-a \leqslant x \leqslant a), \quad (2)$$

where $\delta(t)$ is the punch settlement and $\alpha(t)$ is its tilt angle.

We supplement Eq. (2) with the condition of the punch equilibrium on the foundation

$$\int_{-a}^{a} q(\xi,t)\,d\xi = P(t), \quad \int_{-a}^{a} \xi q(\xi,t)\,d\xi = M(t). \quad (3)$$

Here $M(t) = e(t)P(t)$ denotes the moment of application of the force $P(t)$.

In (2) and (3), we make the change of variables by the formulas

$$\begin{aligned}
&x^* = x/a, \quad \xi^* = \xi/a, \quad t^* = t/\tau_0, \quad \tau^* = \tau/\tau_0,\\
&\tau_1^* = \tau_1/\tau_0, \quad \tau_2^* = \tau_2/\tau_0, \quad \lambda = H/a,\\
&\delta^*(t^*) = \frac{\delta(t)}{a}, \quad \alpha^*(t^*) = \alpha(t), \quad c^*(t^*) = \frac{E_2(t-\tau_2)}{E_1(t-\tau_1)},\\
&m^*(x^*) = \frac{\theta}{1-\nu_2^2}\frac{h(x)}{2a}, \quad q^*(x^*,t^*) = \frac{2(1-\nu_2^2)q(x,t)}{E_2(t-\tau_2)},\\
&P^*(t^*) = \frac{2P(t)(1-\nu_2^2)}{E_2(t-\tau_2)a}, \quad M^*(t^*) = \frac{2M(t)(1-\nu_2^2)}{E_2(t-\tau_2)a^2},\\
&\mathbf{V}_k^* f(x^*,t^*) = \int_1^{t^*} K_k(t^*,\tau^*)f(x^*,\tau^*)\,d\tau^*, \quad k = 1,\,2,\\
&K_1(t^*,\tau^*) = \frac{E_1(t-\tau_1)}{E_1(\tau-\tau_1)}\frac{E_2(\tau-\tau_2)}{E_2(t-\tau_2)}K^{(1)}(t-\tau_1,\tau-\tau_1)\tau_0,\\
&K_2(t^*,\tau^*) = K^{(2)}(t-\tau_2,\tau-\tau_2)\tau_0,\\
&\mathbf{F}^* f(x^*,t^*) = \int_{-1}^{1} k_{\rm pl}^*(x^*,\xi^*)f(\xi^*,t^*)\,d\xi^*,\\
&k_{\rm pl}^*(x^*,\xi^*) = \frac{1}{\pi}k_{\rm pl}\Big(\frac{x-\xi}{H}\Big) = \frac{1}{\pi}k_{\rm pl}\Big(\frac{x^*-\xi^*}{\lambda}\Big).
\end{aligned} \quad (4)$$

Then, omitting the asterisks, we obtain a mixed integral equation in the form

$$c(t)m(x)(\mathbf{I}-\mathbf{V}_1)q(x,t) + (\mathbf{I}-\mathbf{V}_2)\mathbf{F}q(x,t) = \delta(t) + \alpha(t)x \quad (-1 \leqslant x \leqslant 1) \quad (5)$$

with the auxiliary conditions

$$\int_{-1}^{1} q(\xi,t)\,d\xi = P(t), \quad \int_{-1}^{1} \xi q(\xi,t)\,d\xi = M(t). \quad (6)$$

Now we divide Eq. (5) by $\sqrt{m(x)}$ and introduce the notation

$$Q(x,t) = \sqrt{m(x)}q(x,t), \quad k(x,\xi) = \frac{k_{\rm pl}(x,\xi)}{\sqrt{m(x)}\sqrt{m(\xi)}},$$

$$\mathbf{A}Q(x,t) = \int_{-1}^{1} k(x,\xi)Q(\xi,t)\,d\xi.$$

Then integral equation (5) can be reduced to the following integral equation with the Hilbert–Schmidt kernel $k(x,\xi)$ (see, e.g. [13]):

$$c(t)(\mathbf{I}-\mathbf{V}_1)Q(x,t) + (\mathbf{I}-\mathbf{V}_2)\mathbf{A}Q(x,t) = \frac{\delta(t)}{\sqrt{m(x)}} + \frac{\alpha(t)x}{\sqrt{m(x)}} \quad (-1 \leqslant x \leqslant 1). \quad (7)$$

The auxiliary conditions (6) have the form

$$\int_{-1}^{1} \frac{Q(\xi,t)}{\sqrt{m(\xi)}}\,d\xi = P(t), \quad \int_{-1}^{1} \frac{Q(\xi,t)}{\sqrt{m(\xi)}}\xi\,d\xi = M(t). \quad (8)$$

In what follows, we construct the solution of the two-dimensional equation (7), which contains integral operators with constant as well as variable limits of integration, with the auxiliary conditions (8) taken into account.

There exist four different versions of the substitution: 1) the settlement and the tilt angle of the punch are known (i.e., the right-hand side of the integral equation is given); 2) the punch settlement and the force moment are known; 3) the tilt angle of the punch and the indenting force are known; 4) the indenting force and its moment application are known. Each of these statements is a separate problem with its specific integral operator, and it is necessary to construct four systems of eigenfunctions for these problems.

In what follows we will construct the solution of the third problem.

II. Solution for known force and tilt angle

Consider the following statement of a problem: it is necessary to find the eccentricity $e(t)$ of the indenting force $P(t)$ in order to provide the prescribed tilt angle $\alpha(t)$. It is considered that the force $P(t)$ is known.

We assume that function $\alpha(t)$ in (7) is given and the first auxiliary condition (8) holds and the second auxiliary condition (8) leads to the formula

$$e(t) = \frac{1}{P(t)}\int_{-1}^{1} \frac{Q(\xi,t)}{\sqrt{m(\xi)}}\xi\,d\xi. \quad (9)$$

We seek the solution of Eq. (7) under the first condition (8) using (9) in the class of functions continuous in time t in the Hilbert space $L_2[-1,1]$ (e.g., see [4]). To this end, we at first construct an orthonormal system of functions in $L_2[-1,1]$ which contains $1/\sqrt{m(x)}$ and remaining basis functions can be written as the products of functions depending on x and weight function $1/\sqrt{m(x)}$. The system of functions

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which satisfies the above conditions can be obtained by the following formulas [14]:

$$\int_{-1}^{1} p_i(\xi)p_j(\xi)\,d\xi = \delta_{ij}, \quad p_n(x) = \frac{P_n(x)}{\sqrt{m(x)}},$$

$$P_0(x) = \frac{1}{\sqrt{J_0}}, \quad J_n = \int_{-1}^{1} \frac{\xi^n}{m(\xi)}\,d\xi,$$

$$P_n(x) = \frac{1}{\sqrt{\Delta_{n-1}\Delta_n}} \begin{vmatrix} J_0 & J_1 & \cdots & J_n \\ J_1 & J_2 & \cdots & J_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x & \cdots & x^n \end{vmatrix}, \tag{10}$$

$$\Delta_{-1} = 1, \quad \Delta_n = \begin{vmatrix} J_0 & J_1 & \cdots & J_n \\ J_1 & J_2 & \cdots & J_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ J_n & J_{n+1} & \cdots & J_{2n} \end{vmatrix}.$$

Note that if $m(x) = \text{const}$ then the polynomials $p_n(x)$ are the orthonormal Legendre polynomials.

The Hilbert space $L_2[-1,1]$ can be presented as the direct sum of orthogonal subspaces $L_2[-1,1] = L_2^{(1)}[-1,1] \oplus L_2^{(2)}[-1,1]$, where $L_2^{(1)}[-1,1]$ is the Euclidean space with the basis $\{p_0(x)\}$ and $L_2^{(2)}[-1,1]$ is the Hilbert space with the basis $\{p_1(x),\, p_2(x),\, p_3(x), \ldots\}$. The integrand and the right-hand side of (7) can also be presented in the form of the algebraic sum of functions continuous in time t and ranging in $L_2^{(1)}[-1,1]$ and $L_2^{(2)}[-1,1]$, respectively, i.e.,

$$\begin{aligned} &Q(x,t) = Q_1(x,t) + Q_2(x,t), \\ &f(x,t) = f_1(x,t) + f_2(x,t), \\ &Q_1(x,t) = z_0(t)p_0(x), \\ &f_1(x,t) = \frac{\delta(t)}{\sqrt{m(x)}} = \sqrt{J_0}\delta(t)p_0(x), \\ &f_2(x,t) \equiv \frac{\alpha(t)x}{\sqrt{m(x)}}. \end{aligned}$$

Note that the formula for $Q(x,t)$ contains known term $Q_1(x,t)$ which is determined by the first auxiliary condition (1.8)

$$z_0(t) = \frac{P(t)}{\sqrt{J_0}},$$

and the term $Q_2(x,t)$ must be found. Conversely, for the right-hand side, one should find $f_1(x,t)$, while $f_2(x,t) \equiv 0$. These peculiarities permit one to class the resulting problem as a specific case of the generalized projection problem stated in [15].

We can introduce the orthogonal projection operator mapping the space $L_2[-1,1]$ onto subspace $L_2^{(1)}[-1,1]$

$$\mathbf{P}_1\phi(x,t) = \int_{-1}^{1} \phi(\xi,t)[p_0(x)p_0(\xi)]\,d\xi.$$

Obviously, the orthoprojector $\mathbf{P}_2 = \mathbf{I} - \mathbf{P}_1$ maps the space $L_2[-1,1]$ onto $L_2^{(2)}[-1,1]$. In addition, the following relations hold

$$\mathbf{P}_i f(x,t) = f_i(x,t), \quad \mathbf{P}_i Q(x,t) = Q_i(x,t), \quad i = 1,\, 2.$$

Using [15], we apply the orthogonal projection operator $\mathbf{P}_2$ to Eq. (7). As a result, we obtain the equation for determining $Q_2(x,t)$ with a known right-hand side

$$\begin{aligned} c(t)(\mathbf{I} - \mathbf{V}_1)Q_2(x,t) + (\mathbf{I} - \mathbf{V}_2)\mathbf{P}_2\mathbf{A}Q_2(x,t) \\ = -(\mathbf{I} - \mathbf{V}_2)\mathbf{P}_2\mathbf{A}Q_1(x,t). \end{aligned} \tag{11}$$

It is necessary to construct its solution in the form of an expansion in the eigenfunctions of the operator $\mathbf{P}_2\mathbf{A}$ which is a compact, strong positive, and self-adjoint operator from $L_2^{(2)}[-1,1]$ into $L_2^{(2)}[-1,1]$. The system of eigenfunctions of such an operator is a basis in the space $L_2^{(2)}[-1,1]$. The spectral problem for the operator $\mathbf{P}_2\mathbf{A}$ can be written in the form

$$\begin{aligned} &\mathbf{P}_2\mathbf{A}\varphi_k(x) = \gamma_k\varphi_k(x), \\ &\varphi_k(x) = \sum_{i=1}^{\infty} \varphi_i^{(k)} p_i(x), \quad k = 1,\, 2,\, \ldots, \\ &k(x,\xi) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} R_{mn}p_m(x)p_n(\xi), \\ &R_{mn} = \int_{-1}^{1}\int_{-1}^{1} k(x,\xi)p_m(x)p_n(\xi)\,dx\,d\xi, \\ &R_{nm} = R_{mn}, \quad m,\, n = 0,\, 1,\, \ldots, \end{aligned}$$

and hence

$$\sum_{n=1}^{\infty} R_{mn}\varphi_n^{(k)} = \gamma_k\varphi_m^{(k)}, \quad k,\, m = 1,\, 2,\, \ldots$$

We expand the function $Q_2(x,t)$ with respect to the new basis functions $\varphi_k(x)$ ($k = 1,\, 2,\, \ldots$) in $L_2^{(2)}[-1,1]$, i.e.,

$$Q_2(x,t) = \sum_{k=1}^{\infty} z_k(t)\varphi_k(x).$$

Substituting this equation into (11) and taking into account that the unknown expansion functions $z_k(t)$ ($k = 1,\, 2,\, \ldots$) can be determined by the formula

$$\begin{aligned} &z_k(t) = -(\mathbf{I} + \mathbf{W}_k)\frac{-\alpha(t)g_k^{\alpha} + (\mathbf{I} - \mathbf{V}_2)z_0(t)K_k^{\alpha}}{c(t) + \gamma_k}, \\ &K_k^{\alpha} = \sum_{n=1}^{\infty} R_{0n}\varphi_n^{(k)}, \\ &g_k^{\alpha} = \sqrt{\frac{J_0J_2 - J_1^2}{J_0}} \int_{-1}^{1} p_1(\xi)\varphi_k(\xi)\,d\xi = \varphi_1^{(k)}\sqrt{\frac{J_0J_2 - J_1^2}{J_0}}, \\ &\mathbf{W}_k f(x,t) = \int_1^t R_k^*(t,\tau)f(x,\tau)\,d\tau, \end{aligned}$$

where $R_k^*(t,\tau)$ ($k = 1,\, 2,\, \ldots$) is the resolvent of the kernel

$$K_k^*(t,\tau) = \frac{c(t)K_1(t,\tau) + \gamma_k K_2(t,\tau)}{c(t) + \gamma_k}.$$

Note that the final solution has the following structure

$$q(x,t) = \frac{1}{m(x)}\left[z_0(t)P_0(x) + \ldots\right],$$

i.e., one can explicitly write out the weight function $m(x)$ in the solution. Note that the coating thickness function $h(ax)$ is related to $m(x)$ in the relations of change of variables (4). The formulas obtained permit obtaining efficient analytic solutions for the layers with rough coatings which can be described by complicated and rapidly oscillating functions. Such a result can hardly be done by other known methods.

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Hence, we have found the contact pressure $q(x,t)$ under the punch and now can find the load eccentricity using (9):

$$e(t) = \frac{1}{P(t)}\left[\frac{J_1}{\sqrt{J_0}} z_0(t) + \sum_{i=1}^{\infty} g_i^{\alpha} z_i(t)\right] = \frac{J_1}{J_0} + \frac{1}{P(t)} \sum_{i=1}^{\infty} \varphi_1^{(i)} \sqrt{\frac{J_0 J_2 - J_1^2}{J_0}} z_i(t).$$

In particular it is possible to obtain the absence of the tilt of the punch at any instant ($\alpha(t) \equiv 0$).

In order to find the unknown punch settlement we act Eq. (1.7) by operator $\mathbf{P}_1$

$$\delta(t) = \frac{1}{\sqrt{J_0}}\Big\{-\alpha(t)\frac{J_1}{\sqrt{J_0}} + c(t)(\mathbf{I} - \mathbf{V}_1) z_0(t) + (\mathbf{I} - \mathbf{V}_2)\Big[R_{00} z_0(t) + \sum_{k=1}^{\infty} K_k^{\alpha} z_k(t)\Big]\Big\}.$$

III. Solution for a given settlement and moment

We consider one more version of the problem statement. Let the punch settlement $\delta(t)$ and moment $M(t)$ (or eccentricity $e(t)$)are prescribed. It is necessary to find the force $P(t)$, the tilt angle $\alpha(t)$ and the contact pressure $q(x,t)$.

The second auxiliary condition (8) gives the formula for the force

$$P(t) = \int_{-1}^{1} \frac{Q(\xi,t)}{\sqrt{m(\xi)}} \, d\xi, \tag{12}$$

and the first auxiliary condition (8) holds.

The Hilbert space $L_2[-1,1]$ can be represented as the direct sum of orthogonal subspaces $L_2[-1,1] = \tilde{L}_2^{(1)}[-1,1] \oplus \tilde{L}_2^{(2)}[-1,1]$, where $\tilde{L}_2^{(1)}[-1,1]$ is the Euclidean space with basis $\{\tilde{p}_0(x)\}$ and $\tilde{L}_2^{(2)}[-1,1]$ is the Hilbert space with basis $\{\tilde{p}_1(x), \tilde{p}_2(x), \tilde{p}_3(x), \ldots\}$. Here the basis functions can be found as follows:

$$\tilde{p}_0(x) = \frac{x}{\sqrt{J_2 m(x)}},$$
$$\tilde{p}_1(x) = \frac{J_2 - J_1 x}{\sqrt{J_2 (J_0 J_2 - J_1^2) m(x)}},$$
$$\tilde{p}_k(x) \equiv p_k(x), \quad k = 2, 3, \ldots$$

For the integrand and the right-hand side of (7):

$$Q(x,t) = \tilde{Q}_1(x,t) + \tilde{Q}_2(x,t),$$
$$f(x,t) = \tilde{f}_1(x,t) + \tilde{f}_2(x,t),$$
$$\tilde{f}_1(x,t) = \frac{\alpha(t) x}{\sqrt{m(x)}}, \quad \tilde{f}_2(x,t) = \frac{\delta(t)}{\sqrt{m(x)}},$$

where $\tilde{Q}_i(x,t)$, $\tilde{f}_i(x,t)$ are functions continuous in time t and ranging in $\tilde{L}_2^{(1)}[-1,1]$ and $\tilde{L}_2^{(2)}[-1,1]$, respectively.

The representation for $Q(x,t)$ contains the known first term, and the second term is to be found. Conversely, for the right-hand side, one should find $\tilde{f}_1(x,t)$, while $\tilde{f}_2(x,t)$ in unknown.

The orthogonal projection operator, mapping the space $L_2[-1,1]$ onto $\tilde{L}_2^{(1)}[-1,1]$ can be introduced by formulas

$$\tilde{\mathbf{P}}_1 \phi(x,t) = \int_{-1}^{1} \phi(\xi,t) \tilde{p}_0(x) \tilde{p}_0(\xi) \, d\xi.$$

The orthoprojector $\tilde{\mathbf{P}}_2 = \mathbf{I} - \tilde{\mathbf{P}}_1$ maps the space $L_2[-1,1]$ onto $\tilde{L}_2^{(2)}[-1,1]$.

We apply the orthogonal projection operator $\tilde{\mathbf{P}}_2$ to Eq. (7). As a result, we obtain the equation for determining $\tilde{Q}_2(x,t)$ with a known right-hand side. It is necessary to construct its solution in the form of a series in the eigenfunctions of the operator $\tilde{\mathbf{P}}_2\mathbf{A}$. The spectral problem for this operator can be written in the form

$$\tilde{\mathbf{P}}_2 \mathbf{A} \tilde{\varphi}_k(x) = \tilde{\gamma}_k \tilde{\varphi}_k(x),$$
$$\tilde{\varphi}_k(x) = \sum_{i=1}^{\infty} \tilde{\varphi}_i^{(k)} \tilde{p}_i(x), \quad k = 1, 2, \ldots,$$
$$k(x,\xi) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \tilde{R}_{mn} \tilde{p}_m(x) \tilde{p}_n(\xi),$$
$$\tilde{R}_{mn} = \int_{-1}^{1} \int_{-1}^{1} k(x,\xi) \tilde{p}_m(x) \tilde{p}_n(\xi) \, dx \, d\xi,$$
$$\tilde{R}_{nm} = \tilde{R}_{mn}, \quad m, n = 0, 1, \ldots,$$

and, hence,

$$\sum_{n=1}^{\infty} \tilde{R}_{mn} \tilde{\varphi}_n^{(k)} = \tilde{\gamma}_k \tilde{\varphi}_m^{(k)}, \quad k, m = 1, 2, \ldots$$

Final formulas for contact pressure under the punch have the form

$$q(x,t) = \frac{Q(x,t)}{\sqrt{m(x)}},$$
$$Q(x,t) = \tilde{z}_0(t) \tilde{p}_0(x) + \sum_{k=1}^{\infty} \tilde{z}_k(t) \tilde{\varphi}_k(x),$$
$$\tilde{z}_0(t) = \frac{M(t)}{\sqrt{J_2}},$$
$$\tilde{z}_k(t) = -(\mathbf{I} + \widetilde{\mathbf{W}}_k) \frac{-\delta(t) g_k^{\delta} + (\mathbf{I} - \mathbf{V}_2) \tilde{z}_0(t) K_k^{\delta}}{c(t) + \tilde{\gamma}_k},$$
$$K_k^{\delta} = \sum_{n=1}^{\infty} \tilde{R}_{0n} \tilde{\varphi}_n^{(k)},$$
$$g_k^{\delta} = \sqrt{\frac{J_0 J_2 - J_1^2}{J_2}} \int_{-1}^{1} \tilde{p}_1(x) \tilde{\varphi}_k(\xi) \, d\xi = \tilde{\varphi}_1^{(k)} \sqrt{\frac{J_0 J_2 - J_1^2}{J_2}},$$
$$\widetilde{\mathbf{W}}_k f(x,t) = \int_1^t \tilde{R}_k^*(t,\tau) f(x,\tau) \, d\tau.$$

Kernels $\tilde{R}_k^*(t,\tau)$ ($k = 1, 2, \ldots$) are the resolvents of the kernels

$$\tilde{K}_k^*(t,\tau) = \frac{c(t) K_1(t,\tau) + \tilde{\gamma}_k K_2(t,\tau)}{c(t) + \tilde{\gamma}_k}.$$

So, having defined the contact pressure $q(x,t)$ under the punch, using (12) we can find the force

$$P(t) = \frac{J_1}{\sqrt{J_2}} \tilde{z}_0(t) + \sum_{i=1}^{\infty} g_i^{\delta} \tilde{z}_i(t) = \frac{J_1}{J_2} M(t) + \sum_{i=1}^{\infty} \tilde{\varphi}_1^{(i)} \sqrt{\frac{J_0 J_2 - J_1^2}{J_2}} \tilde{z}_i(t). \tag{13}$$

We can obtain exact formula for the force $P(t)$ if the eccentricity $e(t)$ is given and layers are made of elastic

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materials. In this case

$$\begin{aligned}
&c(t) \equiv c,\\
&\mathbf{V}_i = \widetilde{\mathbf{W}}_k = \mathbf{0}, \quad i = 1, 2, \quad k = 1, 2, \ldots,\\
&K_1(t,\tau) = K_2(t,\tau) \equiv 0.
\end{aligned}$$

and we can write formula (13) in the form

$$\begin{aligned}
P(t) &= \frac{J_1}{J_2}M(t) + \sum_{i=1}^{\infty} g_i^\delta \tilde{z}_i(t)\\
&= \frac{J_1}{J_2}M(t) + \sum_{i=1}^{\infty} g_i^\delta \frac{\delta(t)g_i^\delta - \tilde{z}_0(t)K_i^\delta}{c+\tilde{\gamma}_i}\\
&= \frac{J_1}{J_2}M(t) + \delta(t)\sum_{i=1}^{\infty}\frac{(g_i^\delta)^2}{c+\tilde{\gamma}_i} - \frac{M(t)}{\sqrt{J_2}}\sum_{i=1}^{\infty}\frac{g_i^\delta K_i^\delta}{c+\tilde{\gamma}_i}.\\
&= \frac{M(t)}{\sqrt{J_2}}\left[\frac{J_1}{\sqrt{J_2}} - \sum_{i=1}^{\infty}\frac{g_i^\delta K_i^\delta}{c+\tilde{\gamma}_i}\right] + \delta(t)\sum_{i=1}^{\infty}\frac{(g_i^\delta)^2}{c+\tilde{\gamma}_i}.
\end{aligned}$$

Using notation $M(t) = P(t)e(t)$ one can obtain equation for the force $P(t)$

$$P(t) = \frac{P(t)e(t)}{\sqrt{J_2}}\left[\frac{J_1}{\sqrt{J_2}} - \sum_{i=1}^{\infty}\frac{g_i^\delta K_i^\delta}{c+\tilde{\gamma}_i}\right] + \delta(t)\sum_{i=1}^{\infty}\frac{(g_i^\delta)^2}{c+\tilde{\gamma}_i}.$$

Hence, we obtain

$$P(t) = \frac{\delta(t)\displaystyle\sum_{i=1}^{\infty}\frac{(g_i^\delta)^2}{c+\tilde{\gamma}_i}}{1 - e(t)\left[\dfrac{J_1}{J_2} - \dfrac{1}{\sqrt{J_2}}\displaystyle\sum_{i=1}^{\infty}\frac{g_i^\delta K_i^\delta}{c+\tilde{\gamma}_i}\right]}. \qquad (14)$$

This relation allows one to define the magnitude of the indenting force using measured force eccentricity and punch settlement.

Note that the expansion coefficients $\tilde{R}_{mn}$ of the kernel $k(x,\xi)$ can be written through coefficients R_{mn}. Basis functions $\tilde{p}_k(x)$ $(k = 0, 1)$ can be written through $p_k(x)$ $(k = 1, 2)$,

$$\begin{aligned}
&\tilde{p}_0(x) = k_0^{(0)}p_0(x) + k_1^{(0)}p_1(x),\\
&\tilde{p}_1(x) = k_0^{(1)}p_0(x) + k_1^{(1)}p_1(x),\\
&k_0^{(0)} = -k_1^{(1)} = \frac{J_1}{\sqrt{J_0J_2}},\\
&k_1^{(0)} = k_0^{(1)} = \sqrt{1 - \frac{J_1^2}{J_0J_2}},
\end{aligned}$$

and expansion coefficients $\tilde{R}_{mn}$ can be written in the form

$$\begin{aligned}
\tilde{R}_{mn} &= \sum_{i,j=1}^{2} k_i^{(m)}k_j^{(n)}R_{ij} = k_0^{(m)}k_0^{(n)}R_{00}\\
&\quad + [k_0^{(m)}k_1^{(n)} + k_1^{(m)}k_0^{(n)}]R_{01} + k_1^{(m)}k_1^{(n)}R_{11},\\
&\qquad m,\, n = 0,\, 1,\\
\tilde{R}_{nk} &= \tilde{R}_{kn} = \sum_{i=1}^{2} k_i^{(n)}R_{ik} = k_0^{(n)}R_{0k} + k_1^{(n)}R_{1k},\\
&\qquad n = 0,\, 1, \quad k = 2,\, 3,\, \ldots,\\
\tilde{R}_{kl} &= R_{kl}, \quad k,\, l = 2,\, 3,\, \ldots
\end{aligned}$$

Now we can find the unknown tilt angle

$$\begin{aligned}
\alpha(t) = \frac{1}{\sqrt{J_2}}\Bigg\{&-\delta(t)\frac{J_1}{\sqrt{J_2}} + c(t)(\mathbf{I} - \mathbf{V}_1)\tilde{z}_0(t)\\
&+ (\mathbf{I} - \mathbf{V}_2)\left[\tilde{R}_{00}\tilde{z}_0(t) + \sum_{k=1}^{\infty}K_k^\delta \tilde{z}_k(t)\right]\Bigg\}.
\end{aligned}$$

In the case of elastic layers and given eccentricity we obtain the formula

$$\begin{aligned}
\alpha(t) = \Bigg\{&\delta(t)\left[\frac{1}{\sqrt{J_2}}\sum_{k=1}^{\infty}\frac{g_k^\delta K_k^\delta}{c+\tilde{\gamma}_k} - \frac{J_1}{J_2}\right]\\
&+ \frac{P(t)e(t)}{J_2}\left[c + \tilde{R}_{00} - \sum_{k=1}^{\infty}\frac{(K_k^\delta)^2}{c+\tilde{\gamma}_k}\right]\Bigg\}.
\end{aligned}$$

Using equation (14) one can obtain the following formula

$$\alpha(t) = \delta(t)\left\{k_2 + \frac{k_1 e(t)}{J_2[1{+}k_2e(t)]}\left[c + \tilde{R}_{00} - \sum_{k=1}^{\infty}\frac{(K_k^\delta)^2}{c{+}\tilde{\gamma}_k}\right]\right\},$$

where

$$k_1 = \sum_{i=1}^{\infty}\frac{(g_i^\delta)^2}{c+\tilde{\gamma}_i}, \quad k_2 = \frac{1}{\sqrt{J_2}}\sum_{i=1}^{\infty}\frac{g_i^\delta K_i^\delta}{c+\tilde{\gamma}_i} - \frac{J_1}{J_2}$$

IV. SOLUTION FOR A GIVEN SETTLEMENT AND TILT ANGLE

This method allowed us to construct the solution of the problem with given right-hand side, i.e., when the settlement $\delta(t)$ and tilt angle $\alpha(t)$ of the punch are given. It is necessary to define the force $P(t)$ and the eccentricity $e(t)$ (or the moment $M(t)$).

Auxiliary conditions (8) allow to obtain formulas for definition of functions $P(t)$ and $e(t)$:

$$\begin{aligned}
P(t) &= \int_{-1}^{1}\frac{Q(\xi,t)}{\sqrt{m(\xi)}}\,d\xi,\\
e(t) &= \frac{1}{P(t)}\int_{-1}^{1}\frac{Q(\xi,t)}{\sqrt{m(\xi)}}\xi\,d\xi.
\end{aligned} \qquad (15)$$

The right-hand side of Eq. (7) is known and $\hat{\mathbf{P}}_1 = \mathbf{0}$. It is quite clear that $\hat{\mathbf{P}}_2 = \mathbf{I}$. The spectral problem for the operator $\hat{\mathbf{P}}_2\mathbf{A}$ can be written in the form

$$\begin{aligned}
&\hat{\mathbf{P}}_2\mathbf{A}\hat{\varphi}_k(x) = \hat{\gamma}_k\hat{\varphi}_k(x),\\
&\hat{\varphi}_k(x) = \sum_{i=1}^{\infty}\hat{\varphi}_i^{(k)}p_i(x), \quad k = 0,\, 1,\, \ldots,\\
&k(x,\xi) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}R_{mn}p_m(x)p_n(\xi),\\
&R_{mn} = \int_{-1}^{1}\int_{-1}^{1}k(x,\xi)p_m(x)p_n(\xi)\,dx\,d\xi,\\
&R_{nm} = R_{mn}, \quad m,\, n = 0,\, 1,\, \ldots,
\end{aligned}$$

and hence

$$\sum_{n=1}^{\infty}R_{mn}\hat{\varphi}_n^{(k)} = \hat{\gamma}_k\hat{\varphi}_m^{(k)}, \quad k,\, m = 0,\, 1,\, \ldots$$

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We avoid technical details and give the final formulas for contact pressure under the punch

$$
\begin{aligned}
q(x,t) &= \frac{Q(x,t)}{\sqrt{m(x)}},\\
Q(x,t) &= \sum_{k=0}^{\infty} \hat{z}_k(t)\hat{\varphi}_k(x),\\
\hat{z}_k(t) &= (\mathbf{I}+\widehat{\mathbf{W}}_k)\frac{\delta(t)\hat{g}_k^{\delta}+\alpha(t)\hat{g}_k^{\alpha}}{c(t)+\hat{\gamma}_k},\\
\widehat{\mathbf{W}}_k f(x,t) &= \int_1^t \hat{R}_k^*(t,\tau)f(x,\tau)\,d\tau,\\
\hat{g}_k^{\alpha} &= \frac{J_1}{\sqrt{J_0}}\int_{-1}^{1} p_0(\xi)\hat{\varphi}_k(\xi)\,d\xi\\
&\quad + \sqrt{\frac{J_0J_2-J_1^2}{J_0}}\int_{-1}^{1} p_1(\xi)\hat{\varphi}_k(\xi)\,d\xi\\
&= \hat{\varphi}_0^{(k)}\frac{J_1}{\sqrt{J_0}}+\hat{\varphi}_1^{(k)}\sqrt{\frac{J_0J_2-J_1^2}{J_0}},\\
\hat{g}_k^{\delta} &= \sqrt{J_0}\int_{-1}^{1} p_0(\xi)\hat{\varphi}_k(\xi)\,d\xi = \hat{\varphi}_0^{(k)}\sqrt{J_0}.
\end{aligned}
$$

The functions $p_k(x)$ can be calculated using Eq. (10) where $\hat{R}_k^*(t,\tau)$ $(k=0,\,1,\,\ldots)$ is resolvent of the kernel

$$
\hat{K}_k^*(t,\tau) = \frac{c(t)K_1(t,\tau)+\hat{\gamma}_kK_2(t,\tau)}{c(t)+\hat{\gamma}_k}.
$$

Hence, the contact pressure $q(x,t)$ under the punch has been obtained using (15) and one can find the force $P(t)$ and the eccentricity $e(t)$:

$$
\begin{aligned}
P(t) &= \sum_{i=0}^{\infty}\hat{\varphi}_0^{(i)}\sqrt{J_0}\,\hat{z}_i(t),\\
e(t) &= \frac{1}{P(t)}\sum_{i=0}^{\infty}\left(\hat{\varphi}_0^{(i)}\frac{J_1}{\sqrt{J_0}}+\hat{\varphi}_1^{(i)}\sqrt{\frac{J_0J_2-J_1^2}{J_0}}\right)\hat{z}_i(t).
\end{aligned}
$$

V. CONCLUSIONS

In the present paper, we consider the conformal contact of solids which is the generalization of the interaction between a flat punch and a plane surface. We pose and solve plane problems of conformal contact between viscoelastic aging foundations with rough coatings and rigid punches. We show that it is important to take the conformal contact into account. We also demonstrate the efficiency of the projection method for solving multidimentional integral equations of contact mechanics. The solution of the contact problem has been obtained in analytic form and the formula for the contact stress contains the shape function of the coating surface in explicit form. The latter allows one to perform computations for the actual shape of the coating roughness which usually can be described by rapidly oscillating function (Fig. 2).

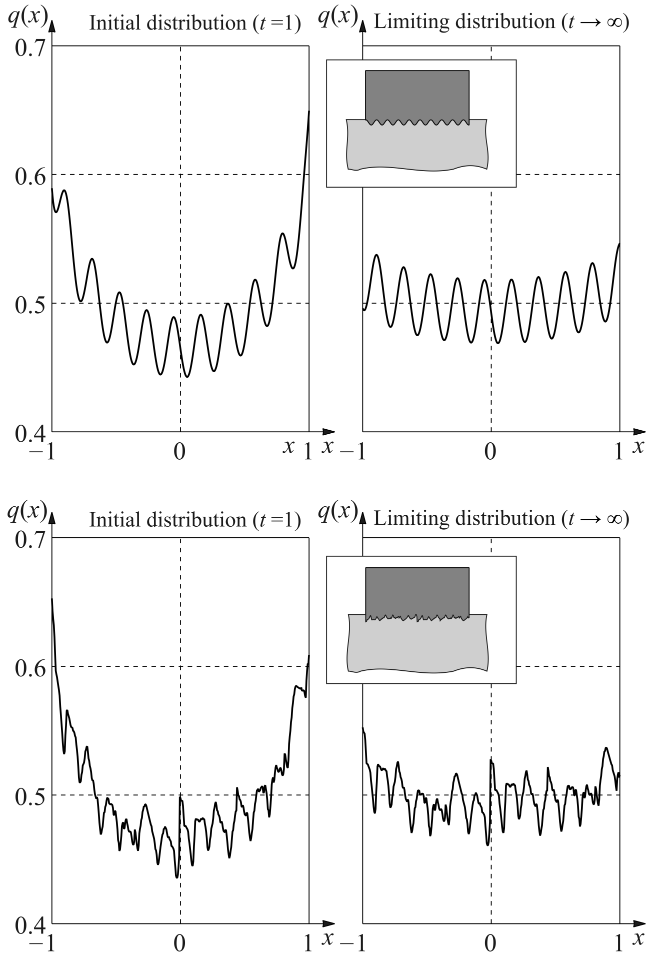

Fig. 2. Contact stress for a model and true roughness of the coating

REFERENCES

[1] V.M. Alexandrov and S.M. Mkhitaryan, *Contact Problems for Bodies with Thin Coatings and Interlayers [in Russian]*. Nauka Publ., Moscow, 1983.

[2] I. G. Goryacheva, *Contact Mechanics in Tribology*. Springer, Dordrecht, 1998.

[3] N. Kh. Arutyunyan, A. V. Manzhirov, and V. E. Naumov *Contact Problems in Mechanics of Growing Bodies [in Russian]*. Nauka Publ., Moscow, 1991.

[4] N. Kh. Arutyunyan and A. V. Manzhirov, *Contact Problems in the Theory of Creep [in Russian]*. Izd-vo Inst. Mekaniki NAN Armenii, Yerevan, 1999.

[5] A. V. Manzhirov, "Contact Problems for Inhomogeneous Aging Viscoelastic Solids," in *Mechanics of Contact Interactions*, Ed. by I. I. Vorovich and V. M. Alexandrov. Fizmatlit, Moscow, 2001, pp. 607–621.

[6] A. V. Manzhirov, "Axisymmetric contact problems for non-uniformly aging layered viscoelastic foundations."*J Appl Math Mech*, vol. 47, no. 4, pp. 558–566, 1983.

[7] A. V. Manzhirov, "On a method of solving two-dimensional integral equations of axisymmetric contact problems for bodies with complex rheology." *J Appl Math Mech*, vol. 49, no. 6, pp. 777–782, 1985.

[8] V. M. Alexandrov and A. V. Manzhirov, "Two-dimensional integral equations in applied mechanics of solids." *J Appl Mech Tech Phys*, vol. 28, no. 5, pp. 781–786, 1987.

[9] A. V. Manzhirov and V. A. Chernysh, "On the interaction of a rigid reinforcing sleeve and inhomogeneous aging high-pressure pipes." *Mech Solids*, vol. 22, no. 6, 1988.

[10] A. V. Manzhirov and V. A. Chernysh, "Contact problem for a layered inhomogeneous aging cylinder reinforced by a rigid ring." *J Appl Mech Tech Phys*, vol. 30, no. 6, pp. 894–900, 1990.

[11] K. E. Kazakov and A. V. Manzhirov, "Conformal contact between layered foundations and punches." *Mech Solids*, vol. 32, no. 3, pp. 512–524, 2008.

[12] K. E. Kazakov, "Modeling of contact interaction for solids with inhomogeneous coatings." *J Phys Conf Ser*, vol. 181(012013), 2009.

[13] E. Goursat, *Cours d'Analyse Mathématique*, Tome III. *Intégrales Infiniment Voisines; Équations aux Dérivées Partielles du Second Ordre; Équations Intégrales; Calcul des Variations*. Gauthier–Villars, Paris, 1927.

[14] G. Szegö, *Orthogonal Polynomials*. Amer. Math. Soc, Providence, 1959.

[15] A.V. Manzhirov and A.D. Polyanin, *Handbook of Mathematics for Engineers and Scientists*. Chapman & Hall/CRC Press, Boca Raton, London, 2007.

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