Stabilization of Equilibrium State of Nonlinear Hamiltonian Systems

Aleksandr S. Andreev, Sergey P. Bezglasnyi

***Abstract*— We consider a controlled nonlinear mechanical system described by the Hamilton's canonical equations. We determine the control *u* acting to the mechanical system which allow to the asymptotic stability are derived. We solve the problem of stabilization by the direct Lyapunov's method and the method of limiting functions and systems. In this case we can use the Lyapunov's functions having nonpositive derivatives.**

***Index Terms*— controlled mechanical system, equilibrium state, limiting functions, stabilization, Lyapunov function**

I. Introduction

THE behavior of many systems of importance in engineering practice is governed by Hamilton's canonical equations [1]

$$\dot{q} = \frac{\partial H(p,q)}{\partial p}, \quad \dot{p} = -\frac{\partial H(p,q)}{\partial q} \tag{1}$$

where $q = (q_1,...,q_n)^{\mathrm{T}}$ is the n-vector of generalized coordinates, $p = (p_1,...,p_n)^{\mathrm{T}}$ is the n-vector of generalized momenta and $H(p,q)$ is the Hamiltonian of the system. Here T denotes transpose of a matrix. We assume that

$$\frac{\partial H(0,0)}{\partial p} = 0, \quad \frac{\partial H(0,0)}{\partial q} = 0 \tag{2}$$

That is, $p = q = 0$ is an equilibrium state of the system (1). It is well known that $p = q = 0$ of (1) is stable in the sense of Lyapunov [2] if $H(p,q)$ is positive definite. Indeed, along any trajectory of (1), we have that

$$\frac{dH(p,q)}{dt} = \left(\frac{\partial H}{\partial q}\right)^{\mathrm{T}} \dot{q} + \left(\frac{\partial H}{\partial p}\right)^{\mathrm{T}} \dot{p} \tag{3}$$

Using (1) in (3), we immediately see that $dH/dt = 0$.

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A. S. Andreev is with the Ulyanovsk State University, Ulyanovsk, 432017 Russia (e-mail: andreevas@ulsu.ru).

S. P. Bezglasnyi is with the Further Mathematics Department, Samara University, Samara, 443086 Russia (corresponding author to provide phone: 8-917-104-1316; e-mail: bezglasnsp@rambler.ru).

Therefore, when $H(p,q)$ is positive definite, the choice of the Lyapunov function $V = H(p,q)$ establishes stability of the origin of (1). Of course, stability of the origin implies that for small perturbations in the initial state $(q(0), p(0))$ the motion of the system (1) remains bounded.

In many engineering problems, mere stability of the origin is not enough, and it is required that the trajectory of the system converge to the origin as time goes to ∞ if the initial state is perturbed from zero. For such cases, additional control signal are provided in the system (1) to give a modified system of the form

$$\dot{q} = \frac{\partial H(p,q)}{\partial p}, \quad \dot{p} = -\frac{\partial H(p,q)}{\partial q} + Bu \tag{4}$$

Where B is constant $n \times m$ matrix and $u(t) \in R^m$ is the control. The form of B depends on the configuration of the actuators in the system.

We are interested in obtaining control law of the form

$$u(t) = u(t, q(t), p(t)) \in R^m, \; u(t,0,0) = 0 \tag{5}$$

such that the equilibrium state $q = p = 0$ of the system (4) and (5) is asymptotically stable. Physically, selection of u may be seen as the problem of proving appropriate damping (active or passive) in the conservative system (1) such that the origin of the modified system becomes asymptotically stable. However, for general complex nonlinear system, it is far from trivial to find an appropriate control law to ensure asymptotic stability.

The problem of stabilization of conservative linear and a restricted class of nonlinear Hamiltonian system has been considered [3, 4]. Using well known invariance principle of LaSalle [5], sufficient conditions for the asymptotic stability and instability of the origin of neoconservative Hamilton's system are derived in [6]. The problem of stability and stabilization of equilibrium position of non-autonomous Lagrang's systems are solved in [7, 8]. In this work we will state and solve the problem of stabilization of zero state (equilibrium position) of non-autonomous Hamilton's systems on the basis of the direct Lyapunov method [2] and the method of limiting functions and systems of equations [9, 10]. This allows to extend the class of the Lyapunov's functions to the functions with nonpositive derivatives. The method of limiting functions and systems is well established. For example, an algorithm for constructing of controls, arbitrarily defined stabilizing the unsteady motion

of nonautonomous mechanical systems was obtained. With its use were solved the following tasks. The problem of the realization of the asymptotically stable program motion of a gyrostat with variable inertia moments was solved in note [11]. The task about orbital maneuvering of the satellite on a circular orbit with the use of tethered space systems was solved in paper [12]. In [13] classes of controls solving the problem of stabilization of the various sets of stationary motions of a gyrostat with the fluid were constructed. The conditions on the system parameters for asymptotical stable motions were found. The gravitational stabilization problem for satellites with movable mass was solved in paper [14].

The organization of the paper is as follows. Sections 2 presents the problem of stabilization of equilibrium state of non-autonomous mechanical system described by the system of differential equations in Hamilton's canonical form and contains additional assumptions and constructions. In section 3 we obtain and prove general theorems about the stabilization of equilibrium position of Hamilton's systems. The obtained theorems generalize and develop and develop results of [6-8, 10, 15].

II. Formulation of the Problem

A. Equation of Motion

We consider a controlled system, the motion of which described by the system of differential equations in Hamilton's form

$$\dot{q} = \frac{\partial H(t,q,p)}{\partial p}, \quad \dot{p} = -\frac{\partial H(t,q,p)}{\partial q} + Bu \tag{6}$$

where $q = (q_1, \ldots, q_n)^{\mathrm{T}}$ is the n-vector of generalized coordinates in the real linear space R^n with norm $\|q\|$, $p = (p_1, \ldots, p_n)^{\mathrm{T}}$ is the n-vector of generalized momenta, $p \in R^n$ with norm $\|p\|$ and $H(t,q,p)$ is the Hamiltonian of the system. The right-hand side in (6) is defined for a class $U = \{u(t,q,p) : u(t,0,0) = 0\}$ of control $u(t,q,p) \in C(G)$, $G = R^+ \times \Gamma$, $R^+ = [0,+\infty]$, $\Gamma = \{\|q\| < L,\ \|p\| < L = const > 0\}$ It is continuous and satisfies the conditions for the existence and uniqueness of solutions in G.

We now state the problem on stabilization of the equilibrium position of system (6). Namely, we have to find the control $u(t,q,p) \in U$ and the conditions on the right-hand side of system (6) making the zero solution $q = p = 0$ of (6) asymptotically stable.

B. Additional Assumptions and Constructions

Along the solution of (6), we have that

$$\frac{dH(t,q,p)}{dt} = \left(\frac{\partial H}{\partial q}\right)^{\mathrm{T}} \dot{q} + \left(\frac{\partial H}{\partial p}\right)^{\mathrm{T}} \dot{p} + \frac{\partial H}{\partial t} = \tag{7}$$
$$= \left(\frac{\partial H}{\partial q}\right)^{\mathrm{T}} \left(\frac{\partial H}{\partial p}\right) + \left(\frac{\partial H}{\partial p}\right)^{\mathrm{T}} \left(-\frac{\partial H}{\partial q} + Bu\right) + \frac{\partial H}{\partial t} = \left(\frac{\partial H}{\partial p}\right)^{\mathrm{T}} Bu + \frac{\partial H}{\partial t}$$

We shall consider three classes of control vectors u^0 as given below

$$u^0(t,q,p) = K(t,q,p)\frac{\partial H(t,q,p)}{\partial p}, \tag{8}$$

$$\frac{\partial H}{\partial t} + pDp^{\mathrm{T}} \le f(t,H) - \mu\|u^0\|^2,\ D = BK + (BK)^{\mathrm{T}} \le 0 \tag{9}$$

($\mu > 0$, function $f(t,v)$ is continuously differentiable with respect to v, $f(t,0) \equiv 0$),

$$u_i^0 = \begin{cases} u_{i1} & if\ -\left(\frac{\partial H}{\partial p}\right)^{\mathrm{T}} b_i < u_{i1} \\ -\left(\frac{\partial H}{\partial p}\right)^{\mathrm{T}} b_i & if\ u_{i1} \le -\left(\frac{\partial H}{\partial p}\right)^{\mathrm{T}} b_i \le u_{i2} \\ u_{i2} & if\ -\left(\frac{\partial H}{\partial p}\right)^{\mathrm{T}} b_i > u_{i2} \end{cases} \tag{10}$$

where $u^0 = (u_1^0, \ldots, u_m^0)^{\mathrm{T}}$ and $B = (b_1, \ldots, b_m)$,

$$u_{i1} \le u_i^0(t) \le u_{i2},\ u_{i1} < 0,\ u_{i2} > 0,$$

$$\frac{\partial H}{\partial t} \le f(t,H) + (1-\mu)\|u^0\|^2.$$

Suppose that for same $u^0(t,q,p)$ the right-hand side of (6) is bounded on each compact set $\mathrm{M} \subset \Gamma$ and satisfies the Lipschitz condition uniformly in *(q,p)* with respect to *t*. Then it satisfies the precompactness conditions in *G* in some functional space F_Φ [9] and with the system of equations (6) one can associate [9] a set of limit systems

$$\dot{q} = \left(\frac{\partial H(t,q,p)}{\partial p}\right)^*, \quad \dot{p} = -\left(\frac{\partial H(t,q,p)}{\partial q}\right)^* + Bu^*, \tag{11}$$

where

$$\left(\frac{\partial H(t,q,p)}{\partial p}\right)^* = \frac{d}{dt}\left(\lim_{t_n \to \infty} \int_0^t \left(\frac{\partial H(t_n + \tau, q, p)}{\partial p}\right) d\tau\right), \tag{12}$$

$$\left(\frac{\partial H(t,q,p)}{\partial q}\right)^* = \frac{d}{dt}\left(\lim_{t_n \to \infty} \int_0^t \left(\frac{\partial H(t_n + \tau, q, p)}{\partial q}\right) d\tau\right), \tag{13}$$

$$u^* = \frac{d}{dt}\left(\lim_{t_n \to \infty} \int_0^t u(t_n + \tau, q, p)\, d\tau\right). \tag{14}$$

Let by analogy function dH/dt (7) satisfies the precompactness conditions G in some functional space F_Ω and one can associate with it a family of limit functions $\Omega(t,q,p)$ defined by

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$$\Omega(t,q,p)=\frac{d}{dt}\left(\lim_{t_n\to\infty}\int_0^t\left\|u(t_n+\tau,q,p)\right\|^2 d\tau\right) \quad (15)$$

Continuous monotonically strictly increasing functions in the section $[0,L]$ such that $h(0)=0$, that is, Hahn type functions [16], will be denoted by $h(\|x\|)$.

III. Basic Results

We shall present a solution of the above problem on stabilization of equilibrium position of mechanical systems based on Lyapunov's direct method.

Theorem 1. Suppose that the following conditions hold for system (6) with control function $u^0(t,q,p)$ (8) - (10):

1) all the functions in the right-hand side of (6) are bounded on the set G for any $L<\infty$ and satisfy the Lipshitz condition with respect to (q,p) for any $L<\infty$;

2) Hamiltonian $H(t,q,p)$ is a positive definite and admits of an infinitesimal upper bound

$$h_1\left(\left\|(q,p)^{\mathrm{T}}\right\|\right)\le H(t,q,p)\le h_2\left(\left\|(q,p)^{\mathrm{T}}\right\|\right);$$

3) the zero solution $q=p=0$ of equations (6) is uniformly stable;

4) for any limit set $\left(\left(\frac{\partial H}{\partial p}\right)^*,\left(\frac{\partial H}{\partial q}\right)^*,u,\Omega\right)$ corresponding to $\left(\frac{\partial H}{\partial p},\frac{\partial H}{\partial q},u^0,\frac{\partial H}{\partial t}\right)$ the set $\{\Omega(t,q,p)=0\}$ does not contain any solutions of the limit system (11), other than $q=p=0$.

Then $u^0(t,q,p)$ (8) - (10) is a stabilizing control function for zero state of system (6). Further, the equilibrium position $q=p=0$ is uniformly asymptotically stable.

Proof. Under condition 1 of theorem 1 equations of (6) with control (8) - (10) are precompact and regular. Then the limiting equations to them have the analogous form (11). Evidently, the zero state $q=p=0$ is the solution of (6) with control (8)-(10). For the derivate of the function $V(t,p,q)=H(t,q,p)$ in view of (6) with (8) - (10) we obtain the estimate

$$\dot V=\dot H_{(8)-(10)}\le f(t,H)-\mu\left\|u^0\right\|^2 \quad (1)$$

for all $(q,p)\in R^{2n}$ where $\dot H_{(8)-(10)}$ denotes the derivates of $H(t,q,p)$ along the solution of (6) with control law (8) - (10). The uniform asymptotic stability of the zero solution follows now form results of paper [17]. The theorem is proved.

Definition 1. The set $\dot H_\infty^{-1}(t,c)$ is a set of points $(q,p)\in R^{2n}$ such that is true $\lim_{t\to+\infty}H(t,q,p)=c$, where c is a constant [10].

Theorem 2. Suppose that the following conditions hold for system (6) with control function $u^0(t,q,p)$ (8) - (10):

1) all the functions in the right-hand side of (6) are bounded on the set G for any $L<\infty$ and satisfy the Lipshitz condition with respect to (q,p) for any $L<\infty$;

2) Hamiltonian $H(t,q,p)$ is a positive definite

$$H(t,q,p)\ge h_1\left(\left\|(q,p)^{\mathrm{T}}\right\|\right);$$

3) the zero solution $q=p=0$ of equations (6) is uniformly stable;

4) there are number L_0 and $L_1\,(0<L_0<L_1)$ such that $\sup(H(t,q,p)$ when $\left\|(q,p)^{\mathrm{T}}\right\|<L_0)<h_1(L_1)$;

5) there is at least one sequence $t_n\to+\infty$ for which the limit set $\left(\left(\frac{\partial H}{\partial p}\right)^*,\left(\frac{\partial H}{\partial q}\right)^*,u^*,\Omega\right)$ corresponding to $\left(\frac{\partial H}{\partial p},\frac{\partial H}{\partial q},u^0,\frac{\partial H}{dt}\right)$ and the corresponding set $\dot H_\infty^{-1}(t,c)$ are such that for any $c=c_0=const>0$ the set $\{\dot H_\infty^{-1}(t,c):c=c_0\}\cap\{\Omega(t,q,p)=0\}$ does not contain the limit system (11) solutions.

Then $u^0(t,q,p)$ (8) - (10) is a stabilizing control function for zero state of system (6). Further, the equilibrium position $q=p=0$ is asymptotically stable uniformly with respect to (q_0,p_0).

Proof. Under condition 1 of theorem 1 equations (6) with control (8) - (10) are precompact and regular. Evidently, the zero state $q=p=0$ is the solution of (6). We have inequality (16) for the derivative of $H(t,q,p)$. By conditions 2 and 4 it follows that the solutions $q=p=0$ is stable. The solution of (6) from the domain $\Gamma_1=\left\{\left\|(q,p)^{\mathrm{T}}\right\|<L_0\right\}$ will be bounded, $\left\|(q(t,t_0,q_0,p_0),p(t,t_0,q_0,p_0))^{\mathrm{T}}\right\|\le L_1$ for all $t\ge t_0$.

Let $(q,p)=(q(t,t_0,q_0,p_0),p(t,t_0,q_0,p_0))$ be a solution of (6) in Γ_1. By condition 2 of the theorem and (16) $H(t)=H(t,q(t,t_0,q_0,p_0),p(t,t_0,q_0,p_0))\to c_0$ as $t\to+\infty$. Let $t_n\to+\infty$ be a sequence determining $\left(\left(\frac{\partial H}{\partial p}\right)^*,\left(\frac{\partial H}{\partial q}\right)^*,u^*,\Omega\right)$ and $H_\infty^{-1}(t,c)$ such that $(q(t_n),p(t_n))\to(q^*,p^*)$ as $t_n\to+\infty$. We form the sequence of functions $(q_{n_k}(t),p_{n_k}(t))=(q(t_{n_k}+t),p(t_{n_k}+t))$.

By [9] the sequence of function $\{(q_{n_k}(t),p_{n_k}(t))\}$ defined for $t_{n_k}\ge t_0$ will converge to a solution

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$(q,p)=\varphi(t):[-\infty,+\infty]\to\Gamma$ of the system (11) uniformly in each interval $[-T,T]$. Taking the limit as $t_{n_k}\to+\infty$, as in [10], we obtain $\varphi(t)\in\{H_\infty^{-1}(t,c):c=c_0\}\cap\{\Omega(t,q,p)=0\}$

By condition 5 of the theorem this is possible only if $c_0=0$. So along each solution $(q(t,t_0,q_0,p_0),p(t,t_0,q_0,p_0)):(q_0,p_0)\in\Gamma_1$ of (6) $H(t,(q(t,t_0,q_0,p_0),p(t,t_0,q_0,p_0)))\to 0$ as $t\to+\infty$.

It follows that the solution $q=p=0$ is asymptotically stable uniformly with respect to (q_0,p_0) [17, 18]. The theorem is proved.

The obtained theorems develop results of [6-8, 10, 15].

IV. Conclusion

In this paper a controlled nonlinear mechanical system described by the Hamilton's canonical equations is considered. The control *u* acting to the mechanical system which allow to the asymptotic stability of state position is determined. The algorithm for constructing of controls, arbitrarily defined stabilizing the equilibrium state of mechanical systems was obtained. The problem of stabilization by the direct Lyapunov's method and the method of limiting functions and systems is solved. Two new theorems are proved. In this case we can use the Lyapunov's functions having nonpositive derivatives.

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