Method of Homogenization for the Study of the Propagation of Electromagnetic Waves in a Composite Part 2: Homogenization

Helene Canot, Emmanuel Frenod

***Abstract*—In this paper we study the two-scale behavior of the electromagnetic field in 3D in and near composite material. It is the continuation of the paper [6] in which we obtain existence and uniqueness results for the problem, we performed an estimate that allows us to approach homogenization. Technique of two-scale convergence is used to obtain the homogenized problem.**

***Index Terms*—Harmonic Maxwell Equations; Electromagnetism; Homogenization; Asymptotic Analysis; Asymptotic Expansion; Two-scale Convergence; Frequencies; Composite Material.**

I. Introduction

We are interested in the time-harmonic Maxwell equations in and near a composite material with boundary conditions modeling electromagnetic field radiated by an electromagnetic pulse (EMP). In the first part, we have presented the model and proved the existence of a unique solution of the problem. Our mathematical context is periodic homogenization. We consider a microscopic scale ε, which represents the ratio between the diameter of the fiber and thickness of the composite material. So, we are trying to understand how the microscopic structure affects the macroscopic electromagnetic field behavior. Homogenization of Maxwell equations with periodically oscillating coefficients was studied in many papers. N. Wellander homogenized linear and nonlinear Maxwell equations with perfect conducting boundary conditions using two-scale convergence in [16] and [17]. N. Wellander and B. Kristensson homogenized the full time-harmonic Maxwell equation with penetrable boundary conditions and at fixed frequency in [18]. The homogenized time-harmonic Maxwell equation for the scattering problem was done in F. Guenneau, S. Zolla and A. Nicolet [10]. Y. Amirat and V. Shelukhin perform two-scale homogenization time-harmonic Maxwell equations for a periodical structure in [4]. They calculate the effective dielectric ε and effective electric conductivity σ. They proved that homogenized Maxwell equations are different in low and high frequencies. The result obtained by two-scale convergence approach takes into account the characteristic sizes of skin thickness and wavelength around the material. We use the Asymptotic expansion and the theory of two-scale convergence introduced by G. Nguetseng [12] and developed by G. Allaire [2].

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II. Homogenization

We recall that our problem is:

$$\nabla\times\nabla\times E^\varepsilon + (-\omega^2\varepsilon^\eta\epsilon^\star + i\ \omega\ \sigma^\varepsilon(x,y,z))E^\varepsilon = 0 \text{ in } \Omega.$$

Equation (1) is provided with the following boundary conditions:

$$\nabla\times E^\varepsilon \times e_2 = -i\omega H_d(x,z)\times e_2 \ on\ R\times\Gamma_d, \tag{2}$$

and

$$\nabla\times E^\varepsilon \times e_2 = 0 \quad on \quad R\times\Gamma_L. \tag{3}$$

We propose an approach based on two-scale convergence. This concept was introduced by G. Nguetseng [13] and specified by G. Allaire [3] which studied properties of the two-scale convergence. M. Neuss-Radu in [11] presented an extension of two-scale convergence method to the periodic surfaces. Many authors applied two-scale convergence approach D. Cionarescu and P. Donato [8], N. Crouseilles, E. Frénod, S. Hirstoaga and A. Mouton [9], Y. Amirat, K. Hamdache and A. Ziani [1] and also A. Back, E. Frénod [5]. This mathematical concept were applied to homogenize the time-harmonic Maxwell equations S. Ouchetto, O. Zouhdi and A. Bossavit [14], H.E. Pak[15].

In our model, the parallel carbon cylinders are periodically distributed in direction x and z, as the material is homogenous in the y direction, we can consider that the material is periodic with a three directional cell of periodicity. In other words, introducing $\mathcal{Z} = [-\frac{1}{2}, \frac{1}{2}]\times[-1,0]^2$, function Σ^ε given in [7] is naturally periodic with respect to (ξ,ζ) with period $[-\frac{1}{2},\frac{1}{2}]\times[-1,0]$ but it is also periodic with respect to $\mathbf{y}$ with period $\mathcal{Z}$.

Now, we review some basis definitions and results about two-scale convergence.

A. Two-scale convergence

We first define the function spaces

$$\begin{aligned}\mathbf{H}(\text{curl},\Omega) &= \{u\in\mathbf{L}^2(\Omega): \quad \nabla\times u\in\mathbf{L}^2(\Omega)\},\\ \mathbf{H}(\text{div},\Omega) &= \{u\in\mathbf{L}^2(\Omega): \quad \nabla\cdot u\in L^2(\Omega)\},\end{aligned} \tag{4}$$

with the usual norms:

$$\begin{aligned}\|u\|^2_{\mathbf{H}(\text{curl},\Omega)} &= \|u\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla\times u\|^2_{\mathbf{L}^2(\Omega)},\\ \|u\|^2_{\mathbf{H}(\text{div},\Omega)} &= \|u\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla\cdot u\|^2_{L^2(\Omega)}.\end{aligned} \tag{5}$$

They are well known Hilbert spaces.

$$\begin{aligned}\mathbf{H}_\#(\text{curl},\mathcal{Z}) &= \{u\in\mathbf{H}(\text{curl},R^3): u \text{ is } \mathcal{Z}\text{-periodic}\}\\ \mathbf{H}_\#(\text{div},\mathcal{Z}) &= \{u\in\mathbf{H}(\text{div},R^3): u \text{ is } \mathcal{Z}\text{-periodic}\}\end{aligned} \tag{6}$$

ISBN: 978-988-14047-4-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

We introduce

$$\mathbf{L}^2_{\#}(\mathcal{Z}) = \{u \in \mathbf{L}^2(R^3), u \text{ is } \mathcal{Z}\text{-periodic}\}, \tag{7}$$

and

$$\mathbf{H}^1_{\#}(\mathcal{Z}) = \{u \in \mathbf{H}^1(R^3), u \text{ is } \mathcal{Z}\text{-periodic}\}, \tag{8}$$

where $\mathbf{H}^1(R^3)$ is the usual Sobolev space on R^3. First, denoting by $\mathbf{C}^0_{\#}(\mathcal{Z})$ the space of functions in $\mathbf{C}^0(R^3)$ and $\mathcal{Z}$-periodic, $\mathbf{C}^0_0(R^3)$ the space of continuous functions over R^3 with compact support, we have the following definitions:

Definition 2.1: A sequence $u^\varepsilon(\mathbf{x})$ in $\mathbf{L}^2(\Omega)$ two-scale converges to $u_0(\mathbf{x},\mathbf{y}) \in \mathbf{L}^2(\Omega, \mathbf{L}^2_{\#}(\mathcal{Z}))$ if for every $V(\mathbf{x},\mathbf{y}) \in \mathbf{C}^0_0(\Omega, \mathcal{C}^0_{\#}(\mathcal{Z}))$

$$\begin{aligned}\lim_{\varepsilon\to 0} \quad & \int_\Omega u^\varepsilon(\mathbf{x}) \cdot V(\mathbf{x}, \mathbf{x}/\varepsilon)\ d\mathbf{x} \\ & = \int_\Omega \int_{\mathcal{Z}} u_0(\mathbf{x},\mathbf{y}) \cdot V(\mathbf{x},\mathbf{y})\ d\mathbf{x}d\mathbf{y}.\end{aligned} \tag{9}$$

Proposition 2.2: If $u^\varepsilon(\mathbf{x})$ two-scale converges to $u_0(\mathbf{x},\mathbf{y}) \in \mathbf{L}^2(\Omega, \mathbf{L}^2_{\#}(\mathcal{Z}))$, we have for all $v(\mathbf{x}) \in C_0(\overline{\Omega})$ and all $w(\mathbf{y}) \in \mathbf{L}^2_{\#}(\mathcal{Z})$

$$\begin{aligned}\lim_{\varepsilon\to 0} \quad & \int_\Omega u^\varepsilon(\mathbf{x}) \cdot v(\mathbf{x}) w(\tfrac{\mathbf{x}}{\varepsilon})\ d\mathbf{x} \\ & = \int_\Omega \int_{\mathcal{Z}} u_0(\mathbf{x},\mathbf{y}) \cdot v(\mathbf{x}) w(\mathbf{y})\ d\mathbf{x}d\mathbf{y}.\end{aligned} \tag{10}$$

Theorem 2.3: (Nguetseng). Let $u^\varepsilon(\mathbf{x}) \in \mathbf{L}^2(\Omega)$. Suppose there exists a constant $c > 0$ such that for all ε

$$\|u^\varepsilon\|_{L^2(\Omega)} \le c. \tag{11}$$

Then there exists a subsequence of ε (still denoted ε) and $u_0(\mathbf{x},\mathbf{y}) \in \mathbf{L}^2(\Omega, \mathbf{L}^2_{\#}(\mathcal{Z}))$ such that:

$$u^\varepsilon(\mathbf{x}) \text{ two-scale converges to } u_0(\mathbf{x},\mathbf{y}). \tag{12}$$

Proposition 2.4: Let $u^\varepsilon(\mathbf{x})$ be a sequence of functions in $\mathbf{L}^2(\Omega)$, which two-scale converges to a limit $u_0(\mathbf{x},\mathbf{y}) \in \mathbf{L}^2(\Omega, \mathbf{L}^2_{\#}(\mathcal{Z}))$.

Then $u^\varepsilon(\mathbf{x})$ converges also to $u(\mathbf{x}) = \displaystyle\int_{\mathcal{Z}} u_0(\mathbf{x},\mathbf{y}) d\mathbf{y}$ in $\mathbf{L}^2(\Omega)$ weakly.

Furthermore, we have

$$\lim_{\varepsilon\to 0} \|u^\varepsilon\|_{\mathbf{L}^2(\Omega)} \ge \|u_0\|_{\mathbf{L}^2(\Omega\times Y)} \ge \|u\|_{\mathbf{L}^2(\Omega)}. \tag{13}$$

Proposition 2.5: Let $u^\varepsilon(\mathbf{x})$ be bounded in $\mathbf{L}^2(\Omega)$. Up to a subsequence, $u^\varepsilon(\mathbf{x})$ two-scale converges to $u_0(\mathbf{x},\mathbf{y}) \in \mathbf{L}^2(\Omega, \mathbf{L}^2_{\#}(\mathcal{Z}))$ such that:

$$u_0(\mathbf{x},\mathbf{y}) = u(\mathbf{x}) + \widetilde{u_0}(\mathbf{x},\mathbf{y}), \tag{14}$$

where $\widetilde{u_0}(\mathbf{x},\mathbf{y}) \in \mathbf{L}^2(\Omega, \mathbf{L}^2_{\#}(\mathcal{Z}))$ satisfies

$$\int_{\mathcal{Z}} \widetilde{u_0}(\mathbf{x},\mathbf{y})\ d\mathbf{y} = 0, \tag{15}$$

and $u(\mathbf{x}) = \displaystyle\int_{\mathcal{Z}} u_0(\mathbf{x},\mathbf{y})\ d\mathbf{y}$ is a weak limit in $\mathbf{L}^2(\Omega)$.

Proof: Due to the a priori estimates (32), $u^\varepsilon(\mathbf{x})$ is bounded in $\mathbf{L}^2(\Omega)$, then by application of Theorem 2.3, u^ε we get the first part of the proposition. Furthermore by defining $\widetilde{u_0}$ as

$$\widetilde{u_0}(\mathbf{x},\mathbf{y}) = u_0(\mathbf{x},\mathbf{y}) - \int_{\mathcal{Z}} u_0(\mathbf{x},\mathbf{y}) d\mathbf{y}, \tag{16}$$

we obtain the decomposition 14 of u_0. ∎

Proposition 2.6: Let any two-scale limit $u_0(\mathbf{x},\mathbf{y})$, given by Proposition (2.5), can be decomposed as

$$u_0(\mathbf{x},\mathbf{y}) = u(\mathbf{x}) + \nabla_{\mathbf{y}}\Phi(\mathbf{x},\mathbf{y}). \tag{17}$$

where $\Phi \in \mathbf{L}^2(\Omega, \mathbf{H}^1_{\#}(\mathcal{Z}))$ is a scalar-valued function and where $u \in \mathbf{L}^2(\Omega)$.

Proof: Proof of (17), integrating by parts, for any $V(\mathbf{x},\mathbf{y}) \in \mathbf{C}^1_0(\Omega, \mathbf{C}^1_{\#}(\mathcal{Z}))$, we have

$$\begin{aligned}\varepsilon \quad & \int_\Omega \nabla\times u^\varepsilon(\mathbf{x}) \cdot V(\mathbf{x}, \tfrac{\mathbf{x}}{\varepsilon})\ d\mathbf{x} \\ & = \varepsilon \int_\Omega u^\varepsilon(\mathbf{x}) \cdot \nabla\times V(\mathbf{x}, \tfrac{\mathbf{x}}{\varepsilon})\ d\mathbf{x} \\ & = \int_\Omega u^\varepsilon(\mathbf{x}) \{\varepsilon\nabla_{\mathbf{x}} \times V(\mathbf{x}, \tfrac{\mathbf{x}}{\varepsilon}) + \nabla_{\mathbf{y}} \times V(\mathbf{x}, \tfrac{\mathbf{x}}{\varepsilon})\}\ d\mathbf{x}.\end{aligned} \tag{18}$$

Taking the two-scale limit as $\varepsilon \to 0$ we obtain

$$0 = \int_\Omega \int_{\mathcal{Z}} u_0(\mathbf{x},\mathbf{y}) \cdot \nabla_{\mathbf{y}} \times V(\mathbf{x},\mathbf{y})\ d\mathbf{x}d\mathbf{y}, \tag{19}$$

which implies that $\nabla_{\mathbf{y}} \times u_0(\mathbf{x},\mathbf{y}) = 0$. To end the proof of the proposition we use the following result:

Proposition 2.7: If $u_0 \in \mathbf{L}^2(\Omega)$ satisfies

$$\nabla_{\mathbf{y}} \times u_0(\mathbf{x},\mathbf{y}) = 0, \tag{20}$$

then there exists $u \in \mathbf{L}^2(\Omega)$ and $\Phi \in \mathbf{L}^2(\Omega, \mathbf{H}^1_{\#}(\mathcal{Z}))$ such that $u_0(\mathbf{x},\mathbf{y}) = u(\mathbf{x}) + \nabla_{\mathbf{y}}\Phi(\mathbf{x},\mathbf{y})$.

Applying this proposition we obtain equality (17) ending the proof of Proposition (2.6). ∎

These results are important properties of the two-scales convergence. We note that the usual concepts of convergence do not preserve information concerning the micro-scale of the function. However, the two-scale convergence preserves information on the micro-scale.

III. Homogenized problem

We will explore in this section the behavior of electromagnetic field E^ε using the asymptotic expansion and the two-scale convergence to determine the homogenized problem. We place in the context of the case 6 with $\delta > L$ and $\overline{\omega} = 10^6 rad.s^{-1}$, then we have $\eta = 5$ and $\Sigma^\varepsilon_a = \varepsilon$, $\Sigma^\varepsilon_r = \varepsilon^4$, $\Sigma^\varepsilon_c = 1$ which gives the following equation:

$$\begin{aligned}&\nabla\times\nabla\times E^\varepsilon - \omega^2\varepsilon^5 k(\epsilon) E^\varepsilon + i\omega[(\mathbf{1}^\varepsilon_C(\tfrac{\mathbf{x}}{\varepsilon}) \\ &+\varepsilon^4 \mathbf{1}^\varepsilon_R(\tfrac{\mathbf{x}}{\varepsilon}))\mathbf{1}_{\{y<0\}} + \varepsilon\mathbf{1}_{\{y>0\}}]E^\varepsilon = 0,\end{aligned} \tag{21}$$

where for a given set $\mathcal{A}$, $\mathbf{1}_{\mathcal{A}}$ stands for the characteristic function of $\mathcal{A}$ and where $\mathbf{1}^\varepsilon_{\mathcal{A}}(\mathbf{x}) = \mathbf{1}_{\mathcal{A}}(\frac{\mathbf{X}}{\varepsilon})$, hence $\mathbf{1}^\varepsilon_C$ and $\mathbf{1}^\varepsilon_R$ are the characteristic functions of the sets filled by carbon fibers and by resin. And where $k(\epsilon) = (\epsilon_c \mathbf{1}^\varepsilon_C(\mathbf{x}) + \epsilon_r \mathbf{1}^\varepsilon_R(\mathbf{x}))\mathbf{1}_{\{y<0\}} + \mathbf{1}_{\{y>0\}}$. First, we will use the classical method of the asymptotic expansion.

A. Asymptotic expansion

We assume that $(E^\varepsilon, H^\varepsilon)$ satisfies the following asymptotic expansion, as $\varepsilon \to 0$:

$$E^\varepsilon(\mathbf{x}) = E_0(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) + \varepsilon E_1(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) + \varepsilon^2 E_2(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) + ..., \tag{22}$$

where for any $k \in N$ $E_k = E_k(\mathbf{x},\mathbf{y})$ are considered as $\mathcal{Z}$-periodic functions with respect to $\mathbf{y}$. Applied to functions $E_k(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})$ the curl operator becomes $\nabla_{\mathbf{x}} \times E_k(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) + \frac{1}{\varepsilon}\nabla_{\mathbf{y}} \times E_k(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})$. Plugging (22) in the formulations (21), gathering the coefficients with the same power of ε, we get:

ISBN: 978-988-14047-4-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$$\left\{\begin{array}{l}\frac{1}{\varepsilon^2}\nabla_{\mathbf{y}}\times\nabla_{\mathbf{y}}\times E_0(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})\\+\frac{1}{\varepsilon}[\nabla_{\mathbf{y}}\times\nabla_{\mathbf{y}}\times E_1(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})\\+\nabla_{\mathbf{y}}\times\nabla_{\mathbf{x}}\times E_0(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})+\nabla_{\mathbf{x}}\times\nabla_{\mathbf{y}}\times E_0(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})]\\+\varepsilon^0[\nabla_{\mathbf{x}}\times\nabla_{\mathbf{y}}\times E_1(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})+\nabla_{\mathbf{x}}\times\nabla_{\mathbf{x}}\times E_0(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})\\+\nabla_{\mathbf{y}}\times\nabla_{\mathbf{y}}\times E_2(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})+\nabla_{\mathbf{y}}\times\nabla_{\mathbf{x}}\times E_1(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})\\+i\omega\mathbf{1}^{\varepsilon}_C(\mathbf{x})\mathbf{1}_{\{y<0\}}E_0(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})]\\+\varepsilon[\nabla_{\mathbf{x}}\times\nabla_{\mathbf{y}}\times E_2(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})\\+\nabla_{\mathbf{x}}\times\nabla_{\mathbf{x}}\times E_1(\mathbf{x},\frac{\mathbf{x}}{\varepsilon}))+\nabla_{\mathbf{y}}\times\nabla_{\mathbf{x}}\times E_2(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})\\+\nabla_{\mathbf{y}}\times\nabla_{\mathbf{y}}\times E_3(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})+i\omega(\mathbf{1}^{\varepsilon}_C(\mathbf{x})\mathbf{1}_{\{y<0\}}E_1(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})\\+\mathbf{1}_{\{y>0\}})E_0(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})]+...)=0.\end{array}\right. \tag{23}$$

In order to write what is in factor of ε in the last equation we used that : $\mathbf{1}_{\{y<0\}}=\mathbf{1}_{\{\frac{y}{\varepsilon}<0\}}$. Since (23) is considered as true for any small ε it gives a cascade of equations, from which we extract the four first equations

$$\left\{\begin{array}{l}\nabla_{\mathbf{y}}\times\nabla_{\mathbf{y}}\times E_0(\mathbf{x},\mathbf{y})=0\\\nabla_{\mathbf{y}}\times\nabla_{\mathbf{y}}\times E_1(\mathbf{x},\mathbf{y})\\+\nabla_{\mathbf{y}}\times\nabla_{\mathbf{x}}\times E_0(\mathbf{x},\mathbf{y})+\nabla_{\mathbf{x}}\times\nabla_{\mathbf{y}}\times E_0(\mathbf{x},\mathbf{y})=0\\\nabla_{\mathbf{x}}\times\nabla_{\mathbf{y}}\times E_1(\mathbf{x},\mathbf{y})+\nabla_{\mathbf{x}}\times\nabla_{\mathbf{x}}\times E_0(\mathbf{x},\mathbf{y})\\+\nabla_{\mathbf{y}}\times\nabla_{\mathbf{y}}\times E_2(\mathbf{x},\mathbf{y})\\+\nabla_{\mathbf{y}}\times\nabla_{\mathbf{x}}\times E_1(\mathbf{x},\mathbf{y})+i\omega\mathbf{1}_C(\mathbf{y})\mathbf{1}_{\{\nu<0\}}E_0(\mathbf{x},\mathbf{y})=0\\\nabla_{\mathbf{x}}\times\nabla_{\mathbf{y}}\times E_2(\mathbf{x},\mathbf{y})+\nabla_{\mathbf{x}}\times\nabla_{\mathbf{x}}\times E_1(\mathbf{x},\mathbf{y})\\+\nabla_{\mathbf{y}}\times\nabla_{\mathbf{x}}\times E_2(\mathbf{x},\mathbf{y})+\nabla_{\mathbf{y}}\times\nabla_{\mathbf{y}}\times E_3(\mathbf{x},\mathbf{y})\\+i\omega\big(\mathbf{1}_C(\mathbf{y})\mathbf{1}_{\{\nu<0\}}E_1(\mathbf{x},\mathbf{y})+\mathbf{1}_{\{\nu>0\}}\big)E_0(\mathbf{x},\mathbf{y})=0\end{array}\right. \tag{24}$$

Applying $\mathrm{div}_{\mathbf{y}}$ in the last two equations in (24), we obtain

$$\nabla_{\mathbf{y}}\cdot\big(i\omega\mathbf{1}_C(\mathbf{y})\mathbf{1}_{\{\nu<0\}}E_0(\mathbf{x},\mathbf{y})\big)=0, \tag{25}$$

and

$$\nabla_{\mathbf{y}}\cdot\big(i\omega(\mathbf{1}_C(\mathbf{y})\mathbf{1}_{\{\nu<0\}}E_1(\mathbf{x},\mathbf{y})+\mathbf{1}_{\{\nu>0\}})E_0(\mathbf{x},\mathbf{y})\big)=0. \tag{26}$$

The boundary condition in (2) write:

$$\left\{\begin{array}{l}(\frac{1}{\varepsilon}\nabla_{\mathbf{y}}\times E_0(\mathbf{x},\mathbf{y})+\nabla_{\mathbf{x}}\times E_0(\mathbf{x},\mathbf{y})\\+\nabla_{\mathbf{y}}\times E_1(\mathbf{x},\mathbf{y}+...)\times n\\=-i\omega H_d\times n,\ \mathbf{x}\in R^3,\ \mathbf{y}\in\mathcal{Z}.\end{array}\right. \tag{27}$$

Now we take the first equation of (24) and the equation (25) to obtain :

$$\left\{\begin{array}{l}\nabla_{\mathbf{y}}\times\nabla_{\mathbf{y}}\times E_0(\mathbf{x},\mathbf{y})=0,\\\nabla_{\mathbf{y}}\cdot\{i\omega(\mathbf{1}_C(\mathbf{y})\mathbf{1}_{\{\nu<0\}})E_0(\mathbf{x},\mathbf{y})\}=0.\end{array}\right. \tag{28}$$

Multiplying the first equation in (28) by E_0 and integrating by parts over $\mathcal{Z}$ leads to

$$\begin{array}{l}\int_{\mathcal{Z}}\nabla_{\mathbf{y}}\times\nabla_{\mathbf{y}}\times E_0(\mathbf{x},\mathbf{y})E_0(\mathbf{x},\mathbf{y})\quad d\mathbf{y}\\=\int_{\mathcal{Z}}|\nabla_{\mathbf{y}}\times E_0(\mathbf{x},\mathbf{y})|^2\quad d\mathbf{y}\\=0.\end{array} \tag{29}$$

We deduce that the equation (28) is equivalent to

$$\nabla_{\mathbf{y}}\times E_0(\mathbf{x},\mathbf{y})=0, \tag{30}$$

for any $\mathbf{y}\in\mathcal{Z}$. Hence from Proposition (2.7) we conclude that $E_0(\mathbf{x},\mathbf{y})$ can be decomposed as

$$E_0(\mathbf{x},\mathbf{y})=E(\mathbf{x})+\nabla_{\mathbf{y}}\Phi_0(\mathbf{x},\mathbf{y}), \tag{31}$$

where $\Phi_0(\mathbf{x},\mathbf{y})\in\mathbf{L}^2(\Omega;\mathbf{H}^1_{\#}(\mathcal{Z}))$ and $E(\mathbf{x})\in\mathbf{L}^2(\Omega)$.

B. Mathematical justification

Now we will show rigorously with two-scale convergence that the solution of problems (1),(2) and (3) converge to the solution of the homogenized problem when ε tends to 0. We recall the following Theorem, we give a proof in [7]:

Theorem 3.1: For any $\varepsilon>0$, for any $\eta\geq 0$, there exists a positive constant ω_0 which does not depend on ε and such that for all $\omega\in(0,\omega_0)$, $E^{\varepsilon}\in X^{\varepsilon}(\Omega)$ solution of (1), (2), (3) satisfies

$$\|E^{\varepsilon}\|_{\mathbf{X}^{\varepsilon}(\Omega)}\leq C \tag{32}$$

with $C=\frac{C_{\gamma_t}C_{\gamma_T}}{\mathcal{C}_0}\|H_d\|_{H(curl,\Omega)}$.

Then, we have the following Theorem:

Theorem 3.2: Under assumptions of Theorem (32), sequence E^{ε} is solution of ((1), (2), (3)). E^{ε} two-scale converges to $E(\mathbf{x})+\nabla_{\mathbf{y}}\Phi_0(\mathbf{x},\mathbf{y})$, where $E\in\mathbf{L}^2(\Omega)$ and $\Phi_0\in\mathbf{L}^2(\Omega,\mathbf{H}^1_{\#}(\mathcal{Z}))$, the unique solution of the homogenized problem:

$$\left\{\begin{array}{ll}\nabla_{\mathbf{x}}\times\nabla_{\mathbf{x}}\times E(\mathbf{x})+i\omega\theta E(\mathbf{x})=0 & \text{in } \Omega,\\-\nabla_{\mathbf{y}}\cdot\{\nabla_{\mathbf{x}}\times\nabla_{\mathbf{x}}\times\nabla_{\mathbf{y}}\Phi_0(\mathbf{x},\mathbf{y})\}&\\-i\omega\nabla_{\mathbf{y}}\cdot\{\mathbf{1}_C(\mathbf{y})\nabla_{\mathbf{y}}\Phi_0(\mathbf{x},\mathbf{y})\}&\\=i\omega E(\mathbf{x})\nabla_{\mathbf{y}}\cdot\{\mathbf{1}_C(\mathbf{y})\} & \text{in } \Omega\times\mathcal{Z},\\\nabla_{\mathbf{x}}\times E(\mathbf{x})\times e_2=-i\omega H_d\times e_2 & \text{on } \Gamma_d,\\\nabla_{\mathbf{x}}\times E(\mathbf{x})\times e_2=0 & \text{on } \Gamma_L,\end{array}\right. \tag{33}$$

where $\theta=\int_{\mathcal{Z}}\mathbf{1}_C(\mathbf{y})\ d\mathbf{y}$ is the volume fraction of carbon fiber.

Proof: **Step 1: Two-scale convergence.** Due to the estimate (32), E^{ε} is bounded in $\mathbf{L}^2(\Omega)$. Hence, up to a subsequence, E^{ε} two-scale converges to $E_0(\mathbf{x},\mathbf{y})$ belonging to $\mathbf{L}^2(\Omega,\mathbf{L}^2_{\#}(\mathcal{Z}))$. That means for any $V(\mathbf{x},\mathbf{y})\in\mathbf{C}^1_0(\Omega,\mathbf{C}^1_{\#}(\mathcal{Z}))$, we have:

$$\lim_{\varepsilon\to 0}\int_{\Omega}E^{\varepsilon}(\mathbf{x})\cdot V(\mathbf{x},\tfrac{\mathbf{x}}{\varepsilon})\ d\mathbf{x}=\int_{\Omega}\int_{\mathcal{Z}}E_0(\mathbf{x},\mathbf{y})\cdot V(\mathbf{x},\mathbf{y})\ d\mathbf{y}d\mathbf{x}. \tag{34}$$

Step 2: Deduction of the constraint equation. We multiply Equation (21) by oscillating test function $V^{\varepsilon}(\mathbf{x})=V(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})$ where $V(\mathbf{x},\mathbf{y})\in\mathbf{C}^1_0(\Omega,\mathbf{C}^1_{\#}(\mathcal{Z}))$:

$$\begin{array}{l}\int_{\Omega}\nabla\times E^{\varepsilon}(\mathbf{x})\cdot(\nabla_{\mathbf{x}}\times V^{\varepsilon}(\mathbf{x},\frac{\mathbf{x}}{\varepsilon}))+\frac{1}{\varepsilon}\nabla_{\mathbf{y}}\times V^{\varepsilon}(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})\\+[-\omega^2\varepsilon^5k(\epsilon)+i\omega\big(\mathbf{1}^{\varepsilon}_C(\frac{\mathbf{x}}{\varepsilon})+\varepsilon^4\mathbf{1}^{\varepsilon}_R(\frac{\mathbf{x}}{\varepsilon})\big)\mathbf{1}_{\{y<0\}}\\+\varepsilon\mathbf{1}_{\{y>0\}}]E^{\varepsilon}\cdot V^{\varepsilon}(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})\ d\mathbf{x}\\=-i\omega\int_{\Gamma_d}H_d\times e_2\cdot(e_2\times V(x,1,z,\xi,\frac{1}{\varepsilon},\zeta))\times e_2\ d\sigma.\end{array} \tag{35}$$

Integrating by parts, we get:

$$\begin{array}{l}\int_{\Omega}E^{\varepsilon}(\mathbf{x})\cdot(\nabla_{\mathbf{x}}\times\nabla_{\mathbf{x}}\times V^{\varepsilon}(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})\\+\frac{1}{\varepsilon}\nabla_{\mathbf{y}}\times\nabla_{\mathbf{x}}\times V^{\varepsilon}(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})+\frac{1}{\varepsilon}\nabla_{\mathbf{x}}\times\nabla_{\mathbf{y}}\times V^{\varepsilon}(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})\\+\frac{1}{\varepsilon^2}\nabla_{\mathbf{y}}\times\nabla_{\mathbf{y}}\times V^{\varepsilon}(\mathbf{x},\frac{\mathbf{x}}{\varepsilon}))+[-\omega^2\varepsilon^5k(\epsilon)\\+i\omega\big(\mathbf{1}^{\varepsilon}_C(\frac{\mathbf{x}}{\varepsilon})+\varepsilon^4\mathbf{1}^{\varepsilon}_R(\frac{\mathbf{x}}{\varepsilon})\big)\mathbf{1}_{\{y<0\}}\\+\varepsilon\mathbf{1}_{\{y>0\}}]E^{\varepsilon}(\mathbf{x})\cdot V^{\varepsilon}(\mathbf{x},\frac{\mathbf{x}}{\varepsilon})\ d\mathbf{x}\\=-i\omega\int_{\Gamma_d}H_d\times e_2\cdot(e_2\times V(x,1,z,\xi,\frac{1}{\varepsilon},\zeta))\times e_2\ d\sigma.\end{array} \tag{36}$$

Now we multiply (36) by ε^2 and we pass to the two-scale limit, applying Theorem 2.3, using (34) we obtain:

$$\int_{\Omega}\int_{\mathcal{Z}}E_0(\mathbf{x},\mathbf{y})\big(\nabla_{\mathbf{y}}\times\nabla_{\mathbf{y}}\times V(\mathbf{x},\mathbf{y})\big)\ d\mathbf{y}d\mathbf{x}=0. \tag{37}$$

We deduce the constraint equation for the profile E_0:

$$\nabla_{\mathbf{y}}\times\nabla_{\mathbf{y}}\times E_0(\mathbf{x},\mathbf{y})=0. \tag{38}$$

ISBN: 978-988-14047-4-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Step 3. Looking for the solutions to the constraint equation. Multiplying Equation (38) by the conjugate of E_0 and integrating by parts over $\mathcal{Z}$ leads to

$$\begin{aligned} &\int_{\mathcal{Z}} \nabla_{\mathbf{y}} \times \nabla_{\mathbf{y}} \times E_0(\mathbf{x},\mathbf{y}) E_0(\mathbf{x},\mathbf{y})\ d\mathbf{y} \\ &= \int_{\mathcal{Z}} |\nabla_{\mathbf{y}} \times E_0(\mathbf{x},\mathbf{y})|^2\ d\mathbf{y} \\ &= 0. \end{aligned} \tag{39}$$

We deduce that equation (39) is equivalent to

$$\nabla_{\mathbf{y}} \times E_0(\mathbf{x},\mathbf{y}) = 0, \tag{40}$$

Moreover since a solution of (40) is also solution of (38), (38) and (40) are equivalent. Hence, from Proposition 2.7 we conclude that $E_0(\mathbf{x},\mathbf{y})$ can be decomposed as

$$E_0(\mathbf{x},\mathbf{y}) = E(\mathbf{x}) + \nabla_{\mathbf{y}}\Phi_0(\mathbf{x},\mathbf{y}). \tag{41}$$

Step 4. Equations for $E(\mathbf{x})$ and $\Phi_0(\mathbf{x},\mathbf{y})$. Now, we seek what E satisfies. For this, we build oscillating test functions satisfying constraint (41) and use them in weak formulation (36). We define test function $V(\mathbf{x},\mathbf{y}) = \alpha(\mathbf{x}) + \nabla_{\mathbf{y}}\beta(\mathbf{x},\mathbf{y})$, $V(\mathbf{x},\mathbf{y}) \in \mathbf{C}_0^1(\Omega) \times \mathbf{C}_0^1(\Omega, \mathbf{C}_\#^1(\mathcal{Z}))$ and we inject in (36) test function $V^\varepsilon = V(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})$, which gives:

$$\begin{aligned} &\int_\Omega E^\varepsilon(\mathbf{x}) \cdot \big(\nabla_{\mathbf{x}} \times \nabla_{\mathbf{x}} \times V(\mathbf{x}, \tfrac{\mathbf{x}}{\varepsilon}) + \tfrac{2}{\varepsilon}\nabla_{\mathbf{x}} \times \nabla_{\mathbf{y}} \times V(\mathbf{x}, \tfrac{\mathbf{x}}{\varepsilon}) \\ &+ \tfrac{1}{\varepsilon^2}\nabla_{\mathbf{y}} \times \nabla_{\mathbf{y}} \times V(\mathbf{x}, \tfrac{\mathbf{x}}{\varepsilon})\big) + [-\omega^2\varepsilon^5 k(\epsilon) + i\omega\big(\mathbf{1}_C^\varepsilon(\tfrac{\mathbf{x}}{\varepsilon}) \\ &+ \varepsilon^4 \mathbf{1}_R^\varepsilon(\tfrac{\mathbf{x}}{\varepsilon})\big)\mathbf{1}_{\{y<0\}} + \varepsilon \mathbf{1}_{\{y>0\}}] E^\varepsilon(\mathbf{x}) \cdot V(\mathbf{x}, \tfrac{\mathbf{x}}{\varepsilon})\ d\mathbf{x} \\ &= -i\omega \int_{\Gamma_d} H_d \times e_2 \cdot (e_2 \times V^\ddagger(x,1,z,\xi,\zeta)) \times e_2\ d\sigma, \end{aligned} \tag{42}$$

with $V(x,1,z,\xi,\nu,\zeta) = V^\ddagger(x,1,z,\xi,\zeta)$ the restriction on V which does not depend on ν. The term containing the constraint, the third one, disappears. Passing to the limit $\varepsilon \to 0$ and replacing the expression of V by the term $\alpha(\mathbf{x}) + \nabla_{\mathbf{y}}\beta(\mathbf{x},\mathbf{y})$, we have

$$\begin{aligned} \nabla_{\mathbf{x}} \times \nabla_{\mathbf{y}} \times V(\mathbf{x},\mathbf{y}) &= \nabla_{\mathbf{x}} \times \nabla_{\mathbf{y}} \times [\alpha(\mathbf{x}) + \nabla_{\mathbf{y}}\beta(\mathbf{x},\mathbf{y})] \\ &= \nabla_{\mathbf{x}} \times \nabla_{\mathbf{y}} \times (\alpha(\mathbf{x})) \\ &\quad + \nabla_{\mathbf{x}} \times \nabla_{\mathbf{y}} \times (\nabla_{\mathbf{y}}\beta(\mathbf{x},\mathbf{y})) \\ &= \nabla_{\mathbf{x}} \times \nabla_{\mathbf{y}} \times (\nabla_{\mathbf{y}}\beta(\mathbf{x},\mathbf{y})). \end{aligned} \tag{43}$$

Since $\nabla_{\mathbf{y}} \times (\nabla_{\mathbf{y}}) = 0$, the term $\frac{2}{\varepsilon}\nabla_{\mathbf{x}} \times \nabla_{\mathbf{y}} \times V(\mathbf{x},\mathbf{y})$ vanishes. Therefore, (42) becomes:

$$\begin{aligned} &\int_\Omega \int_{\mathcal{Z}} E_0(\mathbf{x},\mathbf{y}) \cdot \nabla_{\mathbf{x}} \times \nabla_{\mathbf{x}} \times (\alpha(\mathbf{x}) + \nabla_{\mathbf{y}}\beta(\mathbf{x},\mathbf{y}) \\ &+ i\omega\big(\mathbf{1}_C(\mathbf{y}) E_0(\mathbf{x},\mathbf{y}) \cdot (\alpha(\mathbf{x}) + \nabla_{\mathbf{y}}\beta(\mathbf{x},\mathbf{y}))\ d\mathbf{y}d\mathbf{x} \\ &= -i\omega \int_{\Gamma_d} H_d \times e_2 \cdot (e_2 \times (\alpha(x,1,z) \\ &+ \nabla_{\mathbf{y}}\beta(x,1,z,\xi,\zeta))) \times e_2\ d\sigma. \end{aligned} \tag{44}$$

Now in (44) we replace Expression E_0 giving by (41). We obtain

$$\begin{aligned} &\int_\Omega \int_{\mathcal{Z}} (E(\mathbf{x}) + \nabla_{\mathbf{y}}\Phi_0(\mathbf{x},\mathbf{y})) \cdot \big(\nabla_{\mathbf{x}} \times \nabla_{\mathbf{x}} \times (\alpha(\mathbf{x}) \\ &+ \nabla_{\mathbf{y}}\beta(\mathbf{x},\mathbf{y})) + i\omega \mathbf{1}_C(\mathbf{y})(E(\mathbf{x}) \\ &+ \nabla_{\mathbf{y}}\Phi_0(\mathbf{x},\mathbf{y})) \cdot (\alpha(\mathbf{x}) + \nabla_{\mathbf{y}}\beta(\mathbf{x},\mathbf{y}))\ d\mathbf{y}d\mathbf{x} \\ &= -i\omega \int_{\Gamma_d} H_d \times e_2 \cdot (e_2 \times (\alpha(x,1,z) \\ &+ \nabla_{\mathbf{y}}\beta(x,1,z,\xi,\zeta))) \times e_2\ d\sigma. \end{aligned} \tag{45}$$

Now using the fact that $\int_\Omega \int_{\mathcal{Z}} \nabla_{\mathbf{y}}\Phi_0(\mathbf{x},\mathbf{y}) \cdot \nabla_{\mathbf{x}} \times \nabla_{\mathbf{x}} \times \alpha(\mathbf{x})\ d\mathbf{y}d\mathbf{x} = 0$ we have:

$$\begin{aligned} &\int_\Omega \int_{\mathcal{Z}} E(\mathbf{x}) \cdot \nabla_{\mathbf{x}} \times \nabla_{\mathbf{x}} \times \alpha(\mathbf{x}) \\ &+ E(\mathbf{x}) \cdot \nabla_{\mathbf{x}} \times \nabla_{\mathbf{x}} \times \nabla_{\mathbf{y}}\beta(\mathbf{x},\mathbf{y}) \\ &+ \nabla_{\mathbf{y}}\Phi_0(\mathbf{x},\mathbf{y}) \cdot \nabla_{\mathbf{x}} \times \nabla_{\mathbf{x}} \times \nabla_{\mathbf{y}}\beta(\mathbf{x},\mathbf{y}) \\ &+ i\omega \mathbf{1}_C(\mathbf{y})(E(\mathbf{x})\alpha(\mathbf{x}) + \nabla_{\mathbf{y}}\Phi_0(\mathbf{x},\mathbf{y})\alpha(\mathbf{x}) \\ &+ E(\mathbf{x})\nabla_{\mathbf{y}}\beta(\mathbf{x},\mathbf{y}) + \nabla_{\mathbf{y}}\Phi_0(\mathbf{x},\mathbf{y})\nabla_{\mathbf{y}}\beta(\mathbf{x},\mathbf{y}))\ d\mathbf{y}d\mathbf{x} \\ &= -i\omega \int_{\Gamma_d} H_d \times e_2 \cdot (e_2 \times \alpha(x,1,z)) \times e_2\ d\sigma. \end{aligned} \tag{46}$$

Then taking $\beta(\mathbf{x},\mathbf{y}) = 0$ in (46) and integrating by parts, we get

$$\begin{aligned} &\int_\Omega \big(\nabla_{\mathbf{x}} \times \nabla_{\mathbf{x}} \times E(\mathbf{x}) \cdot \alpha(\mathbf{x}) \\ &+ i\omega\big(\textstyle\int_{\mathcal{Z}} \mathbf{1}_C(\mathbf{y})\ d\mathbf{y}\big) E(\mathbf{x})\alpha(\mathbf{x})\big)\ d\mathbf{x} \\ &= -i\omega \int_{\Gamma_d} H_d \times e_2 \cdot (e_2 \times \alpha(x,1,z)) \times e_2\ d\sigma. \end{aligned} \tag{47}$$

which gives the following well posed problem for $E(\mathbf{x})$

$$\begin{cases} \nabla_{\mathbf{x}} \times \nabla_{\mathbf{x}} \times E(\mathbf{x}) + i\omega(\int_{\mathcal{Z}} \mathbf{1}_C(\mathbf{y})\ d\mathbf{y}) E(\mathbf{x}) = 0 \ \text{ in } \Omega, \\ \nabla_{\mathbf{x}} \times E(\mathbf{x}) \times e_2 = -i\omega H_d \times e_2 \ \text{ on } \Gamma_d, \\ \nabla_{\mathbf{x}} \times E(\mathbf{x}) \times e_2 = 0 \ \text{ on } \Gamma_L. \end{cases} \tag{48}$$

Now taking $\alpha(\mathbf{x}) = 0$ in (46), we obtain

$$\begin{aligned} &\int_\Omega \int_{\mathcal{Z}} \nabla_{\mathbf{x}} \times \nabla_{\mathbf{x}} \times \nabla_{\mathbf{y}}\Phi_0(\mathbf{x},\mathbf{y}) \nabla_{\mathbf{y}}\beta(\mathbf{x},\mathbf{y}) \\ &+ i\omega \mathbf{1}_C(\mathbf{y})(E(\mathbf{x}) \cdot \nabla_{\mathbf{y}}\beta(\mathbf{x},\mathbf{y}) \\ &+ \nabla_{\mathbf{y}}\Phi_0(\mathbf{x},\mathbf{y})\nabla_{\mathbf{y}}\beta(\mathbf{x},\mathbf{y})) d\mathbf{y}d\mathbf{x} = 0. \end{aligned} \tag{49}$$

Integrating by parts

$$\begin{aligned} &\int_\Omega \int_{\mathcal{Z}} -\nabla_{\mathbf{y}} \cdot \{\nabla_{\mathbf{x}} \times \nabla_{\mathbf{x}} \times \nabla_{\mathbf{y}}\Phi_0(\mathbf{x},\mathbf{y})\}\beta(\mathbf{x},\mathbf{y}) \\ &- i\omega\nabla_{\mathbf{y}} \cdot \{\mathbf{1}_C(\mathbf{y})\nabla_{\mathbf{y}}\Phi_0(\mathbf{x},\mathbf{y})\}\beta(\mathbf{x},\mathbf{y})\ d\mathbf{y}d\mathbf{x} \\ &= i\omega \int_\Omega \int_{\mathcal{Z}} \nabla_{\mathbf{y}} \cdot \{\mathbf{1}_C(\mathbf{y})\} E(\mathbf{x})\beta(\mathbf{x},\mathbf{y})\ d\mathbf{y}d\mathbf{x}. \end{aligned} \tag{50}$$

which gives the microscopic problem for $\Phi_0(\mathbf{x},\mathbf{y})$

$$\begin{aligned} &-\nabla_{\mathbf{y}} \cdot \{\nabla_{\mathbf{x}} \times \nabla_{\mathbf{x}} \times \nabla_{\mathbf{y}}\Phi_0(\mathbf{x},\mathbf{y})\} \\ &- i\omega\nabla_{\mathbf{y}} \cdot \{\mathbf{1}_C(\mathbf{y})\nabla_{\mathbf{y}}\Phi_0(\mathbf{x},\mathbf{y})\} \\ &= i\omega E(\mathbf{x})\nabla_{\mathbf{y}} \cdot \{\mathbf{1}_C(\mathbf{y})\}. \end{aligned} \tag{51}$$

This concludes the proof of Theorem 3.2. ■

IV. Conclusion

We presented in this paper the homogenization of time harmonic Maxwell equation by the method of two-scale convergence. We started by studying the time harmonic Maxwell equations with coefficients depending of ε. We remind that λ is the wave length, δ is the skin length, L is thickness of the medium and e the size of the basic cell and then $\varepsilon = \frac{e}{L}$ is the small parameter. We find for low frequencies the macroscopic homogenized Maxwell equations depending on the volume fraction of the carbon fibers and we find also the microscopic equation.

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ISBN: 978-988-14047-4-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)