Application of Homotopy Perturbation and Modified Adomian Decomposition Methods for Higher Order Boundary Value Problems

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***Abstract*—This work considers the numerical solution of higher order boundary value problems using Homotopy perturbation method (HPM) and modified Adomian decomposition method (MADM). HPM is applied without any transformation or calculation of Adomian polynomials. The differential equations are transformed into an infinite number of simple problems without necessarily using the perturbation techniques. Two numerical examples are solved to illustrate the method and the results are compared with the exact and MADM solutions. The accuracy and rapid convergence of HPM in handling the equations without calculating Adomian polynomials reveals its advantage over MADM.**

***Index Terms*—Boundary Value Problems, HPM, MADM, Series Solution**

I. Introduction

Many phenomena in sciences and engineering are modeled by differential equations and are expressed in terms of boundary value problems. Incidentally, most of the boundary value problems do not have closed form solution, this has led to the development of various semi-analytical techniques such as Adomian decomposition method [1-3], differential transform method [4-6] spline method [7-8], Exp-function method [9], generalized differential quadrature rule (GDQR) [10]. Others include variational iteration technique [11], finite-difference method [12] and Runge-kutta method [13]. All these methods have some difficulties in their application. For instance, differential transform method requires transformation of the equations while Adomian decomposition method involves calculations of Adomian polynomials.

The Homotopy Perturbation Method (HPM) applied in this work was proposed by the Chinese researcher J. Huan He [14-15] by coupling the perturbation method and homotopy in topology. The method was developed to eliminate the limitations placed by the traditional perturbation technique viz:

- Presence of small parameters in the equations whereas most non-linear problems do not contain the so-called small parameters.
- Identification of small parameters in equations requires the mastery of some special techniques because wrong choice will affect the results.
- The approximate solution by perturbation method is valid only for the small values of the parameters.

The presence of the so-called small parameters places restrictions on the application of the perturbation method since most of the linear and nonlinear problems have no small parameters. To overcome the drawbacks homotopy perturbation method was developed. According to the method, a nonlinear problem is transformed into an infinite number of simple problems without necessarily using the perturbation techniques. This is done by letting the small parameter float and converges to unity, the problem will be converted into a special perturbation problem. HPM has since then been developed and applied to numerous models. El-Shahed M. [16] in volterra's integro-differential equation, Biazar et al. [17] in Zakharov–Kuznetsov equations, Biazar et al. [18] applied it to hyperbolic partial differential equations, Darvishi et al. [19] and Aminikhaha et al. [20] used it to solve stiff systems of ordinary differential equations.

Furthermore, Chun et al. [21] used it to obtain the solution of two-point boundary value problems in comparison with other methods, Mohyud-Din et al. [22] to solve Flierl–Petviashivili equation and Ganji et al. [23] in Nonlinear Heat Transfer and Porous Media Equations.

The objective of this work is to compare the HPM and MADM for the solution of higher order boundary value problems. Wazwaz [24] presented the modification to Adomian decomposition method which reduces the size of computations involved in the method and thereby enhances the rapidity of its convergence.

II. Test Examples

Example 1: We will first consider a third order three-point boundary value problem

$$y''' - 25y' + 1 = 0 \tag{1}$$

with the following boundary conditions

$$y'(0) = 0, y'(1) = 0, y(0.5) = 0 \tag{2}$$

The exact solution of the above problem is

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$$y(t)=\frac{1}{25^3}\left(\sinh\frac{25}{2}-\sinh 25t\right)+\frac{1}{25^2}\left(t-\frac{1}{2}\right)+ \frac{1}{25^3}\tanh\frac{25}{2}\left(\cosh 25t-\cosh\frac{25}{2}\right) \quad (3)$$

.

Solution by Homotopy Perturbation Method

Transforming equation (1) and the boundary conditions (2) as system of integral equations gives

$$y_1=A+\int_0^x y_2(t)dt;\; y_2=0+\int_0^x y_3(t)dt; \quad y_3=B+\int_0^x\left[25y_1-1\right]dt \quad (4)$$

The equations above can be expressed as

$$\begin{aligned} &y_{10}+py_{11}+p^2y_{12}+p^3y_{13}+\ldots= A+p\int_0^x(y_{20}+py_{21}+p^2y_{22}+p^3y_{23}+\ldots)dt\\ &y_{20}+py_{21}+p^2y_{22}+p^3y_{23}+\ldots= 0+p\int_0^x(y_{30}+py_{31}+p^2y_{32}+p^3y_{33}+\ldots)dt\\ &y_{30}+py_{31}+p^2y_{32}+p^3y_{33}+\ldots=B+ p\int_0^x\left[25\left(y_{10}+py_{11}+p^2y_{12}+p^3y_{13}+\ldots\right)-1\right]dt \end{aligned} \quad (5)$$

Equating the coefficients of equal powers of p, we have the following

$$p^0:\begin{cases}y_{10}=A\\ y_{20}=0\\ y_{30}=B\end{cases};\; p^1:\begin{cases}y_{11}=0\\ y_{21}=Bx\\ y_{31}=-x\end{cases};$$

$$p^2:\begin{cases}y_{12}=\dfrac{Bx^2}{2}\\ y_{22}=-\dfrac{x^2}{2}\\ y_{32}=0\end{cases};\; p^3:\begin{cases}y_{13}=-\dfrac{x^3}{6}\\ y_{23}=0\\ y_{33}=\dfrac{25x^2}{2}\end{cases} \quad (6)$$

and so on

Combining all the first terms, we have

$$y(x)=A+\frac{Bx^2}{2}-\frac{x^3}{6}+\frac{25Bx^4}{24}-\frac{5x^5}{24} +\frac{125Bx^6}{144}-\frac{125x^7}{1008}+\ldots \quad (7)$$

Applying the boundary conditions at $x=1$ for $y'(1)$ and $x=0.5$ for $y(0.5)$ we obtain the system of equations below

$$\begin{aligned}&0.2052915792B+A=-0.02840163584,\\ &14.84064212B=2.928397942\end{aligned} \quad (8)$$

Solving the equations above, we obtain the following

$$A=-0.01210708565,\quad B=0.1973228596 \quad (9)$$

Using (9) in (7) yields the series solution,

$$y(x)=-0.01210708565+ 0.09866142980x^2-\frac{x^3}{6}+0.2055446454x^4- \frac{5x^5}{24}+0.1712872045x^6+\ldots \quad (10)$$

Solution by Modified Adomian Decomposition Method

Writing equation (1) in operator form, yields

$$Ly=25y'-1 \quad (11)$$

Applying L^{-1} the on equation (11), we obtain

$$y=25L^{-1}\left(y'\right)-L^{-1}\left(1\right) \quad (12)$$

Using the boundary conditions we have

$$y(x)=Ax+\frac{x^2B}{2}-\frac{x^3}{6}+25L^{-1}(y') \quad (13)$$

The zeroth component is identified as according the modification made by Wazwaz (2000)

$$y_0=Ax \quad (14)$$

While the remaining recursive relation is written as;

$$y_{n+1}=\frac{x^2B}{2}-\frac{x^3}{6}+25L^{-1}(y_n') \quad (15)$$

$$y_1=\frac{x^2B}{2}-\frac{x^3}{6}+25L^{-1}(y_0')=\frac{x^2B}{2}-\frac{x^3}{6} \quad (16)$$

$$y_2(x)=25L^{-1}(y_1')=\frac{25Bx^4}{24}-\frac{5x^5}{24} \quad (17)$$

$$y_3(x)=25L^{-1}(y_2')=\frac{125Bx^6}{144}-\frac{125x^7}{1008} \quad (18)$$

$$y(x)=A+\frac{Bx^2}{2}-\frac{x^3}{6}+\frac{25Bx^4}{24}-\frac{5x^5}{24}+ \frac{125Bx^6}{144}-\frac{125x^7}{1008}+\frac{3125Bx^8}{8064}+\ldots \quad (19)$$

We now impose the boundary conditions at $x=1$ for $y'(1)$ and $x=0.5$ for $y(0.5)$ to obtain A and B, which gives the system of equations below.

$$\begin{aligned}&A+0.205291579B=0.02840163584\\ &14.84064211B=2.928397941\end{aligned} \quad (20)$$

Solving the system of equations gives the following

$$A=-0.01210708561, B=0.1973228596 \quad (21)$$

Using (21) in (19), we can write the series solution as

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$$y(x) = -0.01210708561 + 0.09866142980x^2 - \frac{x^3}{6} + 0.2055446454x^4 - \frac{5x^5}{24} + 0.1712872045x^6 - \frac{125x^7}{1008} + \ldots \quad (22)$$

Table I: Numerical result for example 1

S/N	EXACT	HPM ERROR	MADM ERROR
0	-0.012107086	3.5211E-11	4.7886E-12
0.1	-0.011268507	3.5368E-11	4.6323E-12
0.2	-0.009222206	3.5897E-11	4.1031E-12
0.3	-0.006466868	3.6995E-11	3.0054E-12
0.4	-0.003320195	3.9038E-11	9.6175E-13
0.5	0	4.267E-11	2.6704E-12
0.6	0.003320195	4.8934E-11	8.934E-12
0.7	0.006466868	5.9484E-11	1.9484E-11
0.8	0.009222206	7.693E-11	3.6929E-11
0.9	0.011268507	1.0542E-10	6.5374E-11
1	0.012107086	1.5215E-10	1.1133E-10

Example 2: We now consider a fifth-order, two-point boundary value problems

$$y^{v}(x) = y - 15e^{x} - 10xe^{x} \quad (23)$$

The boundary conditions are

$$y(0) = 0, y'(0) = 1, y''(0) = 0, \quad y(1) = 0, y'(1) = -e \quad (24)$$

The theoretical solution is given as

$$y(x) = x(1-x)e^{x} \quad (25)$$

Solution by Homotopy Perturbation Method

Transforming equation (23) together with the boundary conditions (24) to a system of integral equations

$$y_1 = 0 + \int_0^x y_2(t)dt; \qquad y_2 = 1 + \int_0^x y_3(t)dt$$

$$y_3 = 0 + \int_0^x y_4(t)dt; \qquad y_4 = A + \int_0^x y_5(t)dt \quad (26)$$

$$y_5 = B + \int_0^x \left(y_1 - 15e^{x} - 10xe^{x}\right)dt$$

Equations (26) can be written in the form

$$y_{10} + py_{11} + p^2y_{12} + p^3y_{13} + \ldots = 0 + p\int_0^x (y_{20} + py_{21} + p^2y_{22} + p^3y_{23} + \ldots)dt;$$

$$y_{20} + py_{21} + p^2y_{22} + p^3y_{23} + \ldots = 1 + p\int_0^x (y_{30} + py_{31} + p^2y_{32} + p^3y_{33} + \ldots)dt;$$

$$y_{30} + py_{31} + p^2y_{32} + p^3y_{33} + \ldots = 0 + p\int_0^x \left(y_{40} + py_{41} + p^2y_{42} + p^3y_{43} + \ldots\right)dt;$$

$$y_{40} + py_{41} + p^2y_{42} + p^3y_{43} + \ldots = A + p\int_0^x \left(y_{50} + py_{51} + p^2y_{52} + p^3y_{53} + \ldots\right)dt;$$

$$y_{50} + py_{51} + p^2y_{52} + p^3y_{53} + \ldots = B + p\int_0^x \left[\begin{array}{l}\left(y_{10} + py_{11} + p^2y_{12} + \ldots\right) - \\ 15e^{x} - 10xe^{x}\end{array}\right]dt \quad (27)$$

Equating the coefficients of like powers of P, we have the following

$$p^0 : \begin{cases} y_{10} = 0 \\ y_{20} = 1 \\ y_{30} = 0 \\ y_{40} = A \\ y_{50} = B \end{cases}; \quad p^1 : \begin{cases} y_{11} = x \\ y_{21} = 0 \\ y_{31} = Ax \\ y_{41} = Bx \\ y_{51} = -15x - \frac{25x^2}{2} - \frac{35x^3}{6} - \frac{15x^4}{8} - \frac{11x^5}{24} - \frac{13x^6}{144} + \ldots \end{cases}$$

$$p^2 : \begin{cases} y_{12} = 0 \\ y_{22} = \frac{Ax^2}{2} \\ y_{32} = \frac{Bx^2}{2} \\ y_{42} = -\frac{15x^2}{2} - \frac{25x^3}{6} - \frac{35x^4}{24} - \frac{3x^5}{8} - \frac{11x^6}{144} + \ldots \\ y_{52} = \frac{x^2}{2}, \text{ etc.} \end{cases} \quad (28)$$

Combining all the first terms, we get

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Table II: Numerical result for example 2

S/N	EXACT	HPM ERROR	MADM ERROR
0	0	0	0
0.1	0.099465383	7.2331E-13	2.8332E-13
0.2	0.195424441	5.17328E-12	2.1333E-12
0.3	0.28347035	1.539E-11	6.7499E-12
0.4	0.358037927	3.15732E-11	1.4933E-11
0.5	0.412180318	5.20873E-11	2.7081E-11
0.6	0.437308512	7.3509E-11	4.3191E-11
0.7	0.422888069	9.09748E-11	6.2854E-11
0.8	0.356086549	9.99623E-11	8.5248E-11
0.9	0.22136428	1.03564E-10	1.0914E-10
1	0	1.37377E-10	1.3289E-10

$$y(x) = x + \frac{Ax^3}{6} + \frac{Bx^4}{24} - \frac{x^5}{8} - \frac{x^6}{30} - \frac{x^7}{144} - \frac{x^8}{896} - \frac{11x^9}{72576} - \frac{x^{10}}{45360} - \dots \quad (29)$$

We proceed as shown above to evaluate the values of A and B by imposing the boundary conditions (56) at $x = 1$, giving rise to system of equations below

$$\begin{aligned} &0.1666914685A + 0.04166942240B = \\ &0.8334297846 \\ &0.5001984148A + 0.1666914683B = \\ &0.1158451633 \end{aligned} \quad (30)$$

The equations yield

$$A = -2.999999995, \quad B = -8.000000019 \quad (31)$$

Applying (31) in (29), gives

$$\begin{aligned} y(x) &= x - 0.499999999x^3 - \\ &0.33333333x^4 - \frac{x^5}{8} - \frac{x^6}{30} - \frac{x^7}{144} - \\ &0.001190476191x^8 - \\ &0.00011736111111x^9 - \frac{x^{10}}{45360} - \dots \end{aligned} \quad (32)$$

Solution by Modified Adomian Decomposition Method

We begin by expressing equation (23) in operator for

$$Ly = y - 15e^x - 10xe^x \quad (33)$$

Applying L^{-1} on equation (65), we obtain

$$y = L^{-1}y - 15L^{-1}\left(e^x\right) - 10L^{-1}\left(xe^x\right) \quad (34)$$

Using the boundary conditions we have

$$\begin{aligned} y(x) &= x + \frac{Ax^3}{6} + \frac{Bx^4}{24} + L^{-1}y - \\ &15L^{-1}\left(e^x\right) - 10L^{-1}\left(xe^x\right) \end{aligned} \quad (35)$$

The zeroth component is identified as according the modification made by Wazwaz (2000)

$$y_0 = x \quad (36)$$

While the remaining recursive relations are written as

$$y_{n+1} = \frac{Ax^3}{6} + \frac{Bx^4}{24} + 25L^{-1}(y_n') \quad (37)$$

$$y_1 = \frac{Ax^3}{6} + \frac{Bx^4}{24} + 25L^{-1}(y_0') \quad (38)$$

$$y_2 = 25L^{-1}(y_1') \quad (39)$$

$$y_3 = 25L^{-1}(y_2') \quad (40)$$

$$\begin{aligned} y(x) &= A + \frac{Bx^2}{2} - \frac{x^3}{6} + \frac{25Bx^4}{24} - \frac{5x^5}{24} + \\ &\frac{125Bx^6}{144} - \frac{125x^7}{1008} + \frac{3125Bx^8}{8064} + \dots \end{aligned} \quad (41)$$

We now impose the boundary conditions at $x = 1$, which gives the system of equations below

$$\begin{aligned} &0.5001984148A + 0.1666914684B = \\ &-0.1158451632 \\ &0.1666914685A + 0.04166942241B = \\ &-0.8334297846 \end{aligned} \quad (42)$$

Solving the equations yields

$$A = -2.999999998, \qquad B = -8.000000003 \quad (43)$$

Substituting for A and B in equation (41), yields

$$\begin{aligned} y(x) &= x - 0.4999999997x^3 - \\ &0.3333333335x^4 - \frac{x^5}{8} - \frac{x^6}{30} - \frac{x^7}{144} - \\ &0.001190476191x^8 - \\ &0.0001736111111x^9 - \frac{x^{10}}{45360} - \dots \end{aligned} \quad (44)$$

III. CONCLUSION

In this work, homotopy perturbation method and modified Adomian decomposition method are applied to obtain the solution of higher order boundary value problems. The homotopy perturbation method iss implemented without linearization, transformation or discretization. Its rapid convergence to the exact solution with few terms without the calculation of Adomian polynomials is its main advantage over modified Adomian decomposition method.

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