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Measures of Entropy based upon Statistical Constants

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***Abstract*---The present article deals with important investigations and developments of some new probabilistic measures of information based upon certain standard statistical constants. The findings of our investigations have been illustrated graphically as well as analytically.**

***Index Terms*--- Entropy, Expansibility, Concavity, Measures of central tendency, Degenerate distributions, Symmetric function.**

1 INTRODUCTION

To quantify the uncertainty content of a random experiment ruled by the probability distribution $P=(p_1, p_2, ..., p_n)$, it was Shannon [15] who for the first time remarked that uncertainty is always associated with every probability distribution and hence it must be a function of probabilities. With this assumption and also with certain desirable postulates, Shannon [15] investigated and developed a measure of uncertainty, today known as entropy function given by

$$H(P) = -\sum_{i=1}^{n} p_i \text{ In } p_i \quad (1)$$

Later on Shannon proved that the entropy function (1.1) is extremely useful in many disciplines of Mathematics and other Sciences. After the invention of Shannon's [15] measure of entropy, Renyi [14] introduced entropy of order α, given by the following expression:

$$H_\alpha(P) = \frac{1}{1-\alpha} In \left(\sum_{i=1}^{n} p_i^{\alpha} \Big/ \sum_{i=1}^{n} p_i \right), \alpha \neq 1, \alpha > 0 \quad (2)$$

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Recently, Parkash and Kakkar [10, 11] investigated and developed the following parametric and non-parametric measures of entropy, the applications of which have been provided to coding theory.

$$M(P) = \prod_{i=1}^{n} \left(\frac{1}{p_i} \right)^{p_i} - 1 \quad (3)$$

$$M_\alpha(P) = \frac{1}{1-\alpha} \left(\prod_{i=1}^{n} \left(\frac{1}{p_i} \right)^{p_i(1-\alpha)} - 1 \right), \alpha > 0, \alpha \neq 1 \quad (4)$$

Moreover, keeping in view the application areas of different measures of entropy, Parkash and Mukesh [12, 13] introduced the following generalized measures of entropy

$$H_\alpha(P) = \frac{1}{\alpha} \sum_{i=1}^{n} \left(1 - p_i^{\alpha p_i} \right), \quad \alpha \neq 0, \alpha > 0 \quad (5)$$

$$H_\alpha(P) = \frac{1}{2(1-\alpha)} \sum_{i=1}^{n} \left[\alpha \left(p_i^{1-\alpha} - 1 \right) + \log p_i^{1-\alpha} \right], \quad \alpha \neq 1, \alpha > 0 \quad (6)$$

And provided their applications in the fields of Statistics and Operations Research.

Some applications of entropy measures in the field of queueing theory have been provided by Buttar [4]. The other establishments regarding the probabilistic entropic models have been provided by Kapur [6, 7], Herremoes [5], Nanda and Paul [9], Sharma and Taneja [16], Cohen and Merlino [3], Chakrabarti [2], Lavenda [8] etc.

It is worth mentioning here that in most of the Biological Sciences, researchers usually measure diversity and equitability of different communities. Some of the pioneers enthusiastically involved in the study are Shannon [15], Renyi [14], Simpson [17], Weiner [18] etc. It has been realized that Shannon's measure is most widely applicable and possesses many interesting and desirable properties but still there is a need for developing new measures to extend the scope of their applications. In section II, we have investigated and introduced some new probabilistic measures of information based upon certain statistical constants.

II SOME NEW MEASURES OF ENTROPY BASED UPON STATISTICAL CONSTANTS

A. Information Measure in terms of Measures of Central Tendency

Let a random variable X takes the values $x_1, x_2, ..., x_n$. Then, geometric mean G, arithmetic mean M and harmonic mean H of these n observations are given by:

$$G = \left(x_1 x_2 x_3 ... x_n\right)^{\frac{1}{n}},\ x_i \geq 0 \tag{7}$$

$$M = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{8}$$

$$H = n / \left(\tfrac{1}{x_1} + \tfrac{1}{x_2} + ... + \tfrac{1}{x_n}\right) \tag{9}$$

Equations (7), (8) and (9) can be rewritten as

$$G = \left(p_1 p_2 p_3 ... p_n\right)^{\frac{1}{n}} . \sum_{i=1}^{n} x_i \tag{10}$$

where $p_i = \dfrac{x_i}{\sum_{i=1}^{n} x_i}$,

Also $nM = \sum_{i=1}^{n} x_i$ (11)

$$H = \frac{n\sum_{i=1}^{n} x_i}{\frac{\sum_{i=1}^{n} x_i}{x_1} + \frac{\sum_{i=1}^{n} x_i}{x_2} + ... + \frac{\sum_{i=1}^{n} x_i}{x_n}}$$

or

$$\frac{M}{H} = \frac{\sum_{i=1}^{n} \frac{1}{p_i}}{n^2} \tag{12}$$

Again equation (10) can be written as

$$\frac{G}{M} = n\left(p_1 p_2 p_3 ... p_n\right)^{\frac{1}{n}}$$

$$\sum_{i=1}^{n} \log p_i = n \log\left(\frac{G}{nM}\right) \tag{13}$$

From equations (12) and (13), we get

$$n \log\left(\frac{G}{nM}\right) - \frac{M}{H} = \sum_{i=1}^{n} \log p_i - \frac{\sum_{i=1}^{n} \frac{1}{p_i}}{n^2} \tag{14}$$

Next, we demonstrate the proposal of an information theoretic measure involving geometric mean G, arithmetic mean M and harmonic mean H. This measure is given by

$$\phi_n(P) = \sum_{i=1}^{n} \log p_i - \frac{\sum_{i=1}^{n} \frac{1}{p_i}}{n^2},\ n \geq 2,\ 0 < p_i < 1 \tag{15}$$

To prove that (15) represents an information measure, we study its essential properties as follows.

(i) $\phi_n(P)$ is permutationally symmetric.

(ii)Since $\dfrac{1}{p_i}$ is continuous function for 0 < p_i < 1 , $\phi_n(P)$ is also continuous everywhere in the same interval.

(iii) Since 0 < p_i < 1, $\phi_n(P) < 0$. This property of negativity is due to the fact that the entropy function (15) involves Burg's[1] entropy which always gives negative value.

(iv) Determination of Maximum Value

Let us consider the following function:

$$L = \sum_{i=1}^{n} \log p_i - \frac{\sum_{i=1}^{n} \frac{1}{p_i}}{n^2} - \lambda\left(\sum_{i=1}^{n} p_i - 1\right)$$

$$\frac{\partial L}{\partial p_1} = \frac{1}{p_1} + \frac{1}{n^2 {p_1}^2} - \lambda = \frac{1}{n^2 {p_1}^2}\left\{1 + n^2 p_1\right\} - \lambda$$

Similarly,

$$\frac{\partial L}{\partial p_2} = \frac{1}{p_2} + \frac{1}{n^2 {p_2}^2} - \lambda = \frac{1}{n^2 {p_2}^2}\left\{1 + n^2 p_2\right\} - \lambda$$

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$$\frac{\partial L}{\partial p_n} = \frac{1}{p_n} + \frac{1}{n^2 {p_n}^2} - \lambda = \frac{1}{n^2 {p_n}^2}\left\{1 + n^2 p_n\right\} - \lambda$$

Thus $\dfrac{\partial L}{\partial p_1} = \dfrac{\partial L}{\partial p_2} = \dfrac{\partial L}{\partial p_3} = ... \dfrac{\partial L}{\partial p_n} = 0$, gives

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$$\frac{1}{n^2 p_1^{\,2}}\left\{1+n^2 p_1\right\}-\lambda=\frac{1}{n^2 p_2^{\,2}}\left\{1+n^2 p_2\right\}-\lambda=\ldots=$$

$$\frac{1}{n^2 p_n^{\,2}}\left\{1+n^2 p_n\right\}-\lambda$$

which further gives

$$\frac{1}{n^2 p_1^{\,2}}\left\{1+n^2 p_1\right\}=\frac{1}{n^2 p_2^{\,2}}\left\{1+n^2 p_2\right\}$$

$$..=\frac{1}{n^2 p_n^{\,2}}\left\{1+n^2 p_n\right\}$$

implying that $p_1 = p_2 = ... = p_n$.

Also using $\sum_{i=1}^{n} p_i = 1$, we get $p_i = \frac{1}{n}$ showing that the maximum value occurs at the uniform distribution. Further, the maximum value is given by

$$\left[\phi_n(P)\right]_{\max} = -\log n - 1 = f(n) \text{ (say)}$$

Also $f^{'}(n) = -\frac{1}{n} < 0 \quad \forall n \geq 2$

Hence, the maximum value is a decreasing function of n and it resembles with Burg's [1] measure of entropy.

(v)We have $\phi_n^{'}(p) = \frac{1}{p_i} + \frac{1}{n^2 p_i^2}$ and

$$\phi_n^{''}(p) = -\left[\frac{1}{p_i^2} + \frac{2}{n^2 p_i^{\,3}}\right] < 0 \;\; \forall i$$

Thus $\phi_n(p)$ is a concave function of $p_1, p_2, ..., p_n$.

Thus, we notice that $\phi_n(P)$ satisfies all the essential properties proving that $\phi_n(P)$ is a new measure of information.

we have presented $\phi_n(P)$ graphically in Fig.-1 which obviously shown that $\phi_n(P)$ is a concave function.

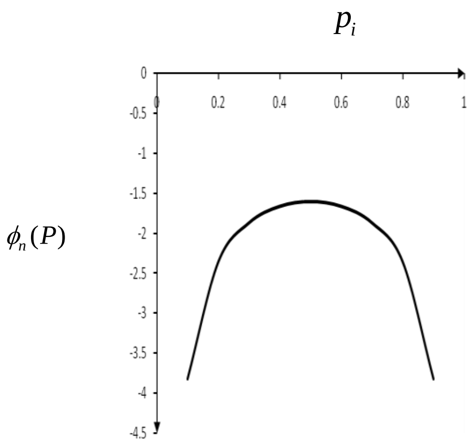

Fig.-1: Concavity of $\phi_n(P)$

B. Information Measure in terms of Measures of Dispersion and Central Tendency

The variance of a discrete distribution of n observations $(x_1, x_2, ..., x_n)$ is defined as

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^{\,2} - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2 \tag{16}$$

The above equation (16) can be rewritten as

$$\sigma^2 = \left[\frac{\left(x_1^{\,2}+x_2^{\,2}+...+x_n^{\,2}\right) - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2}{n\left(\sum_{i=1}^{n} x_i\right)^2}\right]\left(\sum_{i=1}^{n} x_i\right)^2$$

$$=\frac{1}{n}\left[\left(p_1^{\,2}+p_2^{\,2}+...+p_n^{\,2}\right)-\frac{1}{n}\right](nM)^2$$

where $p_i = \frac{x_i}{\sum_{i=1}^{n} x_i}$.

Thus, we have

$$\frac{\sigma^2 + M^2}{nM^2} = \sum_{i=1}^{n} p_i^{\,2} \tag{17}$$

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Adding equations (12) and (17), we get

$$-\left\{\frac{\sigma^2+M^2}{nM^2}+\frac{M}{H}\right\}=-\left\{\sum_{i=1}^{n} p_i^{\,2}+\frac{\sum_{i=1}^{n}\frac{1}{p_i}}{n^2}\right\} \qquad (18)$$

Now, we initiate the development of an information theoretic measure depending upon arithmetic mean M, harmonic mean H and variance σ^2. This measure is given by

$$\psi_n(P)=-\left\{\sum_{i=1}^{n} p_i^{\,2}+\frac{\sum_{i=1}^{n}\frac{1}{p_i}}{n^2}\right\},\ n\geq 2, 0<p_i<1 \qquad (19)$$

We shall prove that the R.H.S. of equation (19) is an information measure.

To prove this, we study its following properties:

(i) Obviously $\psi_n(P)\leq 0$

(ii) $\psi_n(P)$ is continuous function of $p_i \ \forall \ 0<p_i<1$

(iii) $\psi_n(P)$ is permutationally symmetric function of $p_i \ \forall \ 0<p_i<1$

(iv) Maximum Value**:** To obtain, the maximum value of the entropy measure (19), we consider the following Lagrange's function:

$$L=-\left\{\sum_{i=1}^{n} p_i^{\,2}+\frac{\sum_{i=1}^{n}\frac{1}{p_i}}{n^2}\right\}-\lambda\left(\sum_{i=1}^{n} p_i-1\right).$$

Thus, we have

$$\frac{\partial L}{\partial p_1}=-2p_1+\frac{1}{n^2 p_1^{\,2}}-\lambda$$

Similarly,

$$\frac{\partial L}{\partial p_2}=-2p_2+\frac{1}{n^2 p_2^{\,2}}-\lambda,\ldots,$$

$$\frac{\partial L}{\partial p_n}=-2p_n+\frac{1}{n^2 p_n^{\,2}}-\lambda$$

For maximum value, we put

$$\frac{\partial L}{\partial p_1}=\frac{\partial L}{\partial p_2}=\frac{\partial L}{\partial p_3}=\ldots\frac{\partial L}{\partial p_n}=0,$$

which gives

$$-2p_1+\frac{1}{n^2 p_1^{\,2}}=-2p_1+\frac{1}{n^2 p_1^{\,2}}=\ldots=-2p_1+\frac{1}{n^2 p_1^{\,2}}$$

The above relation implies that $p_1=p_2=\ldots\ldots=p_n$.

Also using $\sum_{i=1}^{n} p_i=1$, we get $p_i=\frac{1}{n}$. Thus, the maximum value arises when the distribution is uniform.

(v) Concavity: To study its concavity, we have

$$\psi_n^{'}(P)=-2p_i+\frac{1}{n^2 p_i^{\,2}}$$

Also,

$$\psi_n^{''}(P)=-\frac{2}{n^2 p_i^3}-2<0$$

which shows that $\psi_n(P)$ is concave and with these essential properties, we conclude that $\psi_n(P)$ is another new measure of information.

Next, we have presented the values of $\psi_n(P)$ and obtained the Fig.- 2 which shows that the measure introduced in equation (19) is a concave function.

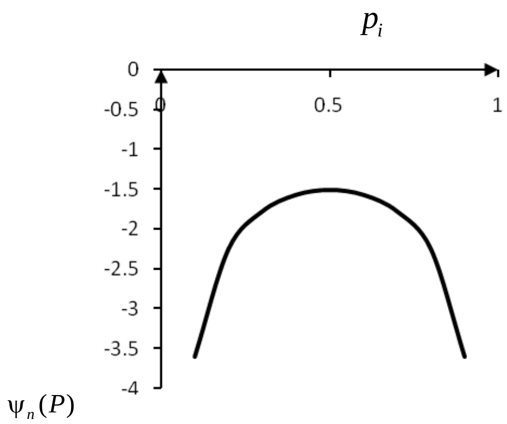

Fig- 2: Concavity of $\psi_n(P)$

III. CONCLUSION

The known fact that information theory deals with a variety of measures of entropy implies that we have to develop those measures which can be successfully applied to a diversity of mathematical disciplines. Moreover, if we have a variety of information measures, we shall be more bendable in applying a standard measure depending upon the situation of its applications. The idea has enforced us for the developed of some new measures and consequently, for the known values of some statistical

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constants, we have investigated and projected many new probabilistic information theoretic measures of entropy.

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